A physical interpretation of Clifford Analysis

Ghislain R. FRANSSENS
Belgian Institute for Space Aeronomy
Ringlaan 3, B-1180 Brussels, Belgium
E-mail: ghislain.franssens@aeronomy.be

18th ICFIDCAA, Aug 13–17, 2010

Abstract

The aim of this talk is to introduce mathematicians, working in Clifford Analysis, to a natural interpretation of their topic in the field of physics. Additionally, it also introduces physicists and engineers to an elegant and powerful mathematical tool, called Clifford Analysis, which lies behind many familiar physics models. Clifford Analysis is a part of mathematical analysis where one studies a selected subset of functions, with domain in R^n and which take values in a particular hypercomplex algebra called a Clifford algebra. A particular Clifford Analysis of functions taking values in the Clifford algebra $Cl_{1,3}$ emerges as a tailor-made function theory describing electromagnetic and quantum fields in Minkowski space.

Elliptic (ECA), Hyperbolic (HCA) and Ultrahyperbolic Clifford Analysis (UCA), involving a first order vector derivation (Dirac) operator, will be discussed and their relevance for physics applications emphasized. Both HCA and UCA are still in an early stage of development and some of the intricacies of their distributional foundation will be reviewed.

1 Clifford Analysis

Clifford Analysis (CA) is an advanced and rather recent part of mathematical analysis, which can be regarded as a particular generalization of the familiar Complex Analysis to an arbitrary number of dimensions, [10]. More precisely, CA is a function theory wherein one studies a subset of functions, with domain in R^n and which take values in a Clifford algebra. Hence a CA is characterized by two choices: (i) a Clifford algebra and (ii) an equation to select the desired subset of functions. The latter equation is itself expressible in the chosen Clifford algebra and plays the role of generalized Cauchy-Riemann equation. Given these two axes of choice, one can speak of Clifford Analyses (in plural), as each choice of (Clifford algebra, selection equation)-pair leads to a different function set.

1.1 Clifford algebras

Over each quadratic inner product space $R^{p,q}$, with canonical quadratic inner product of signature (p,q) and finite dimension $n \triangleq p+q$, one can define $n+1$ real Clifford algebras, one for each signature (p,q) with q running from 0 to n. Let $\{\mathbf{e}^1,\ldots,\mathbf{e}^n\}$ be an orthonormal basis for the linear inner product space $R^{p,q}$, with canonical quadratic form P of signature (p,q). The real Clifford algebra $Cl_{p,q}$ over $R^{p,q}$ is defined by, [17], [8],

$$\left(\mathbf{e}^1\right)^2 = \ldots = \left(\mathbf{e}^p\right)^2 = +1,\ \left(\mathbf{e}^{p+1}\right)^2 = \ldots = \left(\mathbf{e}^n\right)^2 = -1, \tag{1}$$

$$\mathbf{e}^i\mathbf{e}^j + \mathbf{e}^j\mathbf{e}^i = 0, i \neq j, \tag{2}$$

together with linearity over $\mathbb{R}$ and associativity.

Among the infinite series of Clifford algebras one finds several familiar algebras, some of which were independently reinvented in physics. Some are: (i) $Cl_{0,0}$: real numbers ($\mathbb{R}$), (ii) $Cl_{1,0}$: split-complex and $Cl_{0,1}$: complex numbers ($\mathbb{C}$), (iii) $Cl_{1,1}$: split-quaternion, $Cl_{2,0}$: hyperbolic quaternion and $Cl_{0,2}$: quaternion numbers ($\mathbb{H}$), (iv) $Cl_{3,0}$: Pauli numbers and (v) $Cl_{1,3}$: Time-Space algebra and $Cl_{3,1}$: Majorana algebra. Dirac's well-known algebra of gamma matrices is isomorphic to $Cl_4\left(\mathbb{C}\right)$.

