Introducing Clifford Analysis as the Natural Tool for Electromagnetic Research

G. R. Franssens
Belgian Institute for Space Aeronomy, Belgium

Abstract— Clifford algebra and the thereupon based Clifford Analysis are introduced as the natural mathematical tools to formulate and solve electromagnetic problems. This is a new and powerful approach, which permits to make new progress in both the understanding and computation of electromagnetic fields.

1. INTRODUCTION

The model for electromagnetism, still widely in use today in applied sciences, is a virtually unchanged version that goes back to O. Heaviside, who "sorted out" two earlier models by J. Maxwell (one using the real algebra and the other using the quaternion algebra). Its mathematical formulation is not only very outdated, but also has obscured all this time important intrinsic properties of the underlying physical phenomenon. Responsible for this state of affairs is a vector algebra, invented by J. Gibbs and adopted by Heaviside to build his model, that is entirely inappropriate for describing electromagnetism. The Maxwell-Heaviside equations are considered to be the correct model for (classical) electromagnetism, because they predict numerical values for the magnitudes of the field components that are in agreement with experimental values. However, these equations do not correctly model the geometrical content of the electromagnetic field, nor all its physical invariances (e.g., Lorentz invariance, which is traditionally "proved" by mathematical cheating). Modern physical insight requires that a faithful model not only predict correct magnitudes, but also correctly models the geometric content and invariances that correspond to the physical phenomenon. Once one is willing to give attention to these additional requirements, by changing to a model that is also correct in this broader geometrical sense, fascinating new progress becomes possible.

Electromagnetism has a higher degree of intrinsic simplicity and beauty, than what can be inferred from its classical formulation. This can be brought out more clearly by stating electromagnetism in terms of modern mathematical language and concepts. The here presented reformulation is not yet another fancy way of writing Maxwell's equations, but the result of using an advanced mathematical framework that is also capable of solving them. This new framework severely simplifies the solution of electromagnetic problems and allows us to tackle rather advanced problems in a pure analytical way.

The power of the adopted mathematical formalism stems from the fact that it exploits a, so far overlooked, inherent property of electromagnetism: electromagnetic fields in a region without sources possess a generalized form of holomorphy. This property imposes restrictions on an electromagnetic field in such a way that new progress in electromagnetism by analytical methods becomes possible and this to a much larger extent than what is currently believed achievable.

The next two sections contain a condensed introduction to the mathematical framework that allows us to do electromagnetic research with power and elegance: Clifford algebra and the thereupon based Clifford Analysis (CA). A branch of CA, called hyperbolic CA, is shown to be the natural setting to formulate and solve electromagnetic problems (as well as quantum field problems). In particular, the mathematical model for electromagnetism, when stated in CA as in Section 4, reduces to a single as-simple-as-it-gets equation. This equation is equivalent to the Maxwell-Heaviside model in the narrow sense that both models produce the same field component magnitudes, but this new model now also has the geometrical content right. In Section 5, some examples of new insights and results obtained with CA are summarized.

2. CLIFFORD ALGEBRA

The underlying idea of this new approach to electromagnetism is that we should use that mathematical framework which is best adapted to the structure of the universe that we live in. In our 4-dimensional universe this is, in the absence of gravity, (locally) a pseudo-Euclidean space (Minkowski space) or when gravity is present, a pseudo-Riemannian space. An important aspect of such spaces is their quadratic inner product structure (called the metric by physicists), which profoundly shapes the form that the laws of physics take. Over the last century, it has become apparent that the laws of physics essentially express geometrical relationships between quantities, which themselves have geometrical content. It thus makes sense to use a number system that is up to this task. By combining these observations we naturally arrive at what are now called Clifford algebras, and which by W. Clifford himself were called geometrical algebras. Clifford algebras make it possible to easily formulate geometrical relationships between the geometrical objects that can exist in a linear space [1, 4, 13, 15].

