

# Three function decomposition theorems in Clifford analysis with applications in electromagnetism

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**Abstract.** Three theorems are presented which give the, generally non-unique, representation of a  $k$ -vector function in terms of a  $k-1$ -vector function and a  $k+1$ -vector function, in ultrahyperbolic Clifford analysis with general signature  $(p, q)$ . When specialized to the case  $(p, q) = (1, 3)$  and  $k = 2$ , these theorems have a direct application in the domain of electromagnetism. The specialized theorems characterize: (i) the decomposition of an electromagnetic field in terms of an electric field and a magnetic field, (ii) the representation of an electromagnetic field in terms of potential fields and (iii) the representation of an electromagnetic field in terms of source fields.

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## INTRODUCTION

Let  $p, q \in \mathbb{N} : n \triangleq p + q \geq 2$  and  $\Omega \subseteq R^{p,q}$ . We consider in ultrahyperbolic Clifford analysis, [1], [5], [6], the representation of a  $k$ -vector function  $F : \Omega \rightarrow Cl_{p,q}^k$ , with  $0 < k < n$ , in terms of functions with neighboring grades  $k-1$  and  $k+1$ , hence relative to a given 1-vector object. Three theorems are presented, which state the representation of  $k$ -vector functions  $F$  left relative to: (i) a given 1-vector field  $v$ , (ii) the Dirac operator  $\partial$  and (iii) the right inverse (Teodorescu) operator  $\partial^{-1}$  of the Dirac operator. The first theorem is purely algebraic and only uses the real Clifford algebra  $Cl_{p,q}$ , [2], [3], while the other two make use of ultrahyperbolic Clifford analysis. A typical property of the considered representations is the non-uniqueness of the neighboring grade functions. This is a consequence of the fact that the sum of the number of components of the two neighboring grade functions is always higher than the number of components of the given function, when  $n > 2$ .

When specializing to the case:  $(p, q) = (1, 3)$  and  $k = 2$ , these theorems have a direct application in the domain of electromagnetism. Let  $F : \Omega \subseteq R^{1,3} \rightarrow Cl_{1,3}^2$ ,  $S : \Omega \rightarrow Cl_{1,3}^1 \oplus Cl_{1,3}^3$  with  $\text{supp} S \subset \Omega$  and  $\partial \in Cl_{1,3}^1$  the Dirac operator. In this case, the equation

$$\partial F = S \quad (1)$$

is the Clifford algebra model for electromagnetism, [10], [7], [11]. The specialized theorems then characterize: (i) the decomposition of an electromagnetic field in terms of an electric field and a magnetic field, (ii) the representation of an electromagnetic field in terms of potential fields and (iii) the representation of an electromagnetic field in terms of source fields.

## DECOMPOSITION WITH RESPECT TO A VECTOR FUNCTION

**Theorem 1** (i) Any  $F \in (\Omega, Cl_{p,q}^k)$ , with  $0 < k < n$ , can be represented, with respect to a  $v \in (\Omega, Cl_{p,q}^1) : v^2 \neq 0$  in  $\Omega$ , as

$$F = v \wedge E + v \cdot B, \quad (2)$$

with  $E \in (\Omega, Cl_{p,q}^{k-1})$  and  $B \in (\Omega, Cl_{p,q}^{k+1})$ .

(ii) The representation (2) is not unique. The non-uniqueness in  $E$  and  $B$  is of the form

$$(E' + B') - (E + B) = v \wedge \phi + v \cdot \Phi, \quad (3)$$

wherein  $\phi \in (\Omega, Cl_{p,q}^{k-2})$ , for  $2 \leq k$ , and  $\Phi \in (\Omega, Cl_{p,q}^{k+2})$ , for  $k \leq n-2$ , are arbitrary. If  $k = 1$ ,  $\phi = 0$  or if  $k = n-1$ ,  $\Phi = 0$ .

(iii) Iff

$$v \cdot E + v \wedge B = 0, \quad (4)$$

then  $F$  has the unique decomposition given by

$$E + B = v^{-1} F. \quad (5)$$

**Proof.** (i) Since  $v^2 \neq 0$ , exist  $v^{-1} = v/v^2$  such that  $v^{-1}v = 1 = vv^{-1}$ . Using associativity, we have

$$\begin{aligned} F &= (v v^{-1}) F, \\ &= v (v^{-1} F), \\ &= v \cdot (v^{-1} \cdot F + v^{-1} \wedge F) + v \wedge (v^{-1} \cdot F + v^{-1} \wedge F), \\ &= v \wedge (v^{-1} \cdot F) + v \cdot (v^{-1} \wedge F). \end{aligned}$$

Hererin is  $v^{-1} \cdot F$  of grade  $k-1$  and  $v^{-1} \wedge F$  of grade  $k+1$ . Hence, any  $F : \Omega \rightarrow Cl_{p,q}^k$ , with  $0 < k < n$ , can be represented in the form (2).

