Multiplication of the distributions $(x \pm i0)^z$

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Abstract

In previous work of the author, a convolution and multiplication product for the set of Associated Homogeneous Distributions (AHDs) with support in $\mathbb{R}$ was defined and fully investigated. Here this definition is used to calculate the multiplication product of homogeneous distributions of the form $(x \pm i0)^z$, $\forall z \in \mathbb{C}$.

Multiplication products of AHDs generally contain an arbitrary constant if the resulting degree of homogeneity is a negative integer, i.e., is a critical product. However, critical products of the forms $(x+i0)^a . (x+i0)^b$ and $(x-i0)^a . (x-i0)^b$, with $a+b \in \mathbb{Z}_-$, are exceptionally unique. This fact combined with Sokhotskii-Plemelj expressions then leads to linear dependences of the arbitrary constants occurring in products like $\delta^{(k)}.\delta^{(l)}$, $\eta^{(k)}.\delta^{(l)}$, $\delta^{(k)}.\eta^{(l)}$ and $\eta^{(k)}.\eta^{(l)}$, $\forall k,l \in \mathbb{N}$ ($\eta \triangleq \frac{1}{\pi}x^{-1}$). This in turn gives a unique distribution for products like $\delta^{(k)}.\eta^{(l)} + \eta^{(k)}.\delta^{(l)}$ and $\delta^{(k)}.\delta^{(l)} - \eta^{(k)}.\eta^{(l)}$. The latter two products are of interest in quantum field theory, for instance in products of the partial derivatives of the zero-mass two-point Wightman distribution.

Keywords. Generalized function, Associated homogeneous distribution, Multiplication, Wightman distribution, Quantum field theory.

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1 Introduction

In quantum physics one finds the need to evaluate δ^2, when calculating the transition rates of certain particle interactions, [19]. The problem of defining products of distributions is also closely connected with the problem of renormalization in quantum field theory. The propagators of quantum field theory are well-known distributional fundamental solutions, which enter as products in the perturbation expansion of the S-matrix, [8]. It is widely agreed that the so called divergences of the S-matrix in quantum physics are rooted in the lack of a rigorous definition of a multiplication product for these distributions, [21], [5, pp. 266–267, 285–314]. Multiplication of the particular distributions $(x \pm i0)^{-1}$ is of interest in quantum field theory, where they appear for instance in products of the partial derivatives of the zero-mass two-point Wightman distribution, [25, p. 18, p. 79], [6, section 12.1].

In this paper, we investigate, exhibit and explain the somewhat exceptional nature of the multiplication products of the distributions $(x \pm i0)^z$, $\forall z \in \mathbb{C}$, using a new definition for the product of Associated Homogeneous Distributions (AHDs), [20], [26], [9]. Homogeneous Distributions (HDs) are the distributional analogue of homogeneous functions, such as $|x|^z : \mathbb{R} \to \mathbb{C}$. The distributions $(x \pm i0)^z$ are homogeneous with complex degree z. Associated to homogeneous functions are power-log functions, which arise when taking the derivative with respect to the degree of homogeneity z. The set of AHDs with support in the real line $\mathbb{R}$, and which we denote by $\mathcal{H}'(\mathbb{R})$, is the distributional analogue of the set of power-log functions with domain in $\mathbb{R}$. The set $\mathcal{H}'(\mathbb{R})$ is an interesting and important proper subset of the distributions $\mathcal{S}'(\mathbb{R})$ of slow growth (or tempered distributions), [27], [29], [22], and this for two reasons: (i) $\mathcal{H}'(\mathbb{R})$ contains the majority of the (one-dimensional) distributions one typically encounters in physics applications, including the δ and $\eta \triangleq \frac{1}{\pi}x^{-1}$ distributions (the symbol $\triangleq$ denotes definition) and (ii) $\mathcal{H}'(\mathbb{R})$ is closed under Fourier transformation. A comprehensive study of the set $\mathcal{H}'(\mathbb{R})$ by the author has led to a convolution and multiplication algebra for AHDs on $\mathbb{R}$, [9]–[16]. For an easy introduction and overview of this construction, see [17]. We use the notation and definitions set forth in [9].

The author's definition of a multiplication product for AHDs on $\mathbb{R}$ has the following interesting features.

(i) The product of certain distributions of the set $\mathcal{H}'(\mathbb{R})$ exists in the author's sense, while their product does not exist in the sense of the classical method of delta nets and passage to the limit, [1], or the method of analytic regularization and passage to the boundary (a simplified form of the method of hyperfunctions), [4]. For instance, it is now possible to give a rigorous meaning to distributional multiplication products such as δ^2 and η^2.

(ii) It exhibits on the one hand the natural non-uniqueness of critical multiplication products in general and on the other hand the exceptional uniqueness of particular critical multiplication products (e.g., those whose factors are of the form $D_z^m (x \pm i0)^z$ and are of like sign, $\forall m \in \mathbb{N}$ and $\forall z \in \mathbb{C}$). Product values obtained by classical methods are often particular values of our more general multiple-valued results. Exceptional results, such as (42) and (43) below, are of paramount importance since they provide a rigorous basis for the (ad hoc assumed) uniqueness of certain products used in quantum field theory.

(iii) Non-critical multiplication products are always unique and such products are commutativity and associative.

(iv) The adopted definition of a critical product as an extension from a partial distribution to a distribution leads naturally to non-commutative critical multiplication products, [17] and critical triple products are generally non-associative, [13] and [14].

(v) Both the non-commutativity as the non-associativity behaves in a simple and interesting way, [17]. Non-associativity cannot be avoided, this being a consequence of Schwartz' "impossibility theorem", [28], and a similar theorem for the convolution product, [16].

(vi) The author's approach remains entirely within the scope of Schwartz' distributions and thus provides a simpler alternative for the multiplication of AHDs on $\mathbb{R}$ than the larger generalized function algebras, such as e.g., [7].

(vii) Although the author's multiplication product is (so far) only defined for AHDs, this is not a serious drawback concerning applications, as this subset of distributions contains the majority of the physically important (one-dimensional) distributions.

The next section introduces some basic definitions pertaining to AHDs and formally presents the adopted multiplication product. In the third section, multiplication products of the homogeneous distributions $(x \pm i0)^z$ are calculated for all degrees of homogeneity z. We give in the last section a number of particular results for negative integer degrees of homogeneity and compare some of them with results obtained by other authors in the past. Special attention is drawn to the non-uniqueness of the considered critical distributional products in general and to the exceptional uniqueness of certain critical distributional products in particular.

2 Preliminaries

Definition 1 *A distribution $f_m^z \in \mathcal{D}'(\mathbb{R})$ is called an associated (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ and order of association $m \in \mathbb{Z}_+$, iff there exists a sequence of associated homogeneous distributions f_{m-l}^z of degree of homogeneity z and associated order $m-l$, $\forall l \in \mathbb{Z}_{[1,m]}$, not depending on r and with $f_0^z \neq 0$, satisfying,*

$$\langle f_m^z, \varphi(x/r) \rangle = r^{z+1} \left\langle f_m^z + \sum_{l=1}^{m} \frac{(\ln r)^l}{l!} f_{m-l}^z, \varphi(x) \right\rangle, \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Definition 2 *A* partial tempered distribution *is a linear and sequentially continuous functional that is only defined on a proper subset $\mathcal{S}_r(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.*

Definition 3 *An* extension f_e from $\mathcal{S}_r$ to $\mathcal{S}$*, of a partial tempered distribution f, is an element of $\mathcal{H}'(\mathbb{R})$ defined $\forall \varphi \in \mathcal{S}(\mathbb{R})$ and such that $\langle f_e, \psi \rangle = \langle f, \psi \rangle$, $\forall \psi \in \mathcal{S}_r(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.*

Let f^a and g^b be AHDs on $\mathbb{R}$ of degree a and b, respectively. Then, $\mathcal{F}^{-1} f^a$ and $\mathcal{F}^{-1} g^b$ are also AHDs on $\mathbb{R}$ of degree $-(a+1)$ and $-(b+1)$, respectively, [9, Section 4.4 (1)]. Owing to the results obtained in [11] and [12], $(\mathcal{F}^{-1} f^a) * (\mathcal{F}^{-1} g^b)$ exists in $\mathcal{H}'(\mathbb{R})$, either directly as a distribution or as an extension of a partial distribution, and has degree $-a-b-1$. This naturally prompts the following.