Over each quadratic inner product space $R^{p,q}$, i.e., R^n together with the canonical quadratic inner product of signature (p, q) and dimension $n \triangleq p+q$, one can define $n+1$ real Clifford algebras, one for each signature (p, q) with q running from 0 to n. Let $\{\mathbf{e}^1, \ldots, \mathbf{e}^n\}$ be an orthonormal basis for the linear inner product space $R^{p,q}$, with canonical quadratic form P of signature (p, q). The real Clifford algebra $Cl_{p,q}$ over $R^{p,q}$ is defined by [2, 4, 13],

$$\left(\mathbf{e}^1\right)^2 = \ldots = \left(\mathbf{e}^p\right)^2 = +1, \quad \left(\mathbf{e}^{p+1}\right)^2 = \ldots = \left(\mathbf{e}^n\right)^2 = -1, \tag{1}$$

$$\mathbf{e}^i\mathbf{e}^j + \mathbf{e}^j\mathbf{e}^i = 0, \quad i \neq j, \tag{2}$$

together with linearity over $\mathbb{R}$ and associativity.

Among the infinitely many Clifford algebras one finds several familiar algebras, some of which were independently reinvented in physics. For instance: (i) $Cl_{0,0}$: real numbers $\mathbb{R}$, (ii) $Cl_{1,0}$: split-complex and $Cl_{0,1}$: complex numbers $\mathbb{C}$, (iii) $Cl_{1,1}$: split-quaternion, $Cl_{2,0}$: hyperbolic quaternion and $Cl_{0,2}$: quaternion numbers $\mathbb{H}$, (iv) $Cl_{3,0}$: Pauli numbers and (v) $Cl_{1,3}$: Time-Space algebra [12], and $Cl_{3,1}$: Majorana algebra. Dirac's well-known algebra of gamma matrices is isomorphic to the complex Clifford algebra $Cl_4(\mathbb{C})$.

Each Clifford algebra $Cl_{p,q}$ is a unital, non-commutative, associative algebra over $\mathbb{R}$, which naturally forms a graded linear space of dimension 2^n, $Cl_{p,q} = \oplus_{k=0}^n Cl_{p,q}^k$. A real Clifford number (also called "multivector") x is a hypercomplex number over $\mathbb{R}$ with 2^n components of the form (using the implicit summation convention)

$$x = \underbrace{a1}_{1} + \underbrace{a_i\mathbf{e}^i}_{\binom{n}{1}} + \underbrace{\frac{1}{2!}a_{i_1i_2}\left(\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2}\right)}_{\binom{n}{2}} + \ldots + \underbrace{a_{1,\ldots,n}\left(\mathbf{e}^1 \wedge \ldots \wedge \mathbf{e}^n\right)}_{1}. \tag{3}$$

With $[x]_k$ the k-grade projector, $x = \sum_{k=0}^n [x]_k$. A pure grade component $[x]_k$ represents an oriented subspace segment of dimension k, called a k-vector. A Clifford number thus extends the concept of an oriented line segment (i.e., an ordinary vector) by incorporating all possible oriented subspace segments of R^n, with dimensions ranging from 0 to n. E.g., $[x]_2$ represents an oriented plane segment, whereby its $\binom{n}{2}$ components determine its direction and orientation in R^n and its magnitude corresponds to its surface area (shape is not encoded). By an element of $Cl_{p,q}$ one can represent any ensemble of oriented subspace segments of R^n. The Clifford product encodes the natural geometrical constructions that are possible with oriented subspace segments that result in new oriented subspace segments.

3. CLIFFORD ANALYSIS

Clifford Analysis (CA) is an advanced and recent part of mathematical analysis. It can be regarded as a generalization of the familiar 2-dimensional theory of complex holomorphic functions to an arbitrary number of dimensions [2, 3, 7]. More precisely, real CA is a function theory wherein one studies a subset of functions, with domain in R^n and which take values in a real Clifford algebra. In CA, the algebra of complex numbers is replaced with a Clifford algebra $Cl_{p,q}$ and the classical complex Cauchy-Riemann equation is replaced with a $Cl_{p,q}$-valued equation, which preferably has physical relevance. CA is the proper mathematical setting for studying generalized physics (encoded in the chosen generalized Cauchy-Riemann equation) in universes with an arbitrary number p of time dimensions and an arbitrary number q of space dimensions. The particular case, $p = 1$ and $q = 3$, has direct relevance for the physics in our own universe, and this version of CA provides us with a function theory over Minkowski space in which the functions can be designed to represent electromagnetic or quantum fields.

Of all the concepts associated with complex holomorphic functions (such as monogenicity, complex analyticity, conformal map, etc.) only the property of holomorphy extends to higher dimensions. Consequently we have, in any number of dimensions, a generalized Cauchy kernel (generalizing the complex kernel function $\frac{i}{2\pi}\frac{1}{z}$), Cauchy theorem, Cauchy integral formula, residue theorem, etc..