(ii) Applying (i) to grades  $k-1$  and  $k+1$ , any  $E' - E$  and any  $B' - B$  can be represented with respect to  $v$  as

$$\begin{aligned} E' - E &= v \wedge \phi + v \cdot \omega_E, \\ B' - B &= v \wedge \omega_B + v \cdot \Phi, \end{aligned}$$

with  $\phi : \Omega \rightarrow Cl_{p,q}^{k-2}$ , for  $2 \leq k$ ,  $\omega_E, \omega_B : \Omega \rightarrow Cl_{p,q}^k$  and  $\Phi : \Omega \rightarrow Cl_{p,q}^{k+2}$ , for  $k \leq n-2$ . If  $k=1$ ,  $\phi=0$  or if  $k=n-1$ ,  $\Phi=0$ . Invariance of  $F$  requires that

$$\begin{aligned} 0 &= F' - F, \\ &= v \wedge (E' - E) + v \cdot (B' - B), \\ &= v \wedge (v \wedge \phi + v \cdot \omega_E) + v \cdot (v \cdot \Phi + v \wedge \omega_B), \\ &= v \wedge (v \cdot \omega_E) + v \cdot (v \wedge \omega_B), \\ &= v \cdot (v \cdot \omega_E) + v \wedge (v \cdot \omega_E) \\ &\quad + v \cdot (v \wedge \omega_B) + v \wedge (v \wedge \omega_B), \\ &= v(v \cdot \omega_E + v \wedge \omega_B). \end{aligned}$$

Multiplying this equation on the left with  $v^{-1}$  and using associativity, we find that invariance of  $F$  requires that

$$v \cdot \omega_E + v \wedge \omega_B = 0,$$

or, after splitting in grades  $k-1$  and  $k+1$ , that

$$\begin{aligned} v \cdot \omega_E &= 0, \\ v \wedge \omega_B &= 0. \end{aligned}$$

Hence, any non-uniqueness in  $E$  and  $B$ , such that  $F$  remains invariant, can be represented as (2), wherein  $\phi$  and  $\Phi$  are arbitrary.

(iii) The representation (2) can be written as

$$v \cdot E + F + v \wedge B = v(E + B).$$

If (4) holds, then

$$\begin{aligned} v^{-1}F &= v^{-1}(v(E + B)), \\ &= (v^{-1}v)(E + B), \\ &= E + B, \end{aligned}$$

so  $F$  has the unique decomposition given by (5).

If  $F$  has the unique decomposition given by (5), then

$$\begin{aligned} v \cdot E + F + v \wedge B &= v(E + B), \\ &= v(v^{-1}F), \\ &= (vv^{-1})F, \\ &= F, \end{aligned}$$

which implies (4). ■

The internal degree of freedom in the representation (2) can be called an (algebraic) gauge transformation. One could call the particular gauge (4) the orthogonal

or uniqueness gauge. Notice that iff  $n=2$ , then the representation (2) is unique.

From Theorem 1 follows that any function  $E$  and any function  $B$  given by

$$E = v \wedge \phi + v^{-1} \cdot F, \quad (6)$$

$$B = v^{-1} \wedge F + v \cdot \Phi, \quad (7)$$

and wherein  $\phi$  and  $\Phi$  are arbitrary, reproduce a given function  $F$  by the representation (2).