Definition 4 *The multiplication product of $f, g \in \mathcal{H}'(\mathbb{R})$, denoted $f.g$, is given by*

$$f.g \triangleq \mathcal{F}\left((\mathcal{F}^{-1} f) * (\mathcal{F}^{-1} g) \right). \tag{1}$$

Then $f^a.g^b$ is again an AHD on $\mathbb{R}$ of degree of homogeneity $a+b$.

Definition 5 *(i) The convolution product of two AHDs on* $\mathbb{R}$ *of degrees* $a-1$ *and* $b-1$ *is called a* critical convolution product *iff the resulting degree* $a+b-1 \triangleq k \in \mathbb{N}$. *(ii) The multiplication product of two AHDs on* $\mathbb{R}$ *of degrees* a *and* b *is called a* critical multiplication product *iff the resulting degree* $a+b \triangleq l \in \mathbb{Z}_-$.

Defining multiplication in terms of convolution, as in (1), is of course not new, nor is anything gained. The novelty of our approach however lies in the way how closure is achieved for the convolution product in $\mathcal{H}'(\mathbb{R})$.

(i) If $\mathcal{F}^{-1}f^{a-1}$ and $\mathcal{F}^{-1}g^{b-1}$ have both one-sided support, bounded at the same side, then the convolution in (1) is calculated according to the standard definition involving the direct product, e.g., [29, p. 123, eq. (2) and Theorem 5.4-1].

(ii) If $\mathcal{F}^{-1}f^{a-1}$ and $\mathcal{F}^{-1}g^{b-1}$ have general support, but $0 < \mathrm{Re}(a)$, $0 < \mathrm{Re}(b)$ and $\mathrm{Re}(a+b) < 1$, then their convolution still exists as a distribution, which can be obtained from the ordinary convolution integral. This distribution is subsequently further extended by analytic continuation to all $a, b \in \mathbb{C}$: $a+b-1 \notin \mathbb{N}$, [11].

(iii) For any critical convolution product, i.e., a pair $\mathcal{F}^{-1}f^{a-1}$ and $\mathcal{F}^{-1}g^{b-1}$ with $a+b-1 = k \in \mathbb{N}$, their convolution generally results in a partial distribution $\left(\mathcal{F}^{-1}f^{a-1}\right) * \left(\mathcal{F}^{-1}g^{b-1}\right)$, which turns out to be only defined on a proper subset $\mathcal{S}_{\{k\}}(\mathbb{R})$ (functions having zero k-th order moment) of the space $\mathcal{S}(\mathbb{R})$ of rapidly decaying test functions, [12, Section 2]. In this case, $\left(\mathcal{F}^{-1}f^{a-1}\right) * \left(\mathcal{F}^{-1}g^{b-1}\right)$ in (1) is defined as any extension from $\mathcal{S}_{\{k\}}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ of the partial distribution $\left(\mathcal{F}^{-1}f^{a-1}\right) * \left(\mathcal{F}^{-1}g^{b-1}\right)$ to a distribution in $\mathcal{H}'(\mathbb{R})$, [12, eq. (69)]. An extension of a partial distribution to a distribution within $\mathcal{H}'(\mathbb{R})$ is denoted by the subscript e . Such an extension is in general not unique and will involve (for AHDs on $\mathbb{R}$) a single arbitrary constant in the expression of the product.

For instance, the convolution products $x^n * x^m$, $\forall n, m \in \mathbb{N}$, are critical and defined as the extensions $\left(x^n * x^m\right)_e = 0 + cx^{n+m+1}$ of the partial zero distribution 0 defined on $\mathcal{S}_{\{n+m+1\}}(\mathbb{R})$, with $c \in \mathbb{C}$ arbitrary (the details of this example are given in the Appendix and also in [13, Appendix A]). Therefore,

$$x^n * x^m = cx^{n+m+1}. \tag{2}$$

Before continuing, it is interesting to notice that (1) is a natural definition that extends various more specialized and well-known product definitions. It coincides with the convolution theorem in case one of the distributions $\mathcal{F}^{-1}f^a$ or $\mathcal{F}^{-1}g^b$ is in $\mathcal{E}'(\mathbb{R})$, [29, p. 206], or more generally is in $\mathcal{O}'_C(\mathbb{R})$, [27, vol. II, p. 124]. Then, (1) also coincides with the multiplication defined in [9, eq. (24)], since one of the distributions f^a or g^b will be in $\mathcal{Z}_M(\mathbb{R})$ or in $\mathcal{O}_M(\mathbb{R})$. We see from [10, e.g., Theorem 6] that if $-1 < \mathrm{Re}(a)$ and $-1 < \mathrm{Re}(b)$, then f^a and g^b are regular AHDs. If in addition $-1 < \mathrm{Re}(a+b)$, $\mathcal{F}\left(\left(\mathcal{F}^{-1}f^a\right) * \left(\mathcal{F}^{-1}g^b\right)\right)$ is also a regular AHD. Then it follows from the generalized convolution theorem, that definition (1) coincides with definition [9, eq. (23)] for the product of two regular distributions.

On the one hand, the multiplication products $(x-i0)^a \,.\, (x-i0)^b$ and $(x+i0)^a \,.\, (x+i0)^b$ (using (1)) exist exceptionally as distributions, despite being critical products, instead of as partial distributions. This is a consequence of the fact that (i) $\mathcal{F}^{-1}(x-i0)^z \in \mathcal{D}'_R(\mathbb{R})$ (distributions having support bounded on the left) and $\mathcal{F}^{-1}(x+i0)^z \in \mathcal{D}'_L(\mathbb{R})$ (distributions having support bounded on the right) and (ii) that $\left(\mathcal{D}'_{L,R}(\mathbb{R}), *\right)$ are commutative monoids. Hence, no extension process is necessary for products with factors of like sign and such critical products are thus exceptionally unique. In addition, multiplication products of distributions of like sign inherit commutativity and associativity from $\left(\mathcal{D}'_{L,R}(\mathbb{R}), *\right)$. Further, it will be shown in the last section that critical multiplication products of the form $(x+i0)^{-k} \,.\, (x+i0)^{-l}$ and $(x-i0)^{-k} \,.\, (x-i0)^{-l}$, $\forall k, l \in \mathbb{Z}_+$, when combined with Sokhotskii-Plemelj expressions, induce a unique distribution for products like $\delta^{(k-1)}.\eta^{(l-1)} + \eta^{(k-1)}.\delta^{(l-1)}$ and $\delta^{(k-1)}.\delta^{(l-1)} - \eta^{(k-1)}.\eta^{(l-1)}$.