Real CA can be divided in two branches: (i) elliptic CA, for which $p = 0$ or $q = 0$, now about 30 years old [2, 3], and (ii) a) hyperbolic CA, for which $p = 1$ or $q = 1$, and b) ultra-hyperbolic CA, for which $p > 1$ and $q > 1$. The development of branch (ii) is still ongoing. Branch (i) is rooted in ordinary real function theory, while the much harder branch (ii) requires distribution theory [8]. For instance, the generalized Cauchy kernels in non-elliptic CA are rather complicated distributions [11]. Hence, hyperbolic CA of signature $(1, 3)$ is the type of analysis to be used to describe the physics in our universe.

4. ELECTROMAGNETISM

In terms of the Clifford algebra $Cl_{1,3}$, the model for electromagnetism simplifies to one simple equation [12]:

$$\partial F = -J. \tag{4}$$

Herein is: (i) ∂ the Dirac operator (i.e., the 4-dimensional nabla operator), (ii) $F \in Cl^2_{1,3}$ a bi-vector function representing the electromagnetic field and (iii) $J \in Cl^1_{1,3} \oplus Cl^3_{1,3}$ represents the electric (grade 1 part) and (if necessary) magnetic (grade 3 part) monopole charge-current density field, assumed of compact support $\bar{s}$.

The product between ∂ and F is the Clifford product in $Cl_{1,3}$. This product guarantees that automatically the correct partial derivative operator acts on the correct electromagnetic field component and that the result is equated to the correct component of the charge-current density field. The merit of this model not only lies in its extreme compactness, but especially in its analytical tractability by methods of CA, for solving electromagnetic source problems in vacuum and homogeneous isotropic dielectrics. We can regard (4) as a generalized Cauchy-Riemann equation that singles out a subset of Clifford functions, which now by design represent physical electromagnetic fields.

(i) Our current model for electromagnetism (irrespective of its formulation) has no solution in general! It requires that integrability conditions are fulfilled, which turn out to be exactly the local conservation of electric and magnetic monopole charge-current density. This follows almost readily from (4) and it rigorously explains the consequence of the well-known over-determination enforced by the equation and its relation to charge conservation. This point of view is usually not made clear in the classical literature.

(ii) Let $x_0 \in R^n$ denote the calculation point, $\mathcal{D} \triangleq C^\infty_c(R^n, \mathbb{R})$ the set of smooth real-valued functions with compact support in R^n and $\mathcal{D}'$ the linear space of distributions with support in R^n. By definition, the $Cl_{p,q}$-valued vector distribution $C_{x_0} \in \mathcal{D}' \otimes Cl^1_{p,q}$ satisfying

$$\partial C_{x_0} = \delta_{x_0} = C_{x_0}\partial, \tag{5}$$

with δ_{x_0} the delta distribution with support $\{x_0\}$, is called the *Cauchy kernel* relative to x_0. We do not need to specify boundary conditions for C_{x_0}, any fundamental solution of (5) will do. The significance of C_{x_0} lies in the fact that the (left and right) convolution operator $C_{x_0}*$ is a (left and right) $Cl_{p,q}$-valued and functional inverse of the Dirac operator ∂.

(iii) It can be shown [8], that any Cauchy kernel can be obtained from a scalar distribution $f_{x_0} \in \mathcal{D}' \otimes Cl^0_{p,q}$ as

$$C_{x_0} = \partial f_{x_0}. \tag{6}$$

with f_{x_0} a fundamental distribution of the ultra-hyperbolic equation

$$\Box_{p,q} f_{x_0} = \delta_{x_0}, \tag{7}$$

since $\partial^2 = \Box_{p,q}$, the canonical d'Alembertian of signature (p, q).