Each Clifford algebra $Cl_{p,q}$ is a unital, non-commutative, associative algebra over $\mathbb{R}$, which naturally forms a graded linear space of dimension 2^n, $Cl_{p,q} = \oplus_{k=0}^{n} Cl_{p,q}^{k}$. A real Clifford number (also called "multivector" or "cliffor") x is a hypercomplex number over $\mathbb{R}$ with 2^n components of the form (using the implicit summation convention)

$$x = \underbrace{a1}_{1} + \underbrace{a_i \mathbf{e}^i}_{\binom{n}{1}} + \underbrace{\frac{1}{2!} a_{i_1 i_2} \left(\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2}\right)}_{\binom{n}{2}} + \ldots + \underbrace{a_{1,\ldots,n} \left(\mathbf{e}^1 \wedge \ldots \wedge \mathbf{e}^n\right)}_{1}. \quad (3)$$

With $[x]_k$ the k-grade projector, $x = \sum_{k=0}^{n} [x]_k$. A pure grade component $[x]_k$ represents an oriented subspace segment of dimension k, called a k-vector. A Clifford number thus extends the concept of an oriented line segment (i.e., an ordinary vector) by incorporating all possible oriented subspace segments of R^n, with dimensions ranging from 0 to n. E.g., $[x]_2$ represents an oriented plane segment, whereby its $\binom{n}{2}$ components determine its direction in R^n and its magnitude corresponds to its surface area. An element of $Cl_{p,q}$ thus encodes all oriented subspace segments that possibly can exist in R^n. The Clifford product encodes the natural geometrical constructions that are possible with oriented subspace segments and that result in new oriented subspace segments. This is the reason why Clifford algebras are also called geometrical algebras.

1.2 Selection equations

The complex Cauchy-Riemann (CR) condition in $\Omega \subseteq R^2$,

$$\left(\partial_x + i\partial_y\right)\left(u\left(x,y\right) + iv\left(x,y\right)\right) = 0, \quad (4)$$

selects a beautiful subset of functions from $\Omega \to \mathbb{C} : (x,y) \mapsto f = u\left(x,y\right) + iv\left(x,y\right)$, called complex holomorphic functions in Ω. The operator $\partial_x + i\partial_y$ is called the complex Cauchy-Riemann operator, [25].

It is natural to generalize the complex holomorphic function selection eq. (4) to a Clifford algebra valued equation of the form in $\Omega \subseteq R^n$,

$$D_{CR} F = 0, \quad (5)$$

wherein F is a function from $\Omega \subseteq R^n \to Cl_{p,q}$. There are a finite number of possibilities to choose a generalized Cauchy-Riemann operator D_{CR}. Some of the immediate choices are:

(i) *Cauchy-Riemann operator* in R^{n+1}. Define $D_{CR} \triangleq \partial_0 + \mathbf{e}^i \partial_i$ and let $F : \Omega \subseteq R^{n+1} \to Cl_{p,q}$.

(ii) *Dirac operator* ∂ in R^n. Define $D_{CR} \triangleq \partial \triangleq \mathbf{e}^i \partial_i$ and let $F : \Omega \subseteq R^n \to Cl_{p,q}$.

(iii) *General operator* in R^{2^n}. Define $D_{CR} \triangleq \partial_0 + \mathbf{e}^i \partial_i + \left(\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2}\right) \partial_{j(i_1,i_2)} + \ldots + \left(\mathbf{e}^1 \wedge \ldots \wedge \mathbf{e}^n\right) \partial_{2^n - 1}$ and let $F : \Omega \subseteq R^{2^n} \to Cl_{p,q}$. An example of this choice for $n = 2$ is the Fueter (or Moisil-Théodoresco) operator.

Each choice for D_{CR} leads to a different function theory, identified with $\ker D_{CR}$ and which we will call D_{CR}-holomorphic functions (called monogenic functions in [1]). The resulting function theory could be called a D_{CR}-Clifford Analysis (on $Cl_{p,q}$).

1.3 Families of Clifford Analyses

Clifford Analyses can be catalogued by choice of (Clifford algebra, selection equation)-pair.

A. By choice of algebra, we distinguish between:

(i) *Elliptic Clifford Analysis (ECA)*: $Cl_{n,0}$ ($Cl_{0,n}$). This function theory is about 30 years old and now a mature part of Mathematical Analysis, [1], [5].

(ii) *Complex Clifford Analysis*: $Cl_n(\mathbb{C})$. Based on the complex Clifford algebras, [23], [2].

(iii) *Hyperbolic Clifford Analysis (HCA)*: $Cl_{1,q}$ ($Cl_{p,1}$) and *Ultrahyperbolic Clifford Analysis (UCA)*: $Cl_{p,q}$ ($p > 1$ and $q > 1$). These function theories are still in embryonic state. The reason being that their development involves complicated distributions, which are still insufficiently known. A lot of work must still be done before the power and elegance of HCA for physics and engineering applications can be convincingly demonstrated.