On the other hand, multiplication products of mixed sign $(x-i0)^a \,.\, (x+i0)^b$ follow the general rule. In particular, critical multiplication products of the form $(x-i0)^{-k} \,.\, (x+i0)^{-l}$ and $(x+i0)^{-k} \,.\, (x-i0)^{-l}$ are derived, and they provide insight in the origin of the non-commutativity of products like $\delta^{(k-1)}.\eta^{(l-1)}$ and the occurrence of an arbitrary constant in their products.

We end this section by stating the definitions of some particular AHDs, which are closely related to the distributions $(x \pm i0)^z$.

The distributions $(x \pm i0)^z$ play the role of complex basis HDs in the complex representation structure theorem of AHDs, [10, Theorem 2]. They are defined by, [9, eq. (208)], [20, p. 59],

$$(x \pm i0)^z \triangleq x_+^z + e^{\pm i\pi z} x_-^z,$$

wherein $x_\pm^z \triangleq |x|^z\, 1_\pm$ and $1_\pm$ are the regular (Heaviside) step distributions (generated by the characteristic functions of the half-lines). The distributions $(x \pm i0)^z$ are (i) entire in their degree of homogeneity z and (ii) linear dependent iff $z \in \mathbb{N}$ since $(x \pm i0)^k = x^k$, $\forall k \in \mathbb{N}$.

The distributions $\Phi_\pm^z$ play the role of normalized half-lines basis HDs in the normalized half-lines representation structure theorem of AHDs, [10, Theorem 1]. They are defined by, [9, eq. (242)], [20, p. 115],

$$\Phi_\pm^z \triangleq \frac{x_\pm^{z-1}}{\Gamma(z)}.$$

The $\Phi_\pm^{z+1}$ are (i) also entire in their degree of homogeneity z and (ii) linear dependent iff $z \in \mathbb{Z}_-$ since $\Phi_\pm^{-k+1} = (\pm 1)^{k-1}\, \delta^{(k-1)}$, $\forall k \in \mathbb{Z}_+$.

The distributions defined by, $\forall k \in \mathbb{N}$,

$$\eta^{(k)} \triangleq \frac{1}{\pi}(-1)^k k!\, x^{-(k+1)}, \tag{3}$$

are generalized derivatives of the eta distribution $\eta \triangleq \frac{1}{\pi}x^{-1}$, [9, eq. (172)]. The distribution x^{-1}, sometimes written as the pseudo-function Pf $\frac{1}{x}$ or as Cauchy's principal value Pv $\frac{1}{x}$, is defined by

$$\left\langle x^{-1}, \varphi \right\rangle \triangleq \int_0^{+\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx.$$

We will finally need the Fourier transformation pairs ([9, eqs. (277) and (237)], [20, p. 172]), $\forall z \in \mathbb{C}$,

$$\mathcal{F}\, \Phi_\pm^{-z} = (\pm 2\pi i)^z\, (x \mp i0)^z\,, \tag{4}$$

$$\mathcal{F}^{-1}\, (x \pm i0)^z = (\mp 2\pi i)^{-z}\, \Phi_\mp^{-z}. \tag{5}$$

3 Multiplication products of $(x \pm i0)^z$

3.1 Case $(x + i0)^a \,.\, (x + i0)^b$

3.1.1 For all $a, b \in \mathbb{C}$

Applying the definition of multiplication product (1), substituting herein (5), using $\Phi_-^{-a} * \Phi_-^{-b} = \Phi_-^{-(a+b)}$ holding $\forall a, b \in \mathbb{C}$ (see [9, eq. (248)], [20, p. 116]), and backsubstituting (4) gives, $\forall a, b \in \mathbb{C}$,

$$(x + i0)^a \,.\, (x + i0)^b = (x + i0)^{a+b}\,. \tag{6}$$

At $a+b \in \mathbb{Z}_-$, eq. (6) is an example of a critical product which value is exceptionally a unique distribution.

In particular, $\forall z \in \mathbb{C}$ and $\forall n \in \mathbb{Z}_+$, (6) gives

$$\left((x + i0)^z\right)^n = (x + i0)^{nz}\,.$$

This result suggests the natural definition, $\forall z \in \mathbb{C}$,

$$\left((x + i0)^z\right)^0 \triangleq (x + i0)^0 = 1.$$

3.1.2 The particular case $a, b \in \mathbb{Z}_-$

From (6) follows, $\forall k, l \in \mathbb{Z}_+$,

$$(x + i0)^{-k} \,.\, (x + i0)^{-l} = (x + i0)^{-(k+l)}\,,$$

which, after substitution of the Sokhotskii-Plemelj equations (e.g., [9, eq. (213)]),

$$(x \pm i0)^{-n} = \mp i\pi \frac{(-1)^{n-1}}{(n-1)!}\left(\delta^{(n-1)} \pm i\eta^{(n-1)}\right), \tag{7}$$

becomes

$$\begin{aligned} &\frac{\delta^{(k-1)}}{(k-1)!} . \frac{\delta^{(l-1)}}{(l-1)!} + i\left(\frac{\delta^{(k-1)}}{(k-1)!} . \frac{\eta^{(l-1)}}{(l-1)!} + \frac{\eta^{(k-1)}}{(k-1)!} . \frac{\delta^{(l-1)}}{(l-1)!}\right) - \frac{\eta^{(k-1)}}{(k-1)!} . \frac{\eta^{(l-1)}}{(l-1)!} \\ = \;& \frac{1}{\pi i} \frac{\delta^{(k+l-1)}}{(k+l-1)!} + \frac{1}{\pi i} i \frac{\eta^{(k+l-1)}}{(k+l-1)!}. \end{aligned} \tag{8}$$

We now use the results (62), (65), (68) and (71), derived in the Appendix, which we here combine in the form

$$\frac{\delta^{(k-1)}}{(k-1)!}.\frac{\delta^{(l-1)}}{(l-1)!} = \frac{1}{\pi i}c_1\frac{\delta^{(k+l-1)}}{(k+l-1)!}, \tag{9}$$

$$i\left(\frac{\delta^{(k-1)}}{(k-1)!}.\frac{\eta^{(l-1)}}{(l-1)!}+\frac{\eta^{(k-1)}}{(k-1)!}.\frac{\delta^{(l-1)}}{(l-1)!}\right) = \frac{1}{\pi i}c_2^+\frac{\delta^{(k+l-1)}}{(k+l-1)!}, \tag{10}$$

$$\frac{\eta^{(k-1)}}{(k-1)!}.\frac{\eta^{(l-1)}}{(l-1)!} = \frac{1}{\pi i}\left(c_3\frac{\delta^{(k+l-1)}}{(k+l-1)!}-i\frac{\eta^{(k+l-1)}}{(k+l-1)!}\right), \tag{11}$$

with $c_1, c_2^+, c_3 \in \mathbb{C}$ arbitrary. Result (2) (or the equivalent eq. (61)), which underlies (9), was already discussed as example in [13, Appendix A].

Substituting (9)–(11) in (8) yields

$$c_1 + c_2^+ - c_3 = 1, \tag{12}$$

a first relation showing that the arbitrary constants in the multiplication products of the derivatives of delta and eta distributions are linearly related.