Let  $E, B$  and  $E', B'$  be two neighboring grade fields satisfying the conditions (4). Then, since  $v^2 \neq 0$ ,

$$\begin{aligned} 0 &= v \cdot E' - v \cdot E \\ &= v \cdot (v \wedge \phi) = v(v \wedge \phi) \\ &\Leftrightarrow v \wedge \phi = 0, \\ 0 &= v \wedge B' - v \wedge B \\ &= v \wedge (v \cdot \Phi) = v(v \cdot \Phi) \\ &\Leftrightarrow v \cdot \Phi = 0, \end{aligned}$$

which, after substitution in (3), shows that

$$\begin{aligned} E' &= E, \\ B' &= B. \end{aligned}$$

When specializing to  $(p, q) = (1, 3)$  and  $k=2$ , Theorem 1 gives the decomposition of an electromagnetic field  $F$  in terms of an electric field  $E$  and magnetic field  $B$ , typically relative to the time-like unit vector  $v$  (i.e.,  $v^2=1$ ) tangent to a sub-luminal observer's world line. For an observer in the luminal frame (e.g., a photon) is  $v^2=0$  and then  $v^{-1}$  does not exist. Hence, Theorem 1 shows that a decomposition of an electromagnetic field  $F$  in an electric field  $E$  and a magnetic field  $B$  only applies to non-luminal observers. One can show that for observers in a super-luminal frame the roles of electric and magnetic fields are reversed.

## PRELIMINARIES FROM CLIFFORD ANALYSIS

Let  $\bar{\Sigma} \subset \Omega \subseteq R^{p,q}$  be a compact simply connected domain (satisfying a technical condition, see [6]),  $n : \delta\bar{\Sigma} \rightarrow Cl_{p,q}^1$  the outward normal function on the coherently oriented boundary  $\delta\bar{\Sigma}$  of  $\bar{\Sigma}$ ,  $C_x \in \mathcal{D}' \otimes Cl_{p,q}^1$ ,  $\forall x \in \Omega$ , a Cauchy kernel for the Dirac operator  $\partial$ ,  $|_{\delta\bar{\Sigma}}$  restriction to  $\delta\bar{\Sigma}$  and  $\langle, \rangle_S$  Schwartz pairing between a continuous linear functional and a continuous function, [8], [9], [4].

**Definition 2** The (Teodorescu) volume operator  $\partial^{-1} : (\Omega, \mathcal{C}^0 \otimes Cl_{p,q}) \rightarrow (\Omega, \mathcal{C}^1 \otimes Cl_{p,q})$  is defined as

$$-\partial^{-1} \triangleq \langle C_x, \dots \rangle_S, \quad \forall x \in \Omega. \quad (8)$$

**Definition 3** The (Cauchy-type) boundary operator  $\beta : (\Omega, \mathcal{C}^1 \otimes Cl_{p,q}) \rightarrow (\Omega, \mathcal{C}^1 \otimes Cl_{p,q})$  is defined as

$$\beta \triangleq \langle C_x |_{\delta\bar{\Sigma}}, n \dots |_{\delta\bar{\Sigma}} \rangle_S, \quad \forall x \in \Omega. \quad (9)$$

First we need the following generalized Cauchy integral theorem from ultrahyperbolic Clifford analysis, [6].

**Theorem 4** For any  $F : \Omega \rightarrow \mathcal{C}^1 \otimes Cl_{p,q} : \partial F = S$  with  $S : \Omega \rightarrow \mathcal{C}_c^0 \otimes Cl_{p,q}$  and  $\text{supp} S \subset \Sigma$  holds in  $\Sigma$  that

$$F = \partial^{-1} S + \beta F. \quad (10)$$

From Theorem 4 follows

$$\partial \partial^{-1} = \text{Id} \text{ and } \partial \beta = 0, \quad (11)$$

$$\partial^{-1} \partial = \text{Id} - \beta. \quad (12)$$

Hence,  $\partial^{-1}$  is a right inverse of the Dirac operator  $\partial$ .

## REPRESENTATION WITH RESPECT TO THE DIRAC OPERATOR

**Theorem 5** (i) Any  $F \in (\Omega, \mathcal{C}_c^0 \otimes Cl_{p,q}^k)$ , with  $0 < k < n$ , can be represented in  $\Sigma$ , with respect to  $\partial$ , as

$$F = \partial \wedge A + \partial \cdot B, \quad (13)$$

with  $A \in (\Omega, \mathcal{C}^1 \otimes Cl_{p,q}^{k-1})$  and  $B \in (\Omega, \mathcal{C}^1 \otimes Cl_{p,q}^{k+1})$ .