B. By choice of equation, we distinguish between:

(i) *Cauchy-Riemann Clifford Analysis*

(ii) *Dirac Clifford Analysis.*

(iii) *General D_{CR}-type Clifford Analysis.*

We will from now on limit our attention to *Elliptic, Hyperbolic* and *Ultrahyperbolic* Dirac operator Clifford Analysis. Although ECA is a special case of HCA and HCA is a special case of UCA, there are mathematical and physical reasons to consider them as separate families, as we will see further.

2 Electromagnetism

Electromagnetism (EM) is definitely the simplest of Nature's "forces". It is actually a phenomenon of great beauty, but this is unfortunately not always fully appreciated – a state of affairs that can be attributed to the traditional Maxwell-Heaviside formulation involving vector calculus, [24], [19]. Electromagnetism is ubiquitously present in our society and the ease with which this phenomenon can be technologically manipulated has found countless applications in many domains (e.g., telecommunication, optics, power generation, electronics, etc.).

2.1 Models for EM

Maxwell's equations is not the only model for EM. Many other mathematical formulations have been given over the past 140 years. The question then arises: Are they all equivalent?

All such models have in common that they agree on the calculated *magnitudes* of the EM field components, once the magnitudes of the source field components are given. However, these models differ in their *geometrical content*, i.e., in the choice made of how the physical quantities are mathematically represented. The geometrical content of a mathematical model does matter, as it will influence the symmetries (or invariances) of the model. Modern physical insight requires that a good model should not only predict correct magnitudes, but should also model the symmetries of the physical phenomenon correctly. Mathematical models for EM with different geometrical representations not only differ from each other as mathematical constructs, but may possibly also differ (as a model) from the physical reality.

Here, we are only interested in models for EM that are applicable to locally reacting, homogeneous and isotropic dielectrics, with the vacuum as special case. A very limited selection of historical important models are:

(i) J. Maxwell's equations, version 1. This is a system of 20 partial differential equations, formulated in terms of the real algebra (1865).

(ii) J. Maxwell's equations, version 2. This is a system of partial differential equations, formulated under influence of W. Hamilton, in terms of the algebra of quaternions (1873).

(iii) O. Heaviside's equations. This is the well-known system of 4 partial differential equations, formulated in terms of the classical algebra of vector fields (developed by Gibbs and Heaviside) (1884). It is a simplification of Maxwell's equations, version 2.

(iv) The covariant equations, naturally derived immediately after the Special Theory of Relativity, and formulated in terms of the algebra of tensor fields (roughly first half of the 20-th century).

(v) The exterior differential equations. This is a system of two equations formulated in terms of the calculus of exterior differential forms (developed by E. Cartan) (roughly second half of the 20-th century).

(vi) The "equation for electromagnetism". We here refer to the single equation formulated in terms of $Cl_{1,3}$ and which (to my knowledge) first appeared in [16]. Two older formulations, also resulting in a single equation, are [21], using the biquaternions $Cl_2(\mathbb{C}) \simeq Cl_{3,0} \simeq Cl_{1,2}$, and [20], using the Dirac algebra $Cl_4(\mathbb{C}) \simeq Cl_{4,1} \simeq Cl_{2,3} \simeq Cl_{0,5}$. Remark the even subalgebra relations $Cl_{1,2} = Cl^e_{1,3}$ and $Cl_{1,3} = Cl^e_{4,1}$.

Models (i), (ii) and (iii) are incomplete models for EM (in our sense), because they do not contain the mathematical equivalents of real physical invariances (e.g., Lorentz-invariance[1]). These models are only correct in their predictions for field component magnitudes. Model (iii) is commonly taught in applied sciences and, somewhat incorrectly, called Maxwell's equations. Formulations (iv) and (v) are physically more faithful models for EM. Model (vi) stands out as mathematically the most elegant formulation, a consequence of the power of the particular Clifford algebra used in this model. In addition, (vi) reveals a very important property of EM, discussed below, that is not contained in models (i)–(v).

2.2 Clifford algebra model of EM

The Clifford algebra $Cl_{1,3}$ is so tailor-made for EM that the mathematical model, based on this algebra, simplifies to one simple and beautiful $Cl_{1,3}$-valued equation:

$$\partial F = -J. \tag{6}$$

Herein is: (i) ∂ the Dirac operator, (ii) $F \in Cl^2_{1,3}$ a bivector function representing the electromagnetic field and (iii) $J \in Cl^1_{1,3} \oplus Cl^3_{1,3}$ represents the electric (grade 1 part) and (possibly) magnetic monopole (grade 3 part) charge-current density field (assumed smooth, of compact support and satisfying the integrability conditions $[\partial J]_0 = 0 = [\partial J]_4$).