3.2 Case $(x-i0)^a . (x+i0)^b$

Applying our definition of multiplication product (1) and substituting (5) gives, $\forall a, b \in \mathbb{C}$ and with $d \triangleq b-a$,

$$(x-i0)^a . (x+i0)^b = (2\pi)^{-(a+b)}\, e^{+i(\pi/2)d}\mathcal{F}\left(\Phi_+^{-a} * \Phi_-^{-b}\right). \tag{13}$$

3.2.1 $\forall a, b \in \mathbb{C} : a+b \notin \mathbb{Z}$

Substituting the following result, holding $\forall a, b \in \mathbb{C}$ such that $a+b \notin \mathbb{Z}$ (see [11, eq. (13)]),

$$\Phi_+^{-a} * \Phi_-^{-b} = \frac{\sin(\pi a)}{\sin(\pi(a+b))}\Phi_+^{-(a+b)} + \frac{\sin(\pi b)}{\sin(\pi(a+b))}\Phi_-^{-(a+b)},$$

in (13) and backsubstituting (4) gives, $\forall a, b \in \mathbb{C}$ such that $a+b \notin \mathbb{Z}$,

$$(x-i0)^a . (x+i0)^b = \frac{\sin(\pi a)e^{+i\pi b}}{\sin(\pi(a+b))}(x-i0)^{a+b} + \frac{e^{-i\pi a}\sin(\pi b)}{\sin(\pi(a+b))}(x+i0)^{a+b}, \tag{14}$$

$$= x_+^{a+b} + e^{+i\pi d}x_-^{a+b}, \tag{15}$$

$$= e^{+i(\pi/2)d}\left(\cos(d\pi/2)\,|x|^{a+b} - i\sin(d\pi/2)\,|x|^{a+b}\,\mathrm{sgn}\right). \tag{16}$$

The forms (15)–(16) still hold if $a+b \in \mathbb{N}$.

3.2.2 $\forall a, b \in \mathbb{C} : a+b = -n \in \mathbb{Z}_-$ (critical product)

In the result [12, eq. (69) with $m = 0 = n$], together with [12, eqs. (58) and (11)] and [9, eqs. (247) and (270)], holding $\forall a, b \in \mathbb{C} : a+b = n \in \mathbb{Z}_+$ and with $d \triangleq b-a$, interchange $a \leftrightarrow b$ and substitute $-a$ for a and $-b$ for b. Then reverse the order of the factors in the convolution product and use the fact that this can be compensated for by a change of extension (i.e., amounting to a change of the value of the arbitrary constant), [17, Section 3.4.1]. This gives, with now $n = -(a+b)$,

$$\Phi_+^{-a} * \Phi_-^{-b} = \cos(\pi b)\left(\frac{1}{2}\frac{x^{n-1}\,\mathrm{sgn}}{(n-1)!}\right) + \sin(\pi b)\left(\frac{1}{\pi}\frac{x^{n-1}}{(n-1)!}\ln|x|\right) + c_{+,-}^{-a,-b}\left(\frac{1}{\pi}\frac{x^{n-1}}{(n-1)!}\right), \tag{17}$$

with $c_{+,-}^{-a,-b} \in \mathbb{C}$ arbitrary. Taking the Fourier transformation of this expression and using

$$\mathcal{F}\left(\frac{1}{\pi}\frac{x^{n-1}}{(n-1)!}\right) = (-2\pi i)^{-(n-1)}\frac{1}{\pi}\frac{\delta^{(n-1)}}{(n-1)!}, \tag{18}$$

$$\mathcal{F}\left(\frac{1}{2}\frac{x^{n-1}\,\mathrm{sgn}}{(n-1)!}\right) = (-2\pi i)^{-(n-1)}\frac{1}{2i}\frac{\eta^{(n-1)}}{(n-1)!}, \tag{19}$$

and (with ψ the digamma function, [2, p. 258, 6.3.1])

$$\mathcal{F}\left(\frac{1}{\pi}\frac{x^{n-1}}{(n-1)!}\ln|x|\right)=-\frac{1}{\pi}\frac{(-2\pi i)^{-(n-1)}}{(n-1)!}\left(\left(\ln(2\pi)-\psi(n)\right)\delta^{(n-1)}+\frac{\pi}{2}\left(\eta^{(n-1)}\operatorname{sgn}\right)_0\right), \tag{20}$$

we get for (13),

$$\begin{aligned}&(x-i0)^a.(x+i0)^b\\ =\;&\frac{(-1)^{n-1}}{(n-1)!}\pi e^{+i\pi b}\left(c\delta^{(n-1)}+\cos(\pi b)\,\eta^{(n-1)}-i\sin(\pi b)\left(\eta^{(n-1)}\operatorname{sgn}\right)_0\right),\end{aligned} \tag{21}$$

with $c\triangleq 2i\left(c_{+,-}^{-a,-b}-\sin(\pi b)\left(\ln(2\pi)-\psi(n)\right)\right)/\pi$.

Product (21) is in general an AHD of order of association 1, due to the term $\left(\eta^{(n-1)}\operatorname{sgn}\right)_0$. More details about the distribution $\left(\eta^{(n-1)}\operatorname{sgn}\right)_0$ can be found in [9, Section 5.5]. The subscript 0 in (20) and (21) denotes Riesz' analytic finite part. It is obtained by analytic continuation of a distribution f^z in terms of its Laurent series with respect to the degree of homogeneity z and then using the constant term of this series as "regularization" at the singular (i.e., what we call critical) degree of homogeneity (here $z=-n$). The analytic finite part is essentially a particular extension of a partial distribution (and in this case a unique distribution) that can also be obtained by giving a particular value to the arbitrary constant appearing in our extension approach. This type of regularization, by replacing a non-existing distribution with its analytic finite part, is a technique that goes back to M. Riesz and is routinely used in quantum field theory.

In particular, $\forall k,l\in\mathbb{Z}$ such that $k+l=-n\in\mathbb{Z}_-$, (21) reduces to

$$(x-i0)^k.(x+i0)^l=(-1)^l\,2ic_{+,-}^{-k,-l}(-1)^{n-1}\frac{\delta^{(n-1)}}{(n-1)!}+(-1)^{n-1}\pi\frac{\eta^{(n-1)}}{(n-1)!}, \tag{22}$$

or, with a redefined arbitrary constant $c\in\mathbb{C}$,

$$(x-i0)^k.(x+i0)^l=x^{-n}+c\delta^{(n-1)}. \tag{23}$$

Let $a=-k\in\mathbb{Z}_-$ and $b=-l\in\mathbb{Z}_-$ such that $k+l=n\in\mathbb{Z}_+$. From (22) and (7) follows

$$\begin{aligned}&\frac{\delta^{(k-1)}}{(k-1)!}.\frac{\delta^{(l-1)}}{(l-1)!}+i\left(\frac{\delta^{(k-1)}}{(k-1)!}.\frac{\eta^{(l-1)}}{(l-1)!}-\frac{\eta^{(k-1)}}{(k-1)!}.\frac{\delta^{(l-1)}}{(l-1)!}\right)+\frac{\eta^{(k-1)}}{(k-1)!}.\frac{\eta^{(l-1)}}{(l-1)!}\\ =\;&-(-1)^l\frac{2}{\pi}c_{+,-}^{k,l}\frac{1}{i\pi}\frac{\delta^{(k+l-1)}}{(k+l-1)!}-\frac{1}{i\pi}i\frac{\eta^{(k+l-1)}}{(k+l-1)!}.\end{aligned} \tag{24}$$

Substituting

$$i\left(\frac{\delta^{(k-1)}}{(k-1)!}.\frac{\eta^{(l-1)}}{(l-1)!}-\frac{\eta^{(k-1)}}{(k-1)!}.\frac{\delta^{(l-1)}}{(l-1)!}\right)=\frac{1}{\pi i}c_2^-\frac{\delta^{(k+l-1)}}{(k+l-1)!}, \tag{25}$$

wherein $c_2^-\in\mathbb{C}$ is arbitrary, a result following from the difference of 68 and 71, together with (9) and (11) in (24) yields

$$c_1+c_2^-+c_3=-(-1)^l\frac{2}{\pi}c_{+,-}^{k,l}, \tag{26}$$

a second relation for the arbitrary constants in the multiplication products of the derivatives of delta and eta distributions.