(ii) The representation (13) is not unique. The non-uniqueness in  $A$  and  $B$  is of the form

$$(A' + B') - (A + B) = \partial \wedge \phi + \partial \cdot \omega_A + \partial \wedge \omega_B + \partial \cdot \Phi, \quad (14)$$

wherein  $\phi \in (\Omega, \mathcal{C}^1 \otimes Cl_{p,q}^{k-2})$ , for  $2 \leq k$ , and  $\Phi \in (\Omega, \mathcal{C}^1 \otimes Cl_{p,q}^{k+2})$ , for  $k \leq n-2$ , are arbitrary and  $\omega_A, \omega_B \in (\Omega, \mathcal{C}^2 \otimes Cl_{p,q}^k)$  satisfy  $\partial(\partial \cdot \omega_A + \partial \wedge \omega_B) = 0$ . If  $k=1$ ,  $\phi=0$  or if  $k=n-1$ ,  $\Phi=0$ .

(iii) Iff

$$\partial \cdot A + \partial \wedge B = 0, \quad (15)$$

then the representation (13) is unique, with  $A$  and  $B$  given by

$$A + B = \partial^{-1} F + \beta(A + B). \quad (16)$$

**Proof.** (i) Using Theorem 4 (with  $F$  here playing the role of  $S$ ), we have

$$\begin{aligned} F &= \partial(\partial^{-1} F), \\ &= \partial \cdot (\partial^{-1} \cdot F + \partial^{-1} \wedge F) + \partial \wedge (\partial^{-1} \cdot F + \partial^{-1} \wedge F), \\ &= \partial \cdot (\partial^{-1} \cdot F) \\ &\quad + \partial \cdot (\partial^{-1} \wedge F) + \partial \wedge (\partial^{-1} \cdot F) \\ &\quad + \partial \wedge (\partial^{-1} \wedge F). \end{aligned}$$

Equating equal grades yields

$$\begin{aligned} 0 &= \partial \cdot (\partial^{-1} \cdot F), \\ F &= \partial \wedge (\partial^{-1} \cdot F) + \partial \cdot (\partial^{-1} \wedge F), \\ 0 &= \partial \wedge (\partial^{-1} \wedge F). \end{aligned}$$

Herein is  $\partial^{-1} \cdot F$  of grade  $k-1$  and  $\partial^{-1} \wedge F$  of grade  $k+1$ . Hence, any  $F : \Omega \rightarrow \mathcal{C}^0 \otimes Cl_{p,q}^k$ , with  $0 < k < n$ , can be represented in the form (13).

(ii) Applying (i) to grades  $k-1$  and  $k+1$ , any  $A' - A$  and any  $B' - B$  can be represented, with respect to  $\partial$ , as

$$\begin{aligned} A' - A &= \partial \wedge \phi + \partial \cdot \omega_A, \\ B' - B &= \partial \wedge \omega_B + \partial \cdot \Phi, \end{aligned}$$

with  $\phi : \Omega \rightarrow \mathcal{C}^1 \otimes Cl_{p,q}^{k-2}$ , for  $2 \leq k$ ,  $\omega_A, \omega_B : \Omega \rightarrow \mathcal{C}^2 \otimes Cl_{p,q}^k$  and  $\Phi : \Omega \rightarrow \mathcal{C}^1 \otimes Cl_{p,q}^{k+2}$ , for  $k \leq n-2$ . If  $k=1$ ,  $\phi=0$  or if  $k=n-1$ ,  $\Phi=0$ . Invariance of  $F$  requires that

$$\begin{aligned} 0 &= F' - F, \\ &= \partial \wedge (A' - A) + \partial \cdot (B' - B), \\ &= \partial \wedge (\partial \wedge \phi + \partial \cdot \omega_A) + \partial \cdot (\partial \cdot \Phi + \partial \wedge \omega_B), \\ &= \partial \wedge (\partial \cdot \omega_A) + \partial \cdot (\partial \wedge \omega_B), \\ &= \partial \cdot (\partial \cdot \omega_A) + \partial \wedge (\partial \cdot \omega_A) \\ &\quad + \partial \cdot (\partial \wedge \omega_B) + \partial \wedge (\partial \wedge \omega_B), \\ &= \partial(\partial \cdot \omega_A + \partial \wedge \omega_B). \end{aligned}$$

Clearly,  $\phi$  and  $\Phi$  are arbitrary and the invariance of  $F$  requires that  $\partial(\partial \cdot \omega_A + \partial \wedge \omega_B) = 0$ .

