

INTRODUCTION TO CLIFFORD ANALYSIS OVER PSEUDO-EUCLIDEAN SPACE

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Abstract. *An introduction is given to Clifford Analysis (CA) over pseudo-Euclidean space E^n of arbitrary signature. CA over E^n is regarded as a function theory of Clifford-valued functions F satisfying an equation of the form $\partial F = -S$, for a given function S and with ∂ a first order vector-valued differential (Dirac) operator. The formulation of CA over E^n presented here pays special attention to its geometrical setting. This permits to identify tensors which qualify as geometrical Dirac operators and to take a position on the naturalness of contravariant and covariant versions of such a theory. In addition, a formal method is described to construct the general solution to the aforementioned equation in the context of covariant CA over E^n .*

1 INTRODUCTION

The aim of this paper is to give an introduction to Clifford Analysis (CA) over pseudo-Euclidean space E^n . CA over E^n , called Ultrahyperbolic Clifford Analysis (UCA) for short, is a non-trivial extension of the more familiar Euclidean CA over \mathbf{R}^n , [1], [2], [3]. UCA is the proper mathematical setting for studying physics in Minkowski spaces with an arbitrary number of time dimensions p and space dimensions q . The particular case of Hyperbolic Clifford Analysis (HCA), corresponding to $p = 1$ and $q > 1$, has direct relevance to physics, with in particular the case $p = 1, q = 3$ providing a tailor-made function theory, applicable to electromagnetism and quantum physics, [4], [5].

Let Ω be an open region in E^n , $\mathbf{V}^{p,q}$ a real sesquilinear product space of signature (p, q) , $m \triangleq p + q$, and $Cl(\mathbf{V}^{p,q})$ the universal Clifford algebra generated by $\mathbf{V}^{p,q}$. UCA is the study of (e.g., smooth) functions from $\Omega \rightarrow Cl(\mathbf{V}^{p,q})$, acted upon by a first order $Cl(\mathbf{V}^{p,q})$ -valued vector differential operator ∂ , called a Dirac operator. In particular, we want to derive integral representations for functions F satisfying first order equations such as $\partial F = -S$, with S smooth and of compact support. By choosing $S = 0$, a special subset of functions is singled out, called (left) ultrahyperbolic Clifford holomorphic functions, and which can be thought of as a generalization of the familiar complex holomorphic functions in Complex Analysis.

We here give a formulation of UCA which pays attention to its geometrical setting relative to pseudo-Euclidean space E^n of signature (r, s) , $n \triangleq r + s$. More precisely, we focus on how a given abstract space $\mathbf{V}^{p,q}$ is related – or if possible can be unrelated – to the (common) tangent space $\mathbf{E}^{r,s}$ of E^n . To this end, we discuss the concept of soldering, relating a given linear space to the tangent space of a given manifold. We start off with independent signatures for $\mathbf{V}^{p,q}$ and $\mathbf{E}^{r,s}$ and then investigate the feasibility of this choice along the way. Further, we also show how to give an invariant meaning to the vector differential operator ∂ , a matter which is usually not addressed in CA. It is found that a Dirac operator in UCA acquires a natural geometrical meaning if it is defined as an appropriate tensor. As a consequence of this, any such Dirac operator itself defines a soldering, which implies that in a geometrical meaningful UCA, the spaces $\mathbf{V}^{p,q}$ and $\mathbf{E}^{r,s}$ can not be unrelated. In addition, we show that there exists a natural asymmetry between covariant and contravariant Clifford Analysis over E^n . On the one hand, it is found that no geometrical invariant contravariant Clifford Analysis with a *first order* contravariant Dirac operator, taking values in $Cl(\mathbf{E}^{r,s})$, exists. On the other hand, a geometrical invariant covariant Clifford Analysis with a first order covariant Dirac operator, taking values in $Cl(\mathbf{E}^{*r,s})$, arises naturally.

Based on the aforementioned geometrical setting, we then give a formal method to construct an integral representation for covariant Clifford-valued functions F^* satisfying $\partial^* F^* = -S^*$, with ∂^* an anisotropic covariant Dirac operator, taking values in $Cl(\mathbf{V}^{*p,q})$. It is found that, in order to completely characterize this representation, it is necessary to calculate the restriction $C_{x_0}^*|_{\delta\bar{c}}$ of an ultrahyperbolic covariant Cauchy kernel $C_{x_0}^*$ to the boundary $\delta\bar{c}$ of a chosen region c . Any Cauchy kernel $C_{x_0}^*$ in turn is obtained by operating with ∂^* on a fundamental solution of the ultrahyperbolic wave equation. The details of this calculation however lead far into distribution theory and are outside the scope of this introduction.