The product between ∂ and F is the Clifford product in $Cl_{1,3}$. This product guarantees that automatically the correct partial derivative operator acts on the correct EM field component and that the result is equated to the correct component of the charge-current density field. The merit of this model not only lies in its extreme compactness, but especially in its analytical tractability by methods of CA. It is remarkable that Clifford algebras were conceived by W. Clifford in 1876, independently and unrelated to the emerging insights in electromagnetism at about the same time. We can regard eq. (6) as a generalized CR eq. that singles out a subset of Clifford functions, which represent physical EM fields.

We can readily generalize this equation using the following recipe.

(i) Keep the formal form of equation (6).

(ii) Select any Clifford algebra $Cl_{p,q}$ of choice.

(iii) Let F be a $Cl_{p,q}$-valued multivector function.

(iv) Let J be a given $Cl_{p,q}$-valued multivector function.

The result is a model of a generalized EM in a flat universe with p time and q space dimensions!

[1]What is called a Lorentz tranformation in classical EM is not a linear transformation of the **E** and **B** vector fields *separately*, as mathematically would be expected, but the "vector analogue" of the Lorentz tranformation of a second order antisymmetric EM tensor field.

Clifford Analysis	**Generalized Electromagnetism**
Dirac equation	Equation of EM
Clifford-valued functions	Generalized EM fields
Singularities, Residues	Source function (J)
Holomorphy	Holography
Cauchy/Integral theorems	Reciprocity theorems
Boundary Value problems	Radiation problems
Riemann-Hilbert problems	Scattering problems

Table 1: Correspondences between CA and EM.

3 Physical Interpretation

(i) A *Clifford-valued function* F, satisfying $\partial F = -J$ with given J, can be interpreted as a *generalized EM field.*

(ii) A Clifford-valued function F is *holomorphic* in regions where $J = 0$. Holomorphic functions can be interpreted as sourceless (or free) *generalized EM fields.*

(iii) The local *non-holomorphy* of F are caused by and encoded in the (compact support) source function J.

Probably the most intriguing question related to this interpretation is what the property of holomorphy means physically. We have the following situation.

(i) In CA – *Holomorphy*: (Cauchy's integral formula) A Clifford-valued function F, holomorphic in some region Ω, can be reconstructed inside a subregion $\overline{c} \subset \Omega$ from its values on the boundary $\delta\overline{c}$.

(ii) In EM – *Holography*: (Three-dimensional imaging) An EM field F without sources in Ω, reflected off an object inside $\overline{c} \subset \Omega$, produces an interference pattern that is encoded in a photographic film at $\delta\overline{c}$. After removing the object and by illuminating this film with the original incident field, the image of the object (i.e., the field F inside $\overline{c}$) is recreated.

This strongly suggests that Holomorphy $\Leftrightarrow$ Holography. The appellation "holomorphy", given by mathematicians in the 19-th century to the studied complex functions, and the appellation "holography", independently given by physicists in the 20-th century to a related EM phenomenon in our universe, display a remarkable and fortunate resemblance.

A number of immediate correspondences between CA and EM is given in Table 1.

The naturalness of $Cl_{1,3}$ is further demonstrated by the ease with which one derives all the classical Lorentz-invariant equations from classical quantum physics. This is done by factorization (or "square rooting") in $Cl_{1,3}$ of the Klein-Gordon equation $\partial^2\phi = -m^2\phi$, describing a scalar (spin-0) field with mass m.

(i) For $m \neq 0$ we get:

(i.1) Dirac's equation, describing a spin-1/2 field with mass m,

(i.2) Proca's equation, for a spin-1 field with mass m.

(ii) For $m = 0$ we get:

(ii.1) The equation for a massless spin-1/2 field,

(ii.2) Maxwell-Heaviside's eqs., for a massless spin-1 field (EM).

$Cl_{1,3}$ is also useful for describing higher spin fields (Rarita-Schwinger eqs.), e.g., [3].

The fact that a particular Clifford algebra models so elegantly and efficiently the structure of EM and the classical quantum fields, is a clear indication that these fields possess a deeper mathematical number structure. Inversely, one could say that $Cl_{1,3}$ forms a kind of "natural" number system for the universe that we live in. This agreement can be explained in terms of the growing awareness that physical laws are in essence expressing geometrical relationships and that Clifford algebras are in a natural sense geometrical algebras.