(iv) For $(p, q) = (1, 3)$, the causal and anti-causal scalar distributions, $f^+_{x_0}$ and $f^-_{x_0}$ respectively, are obtained from the ordinary wave equation $\Box f^\pm_{x_0} = \delta_{x_0}$ and are given by

$$f^\pm_{x_0} = \frac{\delta_{c^\pm_{x_0}}}{4\pi r}. \tag{8}$$

Herein is $\delta_{c^{\pm}_{x_0}}$ the delta distribution having as support the half null-cone $c^{\pm}_{x_0}$. The action of $f^{\pm}_{x_0}$ on any $\varphi \in \mathcal{D} \triangleq C^{\infty}_c\left(R^4, \mathbb{R}\right)$ is given by [5, p. 249 Eq. (9) and p. 252, Eq. (14') with $k = 0$, and after splitting in causal and anti-causal parts],

$$\left\langle f^{\pm}_{x_0}, \varphi\right\rangle = \frac{1}{4\pi}\int_0^{+\infty}\int_{S^2_{\mathbf{s}_0}} \varphi\left(t_0 \pm r, \mathbf{s}_0 + r\xi\right) dS^2 r dr, \tag{9}$$

with $x_0 = (t_0, \mathbf{s}_0)$, $r \triangleq |\mathbf{s} - \mathbf{s}_0|$ and $\xi \triangleq \frac{\mathbf{s}-\mathbf{s}_0}{|\mathbf{s}-\mathbf{s}_0|} \in S^2_{\mathbf{s}_0}$ the spatial unit 2-sphere centered at $\mathbf{s}_0$. This shows that $f^{\pm}_{x_0}$, $\forall x_0 \in R^4$, are defined $\forall \varphi \in \mathcal{D}$ and are distributions. For our problem, the components of the source function J in (4) play the role of test functions. The causal and anti-causal Cauchy kernels, associated to (8), are

$$C^{\pm}_{x_0} = \frac{\delta^{(1)}_{c^{\pm}_{x_0}}}{4\pi r} dt - \left(\pm\frac{\delta^{(1)}_{c^{\pm}_{x_0}}}{4\pi r} + \frac{\delta_{c^{\pm}_{x_0}}}{4\pi r^2}\right)\xi, \tag{10}$$

with $\delta^{(1)}_{c^{\pm}_{x_0}}$ the generalized derivative of $\delta_{c^{\pm}_{x_0}}$. A rigorous examination reveals that $\delta^{(1)}_{c^{\pm}_{x_0}}$ are partial distributions, only defined for test functions that vanish in a neighborhood of the calculation point x_0. This is a technical complication that is satisfactory handled within distribution theory [11], and then x_0 can also be inside the source region $\bar{s}$.

5. NEW RESULTS

(i) A new integral theorem for the electromagnetic radiation problem with given compact smooth sources in vacuum (or a homogeneous isotropic dielectric) can be derived over an arbitrary region $c \supset \bar{s}$ in time-space (hence also over a moving spatial region enclosing a moving source) and is given by

$$F\left(x_0\right) = \left\langle C^{-}_{x_0}, J\right\rangle + \left\langle C^{-}_{x_0}|_{\delta\bar{c}}, n^*\left(F|_{\delta\bar{c}}\right)\right\rangle_{\delta\bar{c}}, \tag{11}$$

with n^* the dual of the 4-dimensional normal n. Eq. (11) states that the electromagnetic field produced in any point x_0 in c is the result of a contribution from the source field J (given by a "volume" Schwartz pairing) and a contribution from the electromagnetic field present at the boundary $\delta\bar{c}$ (given by a "boundary" Schwartz pairing). The notation $|_{\delta\bar{c}}$ means restriction to the boundary.

(ii) Any electromagnetic field, in a region without sources, is holomorphic. This is expressed by (11) with $J = 0$,

$$F\left(x_0\right) = \left\langle C^{-}_{x_0}|_{\delta\bar{c}}, n^*\left(F|_{\delta\bar{c}}\right)\right\rangle_{\delta\bar{c}}. \tag{12}$$

Holomorphy can be interpreted as an intrinsic form of holography. In physical terms, this means that any such field can be reconstructed inside a region from its values on the boundary of that region. Any field: static, evanescent, transient, etc.!

(iii) The particular solution of the electromagnetic radiation problem follows from (11) as

$$F\left(x_0\right) = \left\langle C^{-}_{x_0}, J\right\rangle. \tag{13}$$

By substituting the expression (10) for $C^{-}_{x_0}$ and splitting the resulting electromagnetic field $F\left(x_0\right)$ in electric and magnetic parts, (13) can be shown to reproduce *Jefimenko's formulas* [6, 14].