2 CLIFFORD ALGEBRAS OVER PSEUDO-EUCLIDEAN SPACE

We only consider real Clifford Algebras of finite dimension, which we wish to situate in a pseudo-Euclidean setting.

2.1 Preliminaries

Let \mathbf{R}^m denote m -dimensional coordinate space over \mathbb{R} . Let $p, q \in \mathbb{N}$ with $m = p + q$ and h a real sesquilinear product structure, of signature (p, q) , and given by a nondegenerate symmetric bilinear function $h : \mathbf{R}^m \times \mathbf{R}^m \rightarrow \mathbb{R}$ such that $(\mathbf{v}, \mathbf{w}) \mapsto h(\mathbf{v}, \mathbf{w})$. Introduce the sesquilinear product space $\mathbf{V}^{p,q} \triangleq (\mathbf{R}^m, h)$ and its dual $\mathbf{V}^{*p,q} \triangleq (\mathbf{R}^m, h^{-1})$.

Define $(\mathbf{V}^{p,q})^{\wedge 0} \triangleq \mathbb{R}$, $(\mathbf{V}^{p,q})^{\wedge 1} \triangleq \mathbf{V}^{p,q}$ and denote by $(\mathbf{V}^{p,q})^{\wedge k}$, $\forall k \in \mathbf{Z}_{[2,m]}$, the linear space of antisymmetric contravariant tensors of grade (i.e., order) k over \mathbb{R} . Elements of $(\mathbf{V}^{p,q})^{\wedge k}$ are called k -vectors (0-vectors and 1-vectors are also called scalars and vectors, respectively) and elements of the graded linear space $\mathbf{M} \triangleq \bigoplus_{k=0}^m (\mathbf{V}^{p,q})^{\wedge k}$ over \mathbb{R} are called multivectors. A k -vector $\mathbf{v}_k \in (\mathbf{V}^{p,q})^{\wedge k}$ has strict components $(v^{i_1 \dots i_k}, 1 \leq i_1 < \dots < i_k \leq m)$ with respect to a basis $B_k \triangleq \{\epsilon_{i_1 \dots i_k}, 1 \leq i_1 < \dots < i_k \leq m\}$ for $(\mathbf{V}^{p,q})^{\wedge k}$ ($B_0 \triangleq \{1\}$) and can be represented as (using the summation convention over unordered indices)

$$\mathbf{v}_k = \frac{1}{k!} v^{i_1 \dots i_k} \epsilon_{i_1 \dots i_k}. \quad (1)$$

Define $(\mathbf{V}^{*p,q})^{\wedge 0} \triangleq \mathbb{R}$, $(\mathbf{V}^{*p,q})^{\wedge 1} \triangleq \mathbf{V}^{*p,q}$ and denote by $(\mathbf{V}^{*p,q})^{\wedge k}$, $\forall k \in \mathbf{Z}_{[2,m]}$, the linear space of antisymmetric covariant tensors of grade k over \mathbb{R} . Elements of $(\mathbf{V}^{*p,q})^{\wedge k}$ are called k -covectors (0-vectors and 1-covectors are also called scalars and covectors, respectively) and elements of the graded linear space $M^* \triangleq \bigoplus_{k=0}^m (\mathbf{V}^{*p,q})^{\wedge k}$ over \mathbb{R} are called multicovectors. A k -covector $\mathbf{v}_k^* \in (\mathbf{V}^{*p,q})^{\wedge k}$ has strict components $(v_{j_1 \dots j_k}, 1 \leq j_1 < \dots < j_k \leq m)$ with respect to a cobasis $B_k^* \triangleq \{\theta^{j_1 \dots j_k}, 1 \leq j_1 < \dots < j_k \leq m\}$ for $(\mathbf{V}^{*p,q})^{\wedge k}$ ($B_0^* \triangleq \{1\}$) and can be represented as

$$\mathbf{v}_k^* = \frac{1}{k!} v_{j_1 \dots j_k} \theta^{j_1 \dots j_k}. \quad (2)$$

Endow $(\mathbf{V}^{p,q})^{\wedge k}$ with a real sesquilinear product $\cdot : (\mathbf{V}^{p,q})^{\wedge k} \times (\mathbf{V}^{p,q})^{\wedge k} \rightarrow \mathbb{R}$ such that, with respect to a basis B_k for $(\mathbf{V}^{p,q})^{\wedge k}$, $\forall k \in \mathbf{Z}_{[1,m]}$,

$$\begin{aligned} (\mathbf{v}_k, \mathbf{w}_k) &\mapsto \mathbf{v}_k \cdot \mathbf{w}_k \triangleq \frac{1}{k!} \frac{1}{k!} v^{i_1 \dots i_k} w^{j_1 \dots j_k} \det [h(\epsilon_{i_a}, \epsilon_{j_b})], \\ &= \frac{1}{k!} \frac{1}{k!} v^{i_1 \dots i_k} w^{j_1 \dots j_k} \delta_{i_1 \dots i_k}^{j_1 \dots j_k} h(\epsilon_{i_1}, \epsilon_{j_1}) \dots h(\epsilon_{i_k}, \epsilon_{j_k}), \\ &= \frac{1}{k!} v^{l_1 \dots l_k} w^{j_1 \dots j_k} h(\epsilon_{l_1}, \epsilon_{j_1}) \dots h(\epsilon_{l_k}, \epsilon_{j_k}). \end{aligned} \quad (3)$$