3.3 Case $(x+i0)^a.(x-i0)^b$

Applying our definition of multiplication product (1) and substituting (5) gives, $\forall a,b\in\mathbb{C}$ and with $d\triangleq b-a$,

$$(x+i0)^a.(x-i0)^b=(2\pi)^{-(a+b)}e^{-i(\pi/2)d}\mathcal{F}\left(\Phi_-^{-a}*\Phi_+^{-b}\right). \tag{27}$$

3.3.1 $\forall a, b \in \mathbb{C} : a + b \notin \mathbb{Z}$

Substituting the following result, holding $\forall a, b \in \mathbb{C}$ such that $a + b \notin \mathbb{Z}$ (see [11, eq. (13)]),

$$\Phi_-^{-a} * \Phi_+^{-b} = \frac{\sin(\pi b)}{\sin(\pi(a+b))} \Phi_+^{-(a+b)} + \frac{\sin(\pi a)}{\sin(\pi(a+b))} \Phi_-^{-(a+b)},$$

in (27) and backsubstituting (4) gives, $\forall a, b \in \mathbb{C}$ such that $a + b \notin \mathbb{Z}$,

$$\begin{aligned}(x+i0)^a . (x-i0)^b &= \frac{\sin(\pi a) e^{-i\pi b}}{\sin(\pi(a+b))} (x+i0)^{a+b} + \frac{e^{+i\pi a}\sin(\pi b)}{\sin(\pi(a+b))} (x-i0)^{a+b}, &(28)\\ &= x_+^{a+b} + e^{-i\pi d} x_-^{a+b}, &(29)\\ &= e^{-i(\pi/2)d} \left(\cos(d\pi/2) |x|^n + i \sin(d\pi/2) |x|^n \operatorname{sgn}\right). &(30)\end{aligned}$$

The forms (29)–(30) still hold if $a + b \in \mathbb{N}$.

3.3.2 $\forall a, b \in \mathbb{C} : a + b = -n \in \mathbb{Z}_-$ (critical product)

In the result [12, eq. (69) with $m = 0 = n$], together with [12, eqs. (58) and (11)] and [9, eqs. (247) and (270)], holding $\forall a, b \in \mathbb{C} : a + b = n \in \mathbb{Z}_+$ and with $d \triangleq b - a$, substitute $-a$ for a and $-b$ for b. This gives, with now $n = -(a+b)$,

$$\Phi_-^{-a} * \Phi_+^{-b} = \cos(\pi a) \left(\frac{1}{2} \frac{x^{n-1} \operatorname{sgn}}{(n-1)!}\right) + \sin(\pi a) \left(\frac{1}{\pi} \frac{x^{n-1}}{(n-1)!} \ln|x|\right) + c_{-,+}^{-a,-b} \left(\frac{1}{\pi} \frac{x^{n-1}}{(n-1)!}\right), \quad (31)$$

with $c_{-,+}^{-a,-b} \in \mathbb{C}$ arbitrary. Taking the Fourier transformation of (31), using (18)–(20), we get for (27),

$$\begin{aligned}&(x+i0)^a . (x-i0)^b\\ = \; &\frac{(-1)^{n-1}}{(n-1)!} \pi e^{+i\pi a} \left(c' \delta^{(n-1)} + \cos(\pi a) \eta^{(n-1)} - i \sin(\pi a) \left(\eta^{(n-1)} \operatorname{sgn}\right)_0\right), \quad (32)\end{aligned}$$

with $c' \triangleq 2i \left(c_{-,+}^{-a,-b} - \sin(\pi a)(\ln(2\pi) - \psi(n))\right)/\pi$. This product is also in general an AHD of order of association 1.

In particular, $\forall k, l \in \mathbb{Z}$ such that $k + l = -n \in \mathbb{Z}_-$, (32) reduces to

$$(x+i0)^k . (x-i0)^l = (-1)^k 2ic_{-,+}^{-k,-l} (-1)^{n-1} \frac{\delta^{(n-1)}}{(n-1)!} + (-1)^{n-1} \pi \frac{\eta^{(n-1)}}{(n-1)!}, \quad (33)$$

or, with a redefined arbitrary constant $c' \in \mathbb{C}$,

$$(x+i0)^k . (x-i0)^l = x^{-n} + c' \delta^{(n-1)}. \quad (34)$$

Let $a = -k \in \mathbb{Z}_-$ and $b = -l \in \mathbb{Z}_-$ such that $k + l = n \in \mathbb{Z}_+$. From (33) and (7) follows

$$\begin{aligned}&\frac{\delta^{(k-1)}}{(k-1)!} . \frac{\delta^{(l-1)}}{(l-1)!} - i \left(\frac{\delta^{(k-1)}}{(k-1)!} . \frac{\eta^{(l-1)}}{(l-1)!} - \frac{\eta^{(k-1)}}{(k-1)!} . \frac{\delta^{(l-1)}}{(l-1)!}\right) + \frac{\eta^{(k-1)}}{(k-1)!} . \frac{\eta^{(l-1)}}{(l-1)!}\\ = \; &-(-1)^k \frac{2}{\pi} c_{-,+}^{k,l} \frac{1}{i\pi} \frac{\delta^{(k+l-1)}}{(k+l-1)!} - \frac{1}{i\pi} i \frac{\eta^{(k+l-1)}}{(k+l-1)!}. \quad (35)\end{aligned}$$

Substituting herein (9), (25) and (11) yields

$$c_1 - c_2^- + c_3 = -(-1)^k \frac{2}{\pi} c_{-,+}^{k,l}, \quad (36)$$

a third relation for the arbitrary constants in the multiplication products of the derivatives of delta and eta distributions.

3.4 Case $(x-i0)^a \,.\, (x-i0)^b$

3.4.1 For all $a, b \in \mathbb{C}$

Applying our definition of multiplication product (1), substituting herein (5), using $\Phi_+^{-a} * \Phi_+^{-b} = \Phi_+^{-(a+b)}$ holding $\forall a, b \in \mathbb{C}$ (see [9, eq. (248)], [20, p. 116]), and backsubstituting (4) gives, $\forall a, b \in \mathbb{C}$,

$$(x-i0)^a \,.\, (x-i0)^b = (x-i0)^{a+b} \,. \tag{37}$$

At $a+b \in \mathbb{Z}_-$, eq. (37) is another example of a critical product which value is exceptionally a unique distribution.