(iii) The representation (13) is equivalent with

$$\partial \cdot A + \partial \wedge B + F = \partial(A + B). \quad (17)$$

If (15) holds, then (17) becomes

$$\partial(A + B) = F.$$

We can solve for  $A + B$ , using Theorem 4, and get in  $\Sigma$ ,

$$A + B = \partial^{-1} F + \beta(A + B),$$

and  $F$  has the unique representation (13) with  $A + B$  given by (16).

If (16) holds in  $\Sigma$ , then by operating on the left of (16) with  $\partial$  we get

$$\partial(A + B) = F.$$

Then, (17) becomes

$$\begin{aligned} \partial \cdot A + \partial \wedge B + F &= \partial(A + B), \\ &= F, \end{aligned}$$

so (15) is necessary. ■

The internal degree of freedom in the representation (13), and characterized by (14), is called a gauge transformation in Physics. Notice that, without the boundary condition for the potentials  $\beta(A+B) = 0$ , the multi-vector potential  $A+B$  is defined up to a left holomorphic multi-vector function.

From Theorem 5 also follows that the non-unique  $k-1$ -vector potential  $A$  and the non-unique  $k+1$ -vector potential  $B$ , which reproduce a given function  $F$  by the representation (13), is given by

$$A = \partial \wedge \phi + \partial \cdot \omega_A + \partial^{-1} \cdot F, \quad (18)$$

$$B = \partial \wedge \omega_B + \partial \cdot \Phi + \partial^{-1} \wedge F, \quad (19)$$

and wherein  $\phi$  and  $\Phi$  are arbitrary and  $\partial \cdot \omega_A + \partial \wedge \omega_B$  is left holomorphic.

Let  $A, B$  and  $A', B'$  be two neighboring grade fields satisfying the conditions (15). Then, from (14) follows

$$\begin{aligned} 0 &= \partial \cdot A' - \partial \cdot A \\ &= \partial \cdot (\partial \wedge \phi) = \partial (\partial \wedge \phi) \\ &\Leftrightarrow \partial \wedge \phi = 0, \\ 0 &= \partial \wedge B' - \partial \wedge B \\ &= \partial \wedge (\partial \cdot \Phi) = \partial (\partial \cdot \Phi) \\ &\Leftrightarrow \partial \cdot \Phi = 0, \end{aligned}$$

which, after substitution in (14) again, shows that

$$\begin{aligned} A' - A &= \partial \cdot \omega_A, \\ B' - B &= \partial \wedge \omega_B. \end{aligned}$$

Adding yields

$$(A' + B') - (A + B) = \partial \cdot \omega_A + \partial \wedge \omega_B,$$

which is a left holomorphic multi-vector function. If  $A, B$  and  $A', B'$  in addition satisfy the boundary condition  $\beta(A+B) = 0$ , holds that

$$\begin{aligned} 0 &= \beta(A' + B') - \beta(A + B) \\ &= \beta(\partial \cdot \omega_A + \partial \wedge \omega_B). \end{aligned}$$

This result combined with the fact that  $\partial \cdot \omega_A + \partial \wedge \omega_B$  is left holomorphic gives

$$\partial \cdot \omega_A + \partial \wedge \omega_B = 0,$$

and then we have uniqueness of the representation (13).

Theorem 5 generalizes Helmholtz' theorem for three dimensional vector fields.

Theorem 5 is also somewhat related to Hodge's decomposition theorem for differential forms. Hodge's decomposition however is formulated with respect to a global  $L^2$ -inner product and requires that the differential form coefficient functions either have compact support

or belong to a suitable Sobolev space, so that the boundary contribution (at infinity) is zero. Theorem 5 differs from Hodge's decomposition in the sense that (i) it is not a decomposition in orthogonal complements with respect to a global inner product, (ii) it is formulated over a compact domain  $\bar{\Sigma}$  and uses the boundary contribution on  $\partial\bar{\Sigma}$  and (iii) it holds in the (ultra-)hyperbolic setting, while Hodge's theorem is restricted to elliptic complexes.

When specializing to  $(p, q) = (1, 3)$  and  $k = 2$ , Theorem 5 gives the representation of the electromagnetic field in terms of a 1-vector potential  $A$  and a 3-vector potential  $B$ .