Herein is δ the k -covariant, k -contravariant generalized Kronecker tensor, [6, p. 142].

Similarly, we endow $(\mathbf{V}^{*p,q})^{\wedge k}$ with the induced real sesquilinear product $\cdot : (\mathbf{V}^{*p,q})^{\wedge k} \times (\mathbf{V}^{*p,q})^{\wedge k} \rightarrow \mathbb{R}$ such that, with respect to a basis B_k^* for $(\mathbf{V}^{*p,q})^{\wedge k}$, $\forall k \in \mathbf{Z}_{[1,m]}$,

$$\begin{aligned} (\mathbf{v}_k^*, \mathbf{w}_k^*) &\mapsto \mathbf{v}_k^* \cdot \mathbf{w}_k^* \triangleq \frac{1}{k!} \frac{1}{k!} v_{i_1 \dots i_k} w_{j_1 \dots j_k} \det [h^{-1}(\theta^{i_a}, \theta^{j_b})], \\ &= \frac{1}{k!} \frac{1}{k!} v_{i_1 \dots i_k} w_{j_1 \dots j_k} \delta_{i_1 \dots i_k}^{j_1 \dots j_k} h^{-1}(\theta^{i_1}, \theta^{j_1}) \dots h^{-1}(\theta^{i_k}, \theta^{j_k}), \\ &= \frac{1}{k!} v_{l_1 \dots l_k} w_{j_1 \dots j_k} h^{-1}(\theta^{l_1}, \theta^{j_1}) \dots h^{-1}(\theta^{l_k}, \theta^{j_k}). \end{aligned} \quad (4)$$

For a fixed contravariant vector $\mathbf{u} \in \mathbf{V}^{p,q}$, the map from $\mathbf{V}^{p,q} \rightarrow \mathbb{R}$ such that $\mathbf{v} \mapsto h(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$, defines a canonical isomorphism between $\mathbf{V}^{p,q}$ and its dual $\mathbf{V}^{*p,q}$. This canonical isomorphism is the map $\flat : \mathbf{V}^{p,q} \rightarrow \mathbf{V}^{*p,q}$ such that $\mathbf{u} \mapsto \mathbf{u}^* \triangleq h(\mathbf{u}, \cdot)$. Its inverse is the map $\sharp : \mathbf{V}^{*p,q} \rightarrow \mathbf{V}^{p,q}$ such that $\mathbf{u}^* \mapsto \mathbf{u} \triangleq h^{-1}(\mathbf{u}^*, \cdot)$. The sesquilinear products (3) and (4) allow to naturally extend the domain of the canonical isomorphism to higher order tensors. The maps \flat and \sharp allow to “raise or lower the indices” of the components of tensors.

The map \flat or the map \sharp generates in a canonical way the bilinear binary function $\langle \cdot, \cdot \rangle : (\mathbf{V}^{*p,q})^{\wedge k} \times (\mathbf{V}^{p,q})^{\wedge k} \rightarrow \mathbb{R}$ such that

$$(\mathbf{u}_k^*, \mathbf{v}_k) \mapsto \langle \mathbf{u}_k^*, \mathbf{v}_k \rangle \triangleq \mathbf{u}_k^*(\mathbf{v}_k) = \sharp \mathbf{u}_k^* \cdot \mathbf{v}_k = \mathbf{u}_k^* \cdot \flat \mathbf{v}_k. \quad (5)$$

The function $\langle \cdot, \cdot \rangle$ may serve to uniquely determine the canonically associated cobasis B_k^* of a given basis B_k , using the relations $\langle \boldsymbol{\theta}^{j_1 \dots j_k}, \boldsymbol{\epsilon}_{i_1 \dots i_k} \rangle = \delta_{i_1 \dots i_k}^{j_1 \dots j_k}, \forall k \in \mathbf{Z}_{[1,m]}$.