In particular, $\forall z \in \mathbb{C}$ and $\forall n \in \mathbb{Z}_+$, (37) gives

$$((x-i0)^z)^n = (x-i0)^{nz} \,.$$

This result suggests again the natural definition, $\forall z \in \mathbb{C}$,

$$((x-i0)^z)^0 \triangleq (x-i0)^0 = 1.$$

3.4.2 The particular case $a, b \in \mathbb{Z}_-$

From (37) follows, $\forall k, l \in \mathbb{Z}_+$,

$$(x-i0)^{-k} \,.\, (x-i0)^{-l} = (x-i0)^{-(k+l)} \,,$$

which becomes, after invoking the Sokhotskii-Plemelj equations (7),

$$\begin{aligned} &\frac{\delta^{(k-1)}}{(k-1)!} . \frac{\delta^{(l-1)}}{(l-1)!} - i \left(\frac{\delta^{(k-1)}}{(k-1)!} . \frac{\eta^{(l-1)}}{(l-1)!} + \frac{\eta^{(k-1)}}{(k-1)!} . \frac{\delta^{(l-1)}}{(l-1)!} \right) - \frac{\eta^{(k-1)}}{(k-1)!} . \frac{\eta^{(l-1)}}{(l-1)!} \\ = \; & -\frac{1}{\pi i} \left(\frac{\delta^{(k+l-1)}}{(k+l-1)!} - i \frac{\eta^{(k+l-1)}}{(k+l-1)!} \right) . \end{aligned} \tag{38}$$

Substituting (9)–(11) in (38) yields

$$c_1 - c_2^+ - c_3 = -1, \tag{39}$$

a fourth relation for the arbitrary constants in the multiplication products of the derivatives of delta and eta distributions.

4 Derived results

A. Eqs. (12) and (39) yield $c_2^+ = 1$ and $c_3 = c_1$. Then, $\forall k, l \in \mathbb{N}$, expression (10) becomes

$$\frac{\delta^{(k-1)}}{(k-1)!} . \frac{\eta^{(l-1)}}{(l-1)!} + \frac{\eta^{(k-1)}}{(k-1)!} . \frac{\delta^{(l-1)}}{(l-1)!} = -\frac{1}{\pi} \frac{\delta^{(k+l-1)}}{(k+l-1)!} \tag{40}$$

and subtracting (11) from (9) gives

$$\frac{\delta^{(k-1)}}{(k-1)!} . \frac{\delta^{(l-1)}}{(l-1)!} - \frac{\eta^{(k-1)}}{(k-1)!} . \frac{\eta^{(l-1)}}{(l-1)!} = +\frac{1}{\pi} \frac{\eta^{(k+l-1)}}{(k+l-1)!} . \tag{41}$$

Putting in (40)–(41) $k = l$ yields the particular results, $\forall k \in \mathbb{Z}_+$,

$$\begin{aligned} \frac{\delta^{(k-1)}}{(k-1)!} . \frac{\eta^{(k-1)}}{(k-1)!} + \frac{\eta^{(k-1)}}{(k-1)!} . \frac{\delta^{(k-1)}}{(k-1)!} &= -\frac{1}{\pi} \frac{\delta^{(2k-1)}}{(2k-1)!} , \\ \left(\frac{\delta^{(k-1)}}{(k-1)!} \right)^2 - \left(\frac{\eta^{(k-1)}}{(k-1)!} \right)^2 &= +\frac{1}{\pi} \frac{\eta^{(2k-1)}}{(2k-1)!} , \end{aligned}$$

which, after substituting (3), become, $\forall k \in \mathbb{Z}_+$,

$$\frac{\delta^{(k-1)}}{(k-1)!} . x^{-k} + x^{-k} . \frac{\delta^{(k-1)}}{(k-1)!} = (-1)^k \frac{\delta^{(2k-1)}}{(2k-1)!} , \tag{42}$$

$$\left(x^{-k} \right)^2 = x^{-2k} + \left(\pi \frac{\delta^{(k-1)}}{(k-1)!} \right)^2 . \tag{43}$$

Eq. (43) is a remarkable identity involving two distributional squares that does not involve an arbitrary constant. The results (42) and (43) for $k=1$ are commonly used in quantum physics (where distributional multiplication is assumed to be commutative). Result (43) for $k=1$ was already given in [24], where it was obtained using delta sequences and passage to the limit. As mentioned in [24], the latter method is able to give meaning to the difference $\left(x^{-1}\right)^2-(\pi\delta)^2$, but not to the individual products $\left(x^{-1}\right)^2$ and $(\delta)^2$.

B. Eqs. (26) and (36), together with $c_3=c_1$, yield

$$c_2^- = \frac{1}{\pi}\left((-1)^k c_{-,+}^{-a,-b} - (-1)^l c_{+,-}^{-a,-b}\right), \tag{44}$$

$$c_1 = -\frac{1}{2\pi}\left((-1)^k c_{-,+}^{-a,-b} + (-1)^l c_{+,-}^{-a,-b}\right) = c_3. \tag{45}$$

The partial distributions $\Phi_-^{-a} * \Phi_+^{-b}$ and $\Phi_+^{-a} * \Phi_-^{-b}$, at a critical degree of homogeneity $-a-b=n\in\mathbb{Z}_+$, are equal (see [12, eq. (11)]), as then the x^{k-1} term is absent and the expression is invariant under an exchange $a\leftrightarrow b$ together with a reflection $x\leftrightarrow -x$. In the expressions (17) and (31) for the extensions of these partial distributions, $c_{-,+}^{-b,-a}\neq c_{+,-}^{-a,-b}$ as there is no natural reason to restrict the extensions of the partial distributions $\Phi_-^{-a} * \Phi_+^{-b}$ and $\Phi_+^{-a} * \Phi_-^{-b}$ to the same branch, so our definition leads to convolution products $\Phi_+^{-a} * \Phi_-^{-b}$ and $\Phi_-^{-a} * \Phi_+^{-b}$ that are not commutative.

Substituting (44)–(45), expression (25) becomes, $\forall k,l\in\mathbb{N}$,

$$\frac{\delta^{(k-1)}}{(k-1)!}.\frac{\eta^{(l-1)}}{(l-1)!} - \frac{\eta^{(k-1)}}{(k-1)!}.\frac{\delta^{(l-1)}}{(l-1)!} = -\frac{1}{\pi^2}\left((-1)^k c_{-,+}^{k,l} - (-1)^l c_{+,-}^{k,l}\right)\frac{\delta^{(k+l-1)}}{(k+l-1)!} \tag{46}$$

and (9) and (11) become, $\forall k,l\in\mathbb{N}$,

$$\frac{\delta^{(k-1)}}{(k-1)!}.\frac{\delta^{(l-1)}}{(l-1)!} = \frac{i}{\pi^2}\frac{1}{2}\left((-1)^k c_{-,+}^{k,l} + (-1)^l c_{+,-}^{k,l}\right)\frac{\delta^{(k+l-1)}}{(k+l-1)!}, \tag{47}$$

$$\frac{\eta^{(k-1)}}{(k-1)!}.\frac{\eta^{(l-1)}}{(l-1)!} = \frac{i}{\pi^2}\frac{1}{2}\left((-1)^k c_{-,+}^{k,l} + (-1)^l c_{+,-}^{k,l}\right)\frac{\delta^{(k+l-1)}}{(k+l-1)!} - \frac{1}{\pi}\frac{\eta^{(k+l-1)}}{(k+l-1)!}. \tag{48}$$

In particular (47)–(48) reduce to, $\forall k\in\mathbb{Z}_+$,

$$\left(\frac{\delta^{(k-1)}}{(k-1)!}\right)^2 = c_k\frac{\delta^{(2k-1)}}{(2k-1)!}, \tag{49}$$

$$\left(\frac{\eta^{(k-1)}}{(k-1)!}\right)^2 = c_k\frac{\delta^{(2k-1)}}{(2k-1)!} - \frac{1}{\pi}\frac{\eta^{(2k-1)}}{(2k-1)!}, \tag{50}$$

wherein $c_k \triangleq (-1)^k\, i\left(c_{-,+}^{k,k}+c_{+,-}^{k,k}\right)/\left(2\pi^2\right)$. Eq. (47) generalizes [30, Theorem 9 for $m=1$], which states that $\delta^{(k)}.\delta^{(l)}=0$ if $k+l$ is even and $\delta^{(k)}.\delta^{(l)}=b\delta^{(k+l+1)}$, for some fixed constant b, if $k+l$ is odd. Eq. (49) is to be compared with [23, Theorems 1 and 2, for integer powers], where also particular branch values are stated for c_k.