4 Elliptic Clifford Analysis

4.1 Interpretation

Here $\partial^2 = \pm\Delta_n$, $\Delta_n g = \delta$ and $C = \pm\partial g$ is a vector *potential* field (+: $R^{n,0}$ or −: $R^{0,n}$).

Example. A holomorphic function of grade 1, i.e., $\mathbf{f} \in Cl^1_{3,0} : \partial\mathbf{f} = 0 = \mathbf{f}\partial$, describes in $R^{3,0}$ an incompressible ($\nabla \cdot \mathbf{f} = 0$) and irrotational ($\nabla \times \mathbf{f} = \mathbf{0}$) static fluid without sources nor sinks.

ECA is useful to solve *elliptic* (i.e., potential) problems: electrostatics, magnetostatics, Newtonian gravity, static fluid problems, etc. Hence, it is the natural mathematical tool to use in a *static* universe. Therefore, Clifford Analysis based on $Cl_{n,0}$ ($Cl_{0,n}$) is a function theory of *holographic static fields in a flat universe with* n *time dimensions (*n *space dimensions)!*

4.2 Cauchy kernel

The distribution $C \in Cl^1_{n,o}$, defined by

$$\partial C = \delta = C\partial, \tag{7}$$

is called *Cauchy kernel*. Its importance stems from the fact that the convolution operator $C*$ ($*C$) is a right (left) inverse of ∂. By Poincaré's lemma, $C = \partial g$, with g a scalar distribution satisfying

$$\Delta g = \delta, \ \left(\Delta \triangleq \partial^2\right). \tag{8}$$

Hence, the Cauchy kernel C is obtained as a derivation of a fundamental solution of Poisson's equation. It is given by the *regular distribution*

$$C \triangleq \frac{1}{A_{n-1}} \frac{\mathbf{x}}{|\mathbf{x}|^n}, \text{ with } A_{n-1} \triangleq \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{9}$$

4.3 Generalized Cauchy integral theorem

Theorem 1 *([1]) For any* $F : \Omega \subseteq R^n \to Cl_{n,0} : \partial F = 0$ *and domain* $\overline{\Sigma} \subset \Omega$ *holds that*

$$F(\mathbf{x}_0) = \int_{\delta\overline{\Sigma}} C(\mathbf{y} - \mathbf{x}_0)\, d\sigma_{\mathbf{y}} F(\mathbf{y}), \forall \mathbf{x}_0 \in \Sigma,$$

wherein $d\sigma_{\mathbf{y}}$ *is the surface element on the boundary* $\delta\overline{\Sigma}$.

Holomorphy: knowledge of F at the boundary $\delta\overline{\Sigma}$ allows to reconstruct F in the interior Σ.

5 Hyperbolic and Ultrahyperbolic Clifford Analysis

5.1 Interpretation

Here $\partial^2 = \Box_{p,q} \triangleq \Delta_p - \Delta_q$, $\Box_{p,q} g_{\mathbf{x}_0} = \delta_{\mathbf{x}_0}$ and $C_{\mathbf{x}_0} = \partial g_{\mathbf{x}_0}$ is a vector *wave* field.

Example. A holomorphic function of grade 1, i.e., $\mathbf{f} = \left(\rho\mathbf{e}^t, \rho\mathbf{v}\right) \in Cl^1_{1,3} : \partial\mathbf{f} = 0 = \mathbf{f}\partial$, describes in $R^{1,3}$ a mass conserved ($\partial_t\rho - \nabla\cdot(\rho\mathbf{v}) = 0$) 4D irrotational ($\left(\mathbf{e}^t\partial_t, \nabla\right) \wedge \left(\rho\mathbf{e}^t, \rho\mathbf{v}\right) = 0$) dynamic fluid without sources nor sinks (ρ: fluid density, $\mathbf{v}$: fluid velocity).