## REPRESENTATION WITH RESPECT TO THE RIGHT INVERSE DIRAC OPERATOR

**Theorem 6** (i) Any  $F \in (\Omega, \mathcal{C}^1 \otimes Cl_{p,q}^k)$  can be represented in  $\Sigma$ , with respect to  $\partial^{-1}$ , as

$$F = \partial^{-1} \wedge J + \partial^{-1} \cdot K + \beta F, \quad (20)$$

with  $J \in (\Omega, \mathcal{C}^0 \otimes Cl_{p,q}^{k-1})$ ,  $K \in (\Omega, \mathcal{C}^0 \otimes Cl_{p,q}^{k+1})$  and  $\text{supp}(J+K) \subset \Sigma$ .

(ii) The representation (20) is not unique. The non-uniqueness in  $J$  and  $K$  is of the form

$$(J' + K') - (J + K) = \partial^{-1} \wedge \phi + \partial^{-1} \cdot \Phi, \quad (21)$$

wherein  $\phi \in (\Omega, \mathcal{C}^0 \otimes Cl_{p,q}^{k-2})$ , for  $2 \leq k$ , and  $\Phi \in (\Omega, \mathcal{C}^0 \otimes Cl_{p,q}^{k+2})$ , for  $k \leq n-2$ , are arbitrary. If  $k = 1$ ,  $\phi = 0$  or if  $k = n-1$ ,  $\Phi = 0$ .

(iii) Iff

$$\partial^{-1} \cdot J + \partial^{-1} \wedge K = 0, \quad (22)$$

then the representation (20) is unique, with  $J$  and  $K$  given by

$$J + K = \partial F. \quad (23)$$

**Proof.** (i) Using Theorem 4, we have

$$\begin{aligned} F &= \partial^{-1} (\partial F) + \beta F, \\ &= \partial^{-1} \cdot (\partial \cdot F + \partial \wedge F) + \partial^{-1} \wedge (\partial \cdot F + \partial \wedge F) \\ &\quad + \beta F, \\ &= \partial^{-1} \cdot (\partial \cdot F) \\ &\quad + \partial^{-1} \cdot (\partial \wedge F) + \partial^{-1} \wedge (\partial \cdot F) + \beta F \\ &\quad + \partial^{-1} \wedge (\partial \wedge F). \end{aligned}$$

Equating equal grades yields

$$\begin{aligned} 0 &= \partial^{-1} \cdot (\partial \cdot F), \\ F &= \partial^{-1} \wedge (\partial \cdot F) + \partial^{-1} \cdot (\partial \wedge F) + \beta F, \\ 0 &= \partial^{-1} \wedge (\partial \wedge F). \end{aligned}$$

Herer is  $\partial^{-1} \cdot F$  of grade  $k-1$  and  $\partial^{-1} \wedge F$  of grade  $k+1$ . Hence, any  $F : \Omega \rightarrow \mathcal{C}^0 \otimes Cl_{p,q}^k$ , with  $0 < k < n$ , can be represented in the form (20).

(ii) Applying (i) to grades  $k-1$  and  $k+1$ , any  $J' - J$  and  $K' - K$  can be represented, with respect to  $\partial^{-1}$ , as

$$\begin{aligned} J' - J &= \partial^{-1} \wedge \phi + \partial^{-1} \cdot \omega_J, \\ K' - K &= \partial^{-1} \wedge \omega_K + \partial^{-1} \cdot \Phi, \end{aligned}$$

with  $\phi : \Omega \rightarrow \mathcal{C}^0 \otimes Cl_{p,q}^{k-2}$ , for  $2 \leq k$ ,  $\omega_J, \omega_K : \Omega \rightarrow \mathcal{C}^0 \otimes Cl_{p,q}^k$  and  $\Phi : \Omega \rightarrow \mathcal{C}^0 \otimes Cl_{p,q}^{k+2}$ , for  $k \leq n-2$ . If  $k=1$ ,  $\phi=0$  or if  $k=n-1$ ,  $\Phi=0$ . The boundary terms  $\beta(J' - J)$  and  $\beta(K' - K)$  are zero, since it is given that  $\text{supp}(J+K) \subset \Sigma$ . Invariance of  $F$  requires that