C. Adding and subtracting the expressions (40) and (46) gives

$$\frac{\delta^{(k-1)}}{(k-1)!}.\frac{\eta^{(l-1)}}{(l-1)!} = \frac{1}{2}\left(-\frac{1}{\pi}-\frac{1}{\pi^2}\left((-1)^k c_{-,+}^{k,l} - (-1)^l c_{+,-}^{k,l}\right)\right)\frac{\delta^{(k+l-1)}}{(k+l-1)!}, \tag{51}$$

$$\frac{\eta^{(l-1)}}{(l-1)!}.\frac{\delta^{(k-1)}}{(k-1)!} = \frac{1}{2}\left(-\frac{1}{\pi}+\frac{1}{\pi^2}\left((-1)^l c_{-,+}^{l,k} - (-1)^k c_{+,-}^{l,k}\right)\right)\frac{\delta^{(k+l-1)}}{(k+l-1)!}, \tag{52}$$

where we exchanged $k\leftrightarrow l$ in the second expression. Since in general $c_{-,+}^{l,k}\neq c_{+,-}^{k,l}$, $\forall k,l\in\mathbb{N}$, we find that, $\forall k,l\in\mathbb{N}$,

$$\delta^{(k)}.\eta^{(l)}\neq\eta^{(l)}.\delta^{(k)},$$

or equivalently,

$$\delta^{(k)}.x^{-l}\neq x^{-l}.\delta^{(k)}.$$

The result obtained in [18], using a commutative definition of multiplication, agrees with (52) for $l=k$ and the choice of constants $c_{-,+}^{k,k}=c_{+,-}^{k,k}$. The result obtained in [24] agrees with (51) for $l=1=k$

and the choice of constants $c^{1,1}_{-,+} = c^{1,1}_{+,-}$. These are two more examples illustrating that the value of a multiplication product, obtained in the literature with some choice of definition for the distributional multiplication product, usually picks out a particular branch value of our more general result.

D. Results (42)–(43) do not involve any arbitrary constants and various definitions for the multiplication of distributions proposed in the literature agree on them, including the author's method. Besides providing another rigorous justification for these results and generalizing them by (40)–(41), the author's distributional algebra construction now also gives the values of the related, but less trivial, products, such as (46)–(48) and (51)–(52), which can not be obtained by the in the introduction mentioned classical methods.

5 Appendix

Let $k, l \in \mathbb{N}$.

5.1 The convolution product $\frac{1}{\pi}\frac{x^k}{k!} * \frac{1}{\pi}\frac{x^l}{l!}$

By [9, eqs. (6.5) and (6.7)], we have

$$\begin{aligned} \frac{x^k}{k!} * \frac{x^l}{l!} &= \left(\Phi_+^{k+1} - (-1)^{k+1}\Phi_-^{k+1}\right) * \left(\Phi_+^{l+1} - (-1)^{l+1}\Phi_-^{l+1}\right), \\ &= \left(\Phi_+^{k+l+2} + (-1)^{k+l+2}\Phi_-^{k+l+2}\right) - \left((-1)^{k+1}\Phi_-^{k+1} * \Phi_+^{l+1} + \Phi_+^{k+1} * (-1)^{l+1}\Phi_-^{l+1}\right). \end{aligned} \tag{53}$$

Using [12, eq. (70)], the mixed-support products herein are generally given by

$$\Phi_-^{k+1} * \Phi_+^{l+1} = \Phi_{-;+,0}^{k+1;l+1} + c_1\,(-1)^{k+1}\frac{x^{k+l+1}}{(k+l+1)!}, \tag{54}$$

$$\Phi_+^{k+1} * \Phi_-^{l+1} = \Phi_{-;+,0}^{l+1;k+1} + c_2(-1)^{l+1}\frac{x^{k+l+1}}{(k+l+1)!}, \tag{55}$$

with arbitrary $c_1, c_2 \in \mathbb{C}$. In (55) we used the fact that the commutation of a critical product at most results in a change of extension (i.e., a change in the arbitrary constant c_2), [17, Section 3.4.1]. With (54)–(55) the convolution product (53) becomes

$$\begin{aligned} \frac{x^k}{k!} * \frac{x^l}{l!} &= \Phi_+^{k+l+2} + (-1)^{k+l+2}\Phi_-^{k+l+2} \\ &\quad - \left((-1)^{k+1}\Phi_{-;+,0}^{k+1;l+1} + (-1)^{l+1}\Phi_{-;+,0}^{l+1;k+1} + (c_1 + c_2)\,\frac{x^{k+l+1}}{(k+l+1)!}\right). \end{aligned} \tag{56}$$

The analytic finite parts in (56) are obtained from [12, eq. (11)] as

$$\Phi_{-;+,0}^{k+1;l+1} = (-1)^{k+1}\left(\frac{1}{2}\frac{x^{k+l+1}\operatorname{sgn}}{(k+l+1)!} + \frac{1}{2}\frac{\alpha - 1/\alpha}{\alpha + 1/\alpha}\frac{x^{k+l+1}}{(k+l+1)!}\right), \tag{57}$$

$$\Phi_{-;+,0}^{l+1;k+1} = (-1)^{l+1}\left(\frac{1}{2}\frac{x^{k+l+1}\operatorname{sgn}}{(k+l+1)!} + \frac{1}{2}\frac{\alpha' - 1/\alpha'}{\alpha' + 1/\alpha'}\frac{x^{k+l+1}}{(k+l+1)!}\right). \tag{58}$$

In [13, Appendix A] it was shown that no extension process is required since $\Phi_-^{k+1} * \Phi_+^{l+1}$ and $\Phi_-^{l+1} * \Phi_+^{k+1}$ exceptionally exists as a distribution instead of as a partial distribution (due to the vanishing of the residues $\Phi_{-;+,-1}^{k+1;l+1}$ and $\Phi_{-;+,-1}^{l+1;k+1}$). This justifies to exceptionally but naturally define $\Phi_-^{k+1} * \Phi_+^{l+1} \triangleq \Phi_{-;+,0}^{k+1;l+1}$ and $\Phi_-^{l+1} * \Phi_+^{k+1} \triangleq \Phi_{-;+,0}^{l+1;k+1}$ (i.e., the "regularization" obtained with $c_1 = 0 = c_2$). Nevertheless, this still does not make these convolution products unique, due to the non-uniqueness of the analytic finite parts $\Phi_{-;+,0}^{k+1;l+1}$ and $\Phi_{-;+,0}^{l+1;k+1}$. As explained in [12], analytic finite parts, occurring in the expression for the convolution product of critical products of AHDs, are generally not unique because they are the result of taking a limit in $\mathbb{C}^2$ and the limit value depends on the direction chosen to calculate the limit. This is reflected in (57)–(58) by the occurrence of the arbitrary complex constant α (α') which parametrizes the direction of the limit $(a, b) \to (k+1, l+1)$ and so gives rise to an indeterminacy proportional to x^{k+l+1}.