HCA (UCA) is useful to solve *(ultra)hyperbolic* (i.e., wave) problems: full EM, acoustics, etc. Hence, it is the natural mathematical tool to use in a *dynamic* universe. Therefore, Clifford Analysis based on $Cl_{p,q}$ is a function theory of *holographic dynamic fields in a flat universe with* p time and q *space dimensions!*

5.2 Cauchy kernel

Again, $C_{\mathbf{x}_0} = \partial g_{\mathbf{x}_0}$, with $g_{\mathbf{x}_0}$ a scalar distribution now satisfying

$$\Box_{p,q} g_{\mathbf{x}_0} = \delta_{\mathbf{x}_0}, \ \left(\Box_{p,q} \triangleq \partial^2 = \Delta_p - \Delta_q \right). \tag{10}$$

A Cauchy kernel $C_{\mathbf{x}_0}$ in UCA is obtained as the vectorial derivation of a fundamental solution of the Ultrahyperbolic Equation (UE). It is now a complicated *singular distribution.* Equation (10) has shown to be quite obstinate to attacks of several famous 20-th century mathematicians. After roughly 50 years of combined efforts, see [15], [22], [4], [9], [6], [7], [14], [18], a formal form for a fundamental solution was obtained. Let $p, q \in \mathbf{N} : p + q \triangleq n \geq 2$. A (real) fundamental solution $g_{\mathbf{x}_0}$ of (10) is (P is the quadratic form):

(i) for $n > 2$,

$$g_{\mathbf{x}_0} = -\frac{1}{(n-2)\, A_{n-1}} \frac{1}{2} \left(\begin{array}{c} e^{iq\frac{\pi}{2}} \left(P(\mathbf{x}-\mathbf{x}_0) + i0\right)^{1-\frac{n}{2}} \\ +e^{-iq\frac{\pi}{2}} \left(P(\mathbf{x}-\mathbf{x}_0) - i0\right)^{1-\frac{n}{2}} \end{array} \right), \tag{11}$$

(ii) for $n = 2$,

$$g_{\mathbf{x}_0} = \frac{\cos\left(q\frac{\pi}{2}\right)}{4} \frac{1}{\pi} \ln\left|P(\mathbf{x}-\mathbf{x}_0)\right| - \frac{\sin\left(q\frac{\pi}{2}\right)}{4} 1_{-}(P(\mathbf{x}-\mathbf{x}_0)). \tag{12}$$

The Cauchy kernel $C_{\mathbf{x}_0}$ depends profoundly on the parity of p and of q. E.g., for $(p,q) \in H \triangleq \{(p,q) \in \mathbb{Z}_{o,+} \times \mathbb{Z}_{o,+} : (p,q) \neq (1,1)\}$ (the Huygens cases), the following form can be obtained,

$$C_{\mathbf{x}_0} = \frac{1}{((q-2)/2)_{((n-2)/2)}} (\mathbf{x}-\mathbf{x}_0)^S \left(\delta^{((n-2)/2)}_{(P(\mathbf{x}-\mathbf{x}_0))}\right)_0. \tag{13}$$

Herein is: (i) $(\ldots)_0$ a regularization of a k-multiplet delta distribution supported on the null-space $P(\mathbf{x}-\mathbf{x}_0) = 0$, (ii) $a_{(m)}$ the falling factorial polynomial of degree m in a and (iii) superscript S means spatial conjugation. Expression (13) can be derived, from the form of the solution for general (p,q), by taking into account the special form to which the distributions $\left(P \pm i\delta^{(k)}_{(P(\mathbf{x}-\mathbf{x}_0))} 0\right)^z$ reduce in general, or it can also be derived directly from the definition of generalized d'Alembertian, [13].

Regularization of $\delta^{(k)}_{(P(\mathbf{x}-\mathbf{x}_0))}$ can not be done in a canonical way (i.e., uniquely). Any two regularizations differ by a term $c\delta_{\mathbf{x}_0}$, $c \in \mathbb{R}$. Hence, the question of the uniqueness of the Cauchy kernel $C_{\mathbf{x}_0}$ arises (mod pseudo-harmonics). Fortunately, due to the presence of the factor $(\mathbf{x}-\mathbf{x}_0)^S$ and since $(\mathbf{x}-\mathbf{x}_0)^S c\delta_{\mathbf{x}_0} = 0$, $\forall c \in \mathbb{R}$, $C_{\mathbf{x}_0}$ is unique for the Huygens cases.

5.3 Generalized Cauchy integral theorem

Theorem 2 *([11]) For any $F : \Omega \subseteq R^n \to Cl_{p,q} : \partial F = -J$ and a compact domain $\overline{\Sigma} \subset \Omega$ (satisfying a technical condition) holds that*

$$F(\mathbf{x}_0) = \langle C_{\mathbf{x}_0}, J \rangle_S + \left\langle C_{\mathbf{x}_0}|_{\delta\overline{\Sigma}}, n\, F|_{\delta\overline{\Sigma}} \right\rangle_S, \forall \mathbf{x}_0 \in \Sigma,$$

wherein (i) n is the outward normal field on the coherently oriented boundary $\delta\overline{\Sigma}$, (ii) $|_{\delta\overline{\Sigma}}$ stands for restriction to $\delta\overline{\Sigma}$ and (iii) $\langle , \rangle_S$ denotes Schwartz pairing.