$$\begin{aligned} 0 &= F' - F, \\ &= \partial^{-1} \wedge (J' - J) + \partial^{-1} \cdot (K' - K) + \beta(F' - F), \\ &= \partial^{-1} \wedge (\partial^{-1} \wedge \phi + \partial^{-1} \cdot \omega_J) \\ &\quad + \partial^{-1} \cdot (\partial^{-1} \cdot \Phi + \partial^{-1} \wedge \omega_K) + \beta(F' - F), \\ &= \partial^{-1} \wedge (\partial^{-1} \cdot \omega_J) + \partial^{-1} \cdot (\partial^{-1} \wedge \omega_K) \\ &\quad + \beta(F' - F), \\ &= \partial^{-1} \cdot (\partial^{-1} \cdot \omega_J) + \partial^{-1} \wedge (\partial^{-1} \cdot \omega_J) \\ &\quad + \partial^{-1} \cdot (\partial^{-1} \wedge \omega_K) + \partial^{-1} \wedge (\partial^{-1} \wedge \omega_K) \\ &\quad + \beta(F' - F), \\ &= \partial^{-1} (\partial^{-1} \cdot \omega_J + \partial^{-1} \wedge \omega_K) + \beta(F' - F). \end{aligned}$$

Clearly,  $\phi$  and  $\Phi$  are arbitrary. Acting with  $\partial$  from the left on this equation yields

$$0 = \partial^{-1} \cdot \omega_J + \partial^{-1} \wedge \omega_K,$$

which is a necessary condition for  $F$  to be invariant. Hence, the non-uniqueness in  $J$  and  $K$  is as stated in (21).

(iii) The representation (20) is equivalent with

$$\partial^{-1} \cdot J + \partial^{-1} \wedge K + F = \partial^{-1} (J + K) + \beta F. \quad (24)$$

(iii.1) If (22) holds, (24) becomes

$$F = \partial^{-1} (J + K) + \beta F.$$

Acting with  $\partial$  from the left on this equation, we get

$$J + K = \partial F,$$

so  $F$  has the unique representation (20), with  $J+K$  given by (23). Hence, (22) are sufficient conditions.

(iii.2) If (23) holds, we can solve for  $F$  using Theorem 4, and get

$$\partial^{-1} (J + K) + \beta F = F.$$

Then (24) becomes

$$\begin{aligned} \partial^{-1} \cdot J + \partial^{-1} \wedge K + F &= \partial^{-1} (J + K) + \beta F, \\ &= F, \end{aligned}$$

so (22) is necessary. ■

The internal degree of freedom in the representation (20), and characterized by (21), could be called a source gauge transformation, to distinguish it from the usual potential gauge transformation.

Obviously, (20) takes the form of Cauchy's integral representation for a function  $F$ , which is holomorphic except at the compact support of  $J+K$ . Where Cauchy's integral representation yields  $F$  for a given source function  $J+K$ , Theorem 6 yields  $J+K$  for a given function  $F$ .

From Theorem 6 also follows that the non-unique  $k-1$ -vector source  $J$  and the non-unique  $k+1$ -vector source  $K$ , which reproduce a given function  $F$  by the representation (20), is given by

$$J = \partial^{-1} \wedge \phi + \partial \cdot F, \quad (25)$$

$$K = \partial^{-1} \cdot \Phi + \partial \wedge F, \quad (26)$$

and wherein  $\phi$  and  $\Phi$  are arbitrary.

Let  $J, K$  and  $J', K'$  be two neighboring grade fields satisfying the conditions (22). Then, from (21) follows

$$\begin{aligned} 0 &= \partial^{-1} \cdot J' - \partial^{-1} \cdot J \\ &= \partial^{-1} \cdot (\partial^{-1} \wedge \phi) = \partial^{-1} (\partial^{-1} \wedge \phi) \\ &\Leftrightarrow \partial^{-1} \wedge \phi = 0, \\ 0 &= \partial^{-1} \wedge K' - \partial^{-1} \wedge K \\ &= \partial^{-1} \wedge (\partial^{-1} \cdot \Phi) = \partial^{-1} (\partial^{-1} \cdot \Phi) \\ &\Leftrightarrow \partial^{-1} \cdot \Phi = 0, \end{aligned}$$

which, after substitution in (21) again, shows that

$$\begin{aligned} J' &= J, \\ K' &= K. \end{aligned}$$

When specializing to  $(p, q) = (1, 3)$  and  $k = 2$ , Theorem 6 gives the representation of the electromagnetic field  $F$  in terms of a 1-vector electric monopole charge-current density  $J$  and a 3-vector magnetic monopole charge-current density  $K$ . Theorem 6 shows that a given electromagnetic field  $F$  can in general be generated by different sources in  $\Sigma$ . To assure that the inverse source problem has a unique solution, additional conditions need to be imposed. These conditions are given by (22), which are the necessary and sufficient conditions under which the inverse source problem acquires a unique solution.

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