We now continue with our calculation. Substituting (57)–(58) gives for (56),

$$\frac{x^k}{k!} * \frac{x^l}{l!} = \left(\Phi_+^{k+l+2} + (-1)^{k+l+2}\Phi_-^{k+l+2}\right) - \left(\frac{x^{k+l+1}\operatorname{sgn}}{(k+l+1)!} - \pi c_{00}\frac{x^{k+l+1}}{(k+l+1)!}\right), \tag{59}$$

wherein we defined $-\pi c_{00} \triangleq c_1 + c_2 + \frac{1}{2}\frac{\alpha-1/\alpha}{\alpha+1/\alpha} + \frac{1}{2}\frac{\alpha'-1/\alpha'}{\alpha'+1/\alpha'}$. By [9, eqs. (6.6)] we have

$$\Phi_+^{k+l+2} + (-1)^{k+l+2}\Phi_-^{k+l+2} = \frac{x^{k+l+1}\,\mathrm{sgn}}{(k+l+1)!}. \tag{60}$$

Substituting (60) in (59) gives

$$\frac{1}{\pi}\frac{x^k}{k!} * \frac{1}{\pi}\frac{x^l}{l!} = c_{00}\frac{1}{\pi}\frac{x^{k+l+1}}{(k+l+1)!}, \tag{61}$$

in which $c_{00} \in \mathbb{C}$ is still arbitrary.

Taking the generalized Fourier transformation of (61), using [9, eq. (4.19)] and (18), finally yields

$$\frac{\delta^{(k)}}{k!}.\frac{\delta^{(l)}}{l!} = \frac{1}{2} i c_{00}\frac{\delta^{(k+l+1)}}{(k+l+1)!}. \tag{62}$$

5.2 The convolution product $\frac{1}{2}\frac{x^k\,\mathrm{sgn}}{k!} * \frac{1}{2}\frac{x^l\,\mathrm{sgn}}{l!}$

By [9, eqs. (6.6) and (6.7)], we have

$$\begin{aligned} \frac{x^k\,\mathrm{sgn}}{k!} * \frac{x^l\,\mathrm{sgn}}{l!} &= \left(\Phi_+^{k+1} + (-1)^{k+1}\Phi_-^{k+1}\right) * \left(\Phi_+^{l+1} + (-1)^{l+1}\Phi_-^{l+1}\right), \\ &= \left(\Phi_+^{k+l+2} + (-1)^{k+l+2}\Phi_-^{k+l+2}\right) + \left((-1)^{k+1}\Phi_-^{k+1} * \Phi_+^{l+1} + \Phi_+^{k+1} * (-1)^{l+1}\Phi_-^{l+1}\right) \end{aligned} \tag{63}$$

Similarly as in the preceding subsection, we get for (63) (with $c_1 = 0 = c_2$)

$$\frac{x^k\,\mathrm{sgn}}{k!} * \frac{x^l\,\mathrm{sgn}}{l!} = \frac{x^{k+l+1}\,\mathrm{sgn}}{(k+l+1)!} + \left(\frac{x^{k+l+1}\,\mathrm{sgn}}{(k+l+1)!} + 4c_{11}\frac{1}{\pi}\frac{x^{k+l+1}}{(k+l+1)!}\right),$$

with $c_{11} \in \mathbb{C}$ arbitrary, or

$$\frac{1}{2}\frac{x^k\,\mathrm{sgn}}{k!} * \frac{1}{2}\frac{x^l\,\mathrm{sgn}}{l!} = \frac{1}{2}\frac{x^{k+l+1}\,\mathrm{sgn}}{(k+l+1)!} + c_{11}\frac{1}{\pi}\frac{x^{k+l+1}}{(k+l+1)!}. \tag{64}$$

Taking the generalized Fourier transformation of (64), using [9, eq. (4.19)] and (19), yields

$$\frac{\eta^{(k)}}{k!}.\frac{\eta^{(l)}}{l!} = -\frac{1}{\pi}\frac{\eta^{(k+l+1)}}{(k+l+1)!} + \frac{1}{\pi i}2c_{11}\frac{1}{\pi}\frac{\delta^{(k+l+1)}}{(k+l+1)!}. \tag{65}$$

5.3 The convolution product $\frac{1}{\pi}\frac{x^k}{k!} * \frac{1}{2}\frac{x^l\,\mathrm{sgn}}{l!}$

By [9, eqs. (6.5)–(6.7)], we have

$$\begin{aligned} \frac{x^k}{k!} * \frac{x^l\,\mathrm{sgn}}{l!} &= \left(\Phi_+^{k+1} - (-1)^{k+1}\Phi_-^{k+1}\right) * \left(\Phi_+^{l+1} + (-1)^{l+1}\Phi_-^{l+1}\right), \\ &= \left(\Phi_+^{k+l+2} - (-1)^{k+l+2}\Phi_-^{k+l+2}\right) - \left((-1)^{k+1}\Phi_-^{k+1} * \Phi_+^{l+1} - \Phi_+^{k+1} * (-1)^{l+1}\Phi_-^{l+1}\right). \end{aligned} \tag{66}$$

Due to (54)–(55), (57)–(58) and [9, eqs. (6.5)], we now get for (66)

$$\frac{1}{\pi}\frac{x^k}{k!} * \frac{1}{2}\frac{x^l\,\mathrm{sgn}}{l!} = c_{01}\frac{1}{\pi}\frac{x^{k+l+1}}{(k+l+1)!}, \tag{67}$$

with $c_{01} \in \mathbb{C}$ arbitrary.

Taking the generalized Fourier transformation of (67), using [9, eq. (4.19)] and (18)–(19), yields

$$\frac{\delta^{(k)}}{k!}.\frac{\eta^{(l)}}{l!} = i c_{01}\frac{1}{\pi}\frac{\delta^{(k+l+1)}}{(k+l+1)!}. \tag{68}$$

5.4 The convolution product $\frac{1}{\pi}\frac{x^k \operatorname{sgn}}{k!} * \frac{1}{2}\frac{x^l}{l!}$

By [9, eqs. (6.5)–(6.7)], we have

$$\begin{aligned} \frac{x^k \operatorname{sgn}}{k!} * \frac{x^l}{l!} &= \left(\Phi_+^{k+1} + (-1)^{k+1}\Phi_-^{k+1}\right) * \left(\Phi_+^{l+1} - (-1)^{l+1}\Phi_-^{l+1}\right), \\ &= \left(\Phi_+^{k+l+2} - (-1)^{k+l+2}\Phi_-^{k+l+2}\right) + \left((-1)^{k+1}\Phi_-^{k+1} * \Phi_+^{l+1} - \Phi_+^{k+1} * (-1)^{l+1}\Phi_-^{l+1}\right). \end{aligned} \tag{69}$$

Similarly as in the previous subsection, we get for (69)

$$\frac{1}{2}\frac{x^k \operatorname{sgn}}{k!} * \frac{1}{\pi}\frac{x^l}{l!} = c_{10}\frac{1}{\pi}\frac{x^{k+l+1}}{(k+l+1)!}, \tag{70}$$

with $c_{10} \in \mathbb{C}$ arbitrary.

Taking the generalized Fourier transformation of (70), using [9, eq. (4.19)] and (18)–(19), yields

$$\frac{\eta^{(k)}}{k!} . \frac{\delta^{(l)}}{l!} = ic_{10}\frac{1}{\pi}\frac{\delta^{(k+l+1)}}{(k+l+1)!}. \tag{71}$$

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