Holomorphy: if $J = 0$, knowledge of F at the boundary $\delta\overline{\Sigma}$ allows to reconstruct F in the interior Σ.

The technical condition amounts to the following.

(i) Restrict $\overline{\Sigma}$ such that its boundary $\delta\overline{\Sigma}$ can be represented as an injective *immersion* $T : \delta\overline{\Sigma} \to R^n$.

(ii) The restriction of the Cauchy kernel $C_{\mathbf{x}_0}$ to the boundary $\delta\overline{\Sigma}$ is defined as the *pullback* along T, $C_{\mathbf{x}_0}|_{\delta\overline{\Sigma}} = T^* C_{\mathbf{x}_0}$.

(iii) The pullback of a distribution along an immersion exists and is unique iff, [18, p. 263],

$$WF\left(C_{\mathbf{x}_0}\right) \cap N_T = \varnothing, \tag{14}$$

with N_T the *set of normals* of the function T and $WF\left(C_{\mathbf{x}_0}\right)$ the *wave front set* of $C_{\mathbf{x}_0}$.

5.4 Boundary Value Problem (BVP)

A BVP consists of the equation $\partial F = -J$ for F in Σ, together with a boundary condition for F on $\delta\overline{\Sigma}$.

A physical interpretation in CA(1,3) of a BVP is the generation of an EM field by a given source density J in open space or in a cavity.

The basis for its solution is the generalized Cauchy integral formula. It further requires knowledge of the limit value $\lim_{\mathbf{x}_0 \to \delta\overline{\Sigma}} C_{\mathbf{x}_0}|_{\delta\overline{\Sigma}}$. The explicit expression for this limit distribution is a far generalization to UCA of the Sokhotsky-Plemelj equation. $\lim_{\mathbf{x}_0 \to \delta\overline{\Sigma}} C_{\mathbf{x}_0}|_{\delta\overline{\Sigma}}$ will produce the kernel of the generalized Hilbert transformation in UCA. The analytic solution of the BVP in UCA will depend on the properties of this Hilbert kernel.

5.5 Generalized Riemann-Hilbert Problem (RHP)

A RHP typically consists of 2 coupled BVPs, with a discontinuity condition to be satisfied on a common boundary.

A physical interpretation in CA(1,3) of such a problem is the scattering of an EM field by a homogeneous dielectric obstacle.

Such problems can be reduced to a generalized and singular integral equation holding over the common boundary. The proper formulation of its solution again requires knowledge of the properties of $\lim_{\mathbf{x}_0 \to \delta\overline{\Sigma}} C_{\mathbf{x}_0}|_{\delta\overline{\Sigma}}$. The distribution $\lim_{\mathbf{x}_0 \to \delta\overline{\Sigma}} C_{\mathbf{x}_0}|_{\delta\overline{\Sigma}}$ is expected to be an element of a convolution algebra. It is conjectured that this algebra might provide a route towards the analytical solution (in an algebraic way) of these types of problems.

5.6 Distributional complications and challenges

(i) The functions $f_\pm(z) \triangleq \langle (P \pm i0)^z, \varphi \rangle$, $\forall \varphi \in \mathcal{D}(R^n)$, are complex analytic in z, except at $z = -n/2 - k$, $\forall k \in \mathbb{N}$, where they have simple poles. Any Cauchy kernel will contain the expressions $(P \pm i0)^{-n/2}$, which are non-existing distributions!

(ii) The quantities $(P \pm i0)^{-n/2}$ fortunately exist as partial distributions (i.e., as a linear continuous functional only defined over a proper subset of test functions). They can be regularized to a distribution $\left((P \pm i0)^{-n/2}\right)_e$, by a functional extension process involving a test function projector operator. Such operators however are not unique!.

(iii) Hence, one must check if the non-uniqueness of the regularization propagates into the Cauchy kernel or not.

(iv) The distribution $g_{\mathbf{x}_0}$ can also be obtained as the pullback along P of a one-dimensional Associated Homogeneous Distribution. If $\mathbf{x}_0 \in \operatorname{supp} J$, the generalized chain rule can not be applied, when calculating $\partial g_{\mathbf{x}_0}$, and we need to use the definition of generalized derivative, which is a much more cumbersome calculation.