We can easily obtain a second expression for $F\left(x_0\right)$ by substituting $C^{-}_{x_0} = \partial f^{-}_{x_0}$ in (13) and using the definition of the distributional derivative. We get

$$F\left(x_0\right) = -\left\langle f^{-}_{x_0}, \partial J\right\rangle. \tag{14}$$

Applying (9) with $\varphi = dJ = \partial \wedge J$, now results in the following equivalent expression for (14),

$$F\left(x_0\right) = -\frac{1}{4\pi}\int_0^{+\infty}\int_{S^2_{\mathbf{s}_0}} \left(\partial \wedge J\right)\left(t_0 - r, \mathbf{s}_0 + r\xi\right) dS^2_{\mathbf{s}_0} r dr. \tag{15}$$

This is a simple integral representation for the electromagnetic field, generated by a C^1 spatially compact monopole charge-current density source field J. Contrary to the form (13), no special care is required to evaluate the integrals in (15) when the observation point x_0 lies inside the source region $\bar{s}$. Expression (15) appears to be a new form for the particular solution of the electromagnetic radiation problem.

(iv) A radiation boundary condition can be formulated, by using a Cauchy kernel with appropriate causality. A practical application for such a condition is, e.g., the construction of a perfectly reflection-less grid boundary (without having to introduce locally a dissipating medium), useful for simulating wave propagation in time-space over an arbitrarily truncated discretization mesh.

(v) CA is also applicable to obtain an integral theorem for the electromagnetic radiation problem with given compact smooth sources in a curved background vacuum [10].

6. PROSPECTS

(i) Although it is not yet definitively proved, CA strongly suggests that the problem of electromagnetic scattering by arbitrarily shaped (homogeneous and isotropic) dielectric bodies (i.e., a generalization of Mie's problem) might be analytically solvable in CA.

(ii) A non-homogeneous medium can be taken into account with the current method if it is equivalent to a curved vacuum.

(iii) The incorporation of arbitrary bi-anisotropic media requires a substantial extension of the present method.

REFERENCES

1. Bolinder, E. F., "Clifford algebra, what is it?," *IEEE Ant. and Prop. Soc. Newsletter*, 18–23, 1987.
2. Brackx, F., R. Delanghe, and F. Sommen, *Clifford Analysis*, Pitman, London, 1982.
3. Delanghe, R., F. Sommen, and V. Souček, *Clifford Algebra and Spinor-valued Functions*, Kluwer, Dordrecht, 1992.
4. Doran, C. and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge, 2003.
5. Gel'fand, I. M. and G. E. Shilov, *Generalized Functions*, Vol. I, Academic Press, 1964.
6. Franssens, G. R., "The electromagnetic radiation problem in an arbitrary gravitational background vacuum," *WSEAS Transactions on Mathematics*, Vol. 6, 838–851, 2007.
7. Franssens, G. R., "Introduction to Clifford analysis," *Proceedings of the IKM 2009*, Weimar, Germany, July 2009, http://euklid.bauing.uni-weimar.de/ikm2009/paper.php.
8. Franssens, G. R., "Introduction to Clifford analysis over pseudo-euclidean space," *Proceedings of the IKM 2009*, Weimar, Germany, July 2009, http://euklid.bauing.uni-weimar.de/ikm2009/paper.php.
9. Franssens, G. R., "A physical interpretation of Clifford analysis," *Proceedings of the 18th ICFIDCAA*, Macau SAR, China, August 13–17, 2010.
10. Franssens, G. R., "Clifford analysis solution of the electromagnetic boundary value problem in a gravitational background vacuum," *Advances in Applied Clifford Algebras*, Vol. 20, 587–616, 2010.
11. Franssens, G. R., "The Cauchy kernels of ultrahyperbolic Clifford analysis," *Proceedings of the 9th International Conference on Clifford Algebras (ICCA'11)*, Weimar, Germany, July 2011.
12. Hestenes, D., *Space-time Algebra*, Gordon and Breach, New York, 1966.
13. Hestenes, D. and G. Sobczyk, *Clifford Algebra to Geometrical Calculus*, Reidel, Dordrecht, 1984.
14. Jefimenko, O. D., *Electricity and Magnetism: An Introduction to the Theory of Electric and Magnetic Fields*, Electret Scientific, Star City, 1989.
15. Lounesto, P., *Clifford Algebras and Spinors*, Cambridge University Press, Cambridge, 2001.