(v) To apply Cauchy integral formula in UCA requires complete characterization of $C_{\mathbf{x}_0}|_{\delta\overline{\Sigma}}$.

(vi) Solution of BV and RH problems in UCA also requires complete characterization of $\lim_{\mathbf{x}_0 \to \delta\overline{\Sigma}} C_{\mathbf{x}_0}|_{\delta\overline{\Sigma}}$.

(vii) The technical condition given above (arising in HCA and UCA only), is essentially an existence and uniqueness condition for the considered BVP or RHP and will restrict the admissible boundaries.

It will now be clear that the proper foundation for UCA requires *distribution theory.* In particular, the theory of *Associated quasiHomogeneous Distributions*. Many technical *lacunas* for these distributions have still to be filled in, in order that formal expressions in UCA are turned into explicit and justified formulas.

References

[1] F. Brackx, R. Delanghe and F. Sommen, *Clifford analysis*, Pitman, London, 1982.

[2] J. Bureš and V. Souček, Generalized hypercomplex analysis and its integral formulae. *Complex Variables: Theory and Application*, 5, 53–70, 1985.

[3] J. Bureš, F. Sommen, V. Souček, P. Van Lancker, "Rarita-Schwinger type operators in Clifford analysis", J. Funct. Anal., 185, 425–455, 2001.

[4] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Vol. II, Wiley&Sons, p. 697, 1962.

[5] R. Delanghe, F. Sommen and V. Souček, *Clifford Algebra and Spinor-Valued Functions*, Kluwer, Dordrecht, 1992.

[6] G. de Rham, "Solution élémentaire d'équations aux dérivées partielles du second ordre à coefficients constants", Colloque Henri Poincaré, Paris, Oct. 1954.

[7] G. de Rham, "Solution élémentaire d'opérateurs différentiels du second ordre", Ann. Inst. Fourier, 8, 337–366, 1958.

[8] C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge Univ. Press, Cambridge, 2003.

[9] Y. Fourès-Bruhat, "Solution élémentaire d'équations ultra-hyperboliques", J. Math. Pures Appl., 35, 277–288, 1956.

[10] G. Franssens, "Introduction to Clifford Analysis", Proc. IKM '09, Weimar, 2009.

[11] G. Franssens, "Introduction to Clifford Analysis over pseudo-Euclidean space", Proc. IKM '09, Weimar, 2009.

[12] G. Franssens, "Delta distributions supported on quadratic $O(p, q)$-invariant surfaces", J. Math. Phys., Vol. 50, Issue 5, 2009.

[13] G. Franssens, "Cauchy kernels of Ultrahyperbolic Clifford Analysis – Huygens cases", preprint.

[14] I.M. Gel'fand, G.E. Shilov, *Generalized Functions*, Vol. I, Academic Press, 1964.

[15] J. Hadamard, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Hermann et Cie, Paris, 1932.

[16] D. Hestenes, *Space-Time Algebra*, Gordon and Breach, New York, 1966.

[17] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometrical Calculus*, Reidel, Dordrecht, 1984.

[18] L. Hörmander, *The analysis of linear partial differential operators I*, Grundl. Math. Wissenschaft., 256, Springer, 1983.

[19] J.D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1999.

[20] G. Juvet, A. Schidlof, “Sur les nombres hypercomplexes de Clifford et leurs applications à l’analyse vectorielle ordinaire, à l’électromagnetisme de Minkowski et à la théorie de Dirac”, Bull. Soc. Neuchat. Sci. Nat., 57, 127–147, 1932.

[21] C. Lanczos, *Die Funktionentheoretischen Beziehungen der Maxwellschen Aethergleichungen – Ein Beitrag zur Relativitäts und Elektronentheorie*, PhD Thesis, 1919. (http://arxiv.org/PS_cache/physics/pdf/0408/0408079v1.pdf)

[22] M. Riesz, “L’intégrale de Riemann-Liouville et le problème de Cauchy”, Acta Math., 81, 1–223, 1949.

[23] J. Ryan, Complexified Clifford analysis. *Complex Variables: Theory and Application,* 1, 119–149, 1982.

[24] J.A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941.

[25] E.T. Whittaker, G.N. Watson, *A Course of Modern Analysis*, 4th Ed., Cambridge Univ. Press, Cambridge, 1927.