2.2 Clifford Algebra over $\mathbf{V}^{p,q}$

Define the bilinear associative exterior product $\wedge : (\mathbf{V}^{p,q})^{\wedge k} \times (\mathbf{V}^{p,q})^{\wedge l} \rightarrow (\mathbf{V}^{p,q})^{\wedge(k+l)}, \forall k, l \in \mathbf{Z}_{[0,m]}$, such that: (i) for $k + l \leq m$,

$$(\mathbf{v}_k, \mathbf{w}_l) \mapsto \mathbf{v}_k \wedge \mathbf{w}_l \triangleq \frac{1}{k!} \frac{1}{l!} v^{i_1 \dots i_k} w^{i_{k+1} \dots i_{k+l}} \delta_{i_1 \dots i_{k+l}}^{j_1 \dots j_{k+l}} \boldsymbol{\epsilon}_{j_1 \dots j_{k+l}}, \quad (6)$$

and (ii) for $m < k + l$, $\mathbf{v}_k \wedge \mathbf{w}_l \triangleq 0$. The exterior product extends to $\wedge : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$ by distributivity over the direct sum. The exterior product is independent of any choice of bases.

Define a (left) contraction product $\lrcorner : \mathbf{V}^{p,q} \times (\mathbf{V}^{p,q})^{\wedge k} \rightarrow (\mathbf{V}^{p,q})^{\wedge k-1}$ such that (i) for $\forall k \in \mathbf{Z}_{[1,m]}$,

$$(\mathbf{v}, \mathbf{w}_k) \mapsto \mathbf{v} \lrcorner \mathbf{w}_k \triangleq \frac{1}{k!} \sum_{l=1}^k (-1)^{l-1} v^i w^{j_1 \dots j_k} h(\boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}_{j_l}) \boldsymbol{\epsilon}_{j_1 \dots \widehat{j_l} \dots j_k}, \quad (7)$$

wherein $\widehat{j_l}$ denotes the absence of j_l , and (ii) for $k = 0$, $\mathbf{v} \lrcorner \mathbf{w}_k \triangleq 0$. The contraction product extends to $\lrcorner : \mathbf{V}^{p,q} \times \mathbf{M} \rightarrow \mathbf{M}$ by distributivity over the direct sum and reduces for $k = 1$ to the sesquilinear product of $\mathbf{V}^{p,q}$. The contraction product is independent of any choice of bases.

Define a Clifford product, denoted by juxtaposition, first between a vector \mathbf{v} and a k -vector \mathbf{w}_k as

$$\mathbf{v} \mathbf{w}_k \triangleq +\mathbf{v} \lrcorner \mathbf{w}_k + \mathbf{v} \wedge \mathbf{w}_k, \quad (8)$$

$$(-1)^k \mathbf{w}_k \mathbf{v} \triangleq -\mathbf{v} \lrcorner \mathbf{w}_k + \mathbf{v} \wedge \mathbf{w}_k, \quad (9)$$

and then extend it, by distributivity over the direct sum and associativity, to $\mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$. The linear space M together with the Clifford product defined on M is the contravariant universal real Clifford Algebra $Cl(\mathbf{V}^{p,q})$ generated by $\mathbf{V}^{p,q}$.

Practical calculations are considerably simplified by choosing orthonormal bases $B_k \triangleq \{\boldsymbol{\epsilon}_{i_1 \dots i_k}, \forall i_1 < \dots < i_k\}$, with respect to h , for each $(\mathbf{V}^{p,q})^{\wedge k}$, as then $\boldsymbol{\epsilon}_{i_1 \dots i_k} = \boldsymbol{\epsilon}_{i_1} \dots \boldsymbol{\epsilon}_{i_k} = \boldsymbol{\epsilon}_{i_1} \wedge \dots \wedge \boldsymbol{\epsilon}_{i_k}$ and $\boldsymbol{\epsilon}_i \boldsymbol{\epsilon}_i = \pm 1$.

2.3 Clifford Algebra over $\mathbf{V}^{*p,q}$

Define the bilinear associative exterior product $\wedge : (\mathbf{V}^{*p,q})^{\wedge k} \times (\mathbf{V}^{*p,q})^{\wedge l} \rightarrow (\mathbf{V}^{*p,q})^{\wedge(k+l)}, \forall k, l \in \mathbf{Z}_{[0,m]}$, such that: (i) for $k + l \leq m$,

$$(\mathbf{v}_k^*, \mathbf{w}_l^*) \mapsto \mathbf{v}_k^* \wedge \mathbf{w}_l^* \triangleq \frac{1}{k!} \frac{1}{l!} v_{i_1 \dots i_k} w_{i_{k+1} \dots i_{k+l}} \delta_{j_1 \dots j_{k+l}}^{i_1 \dots i_{k+l}} \boldsymbol{\theta}^{j_1 \dots j_{k+l}}, \quad (10)$$

and (ii) for $m < k + l$, $\mathbf{v}_k^* \wedge \mathbf{w}_l^* \triangleq 0$. The exterior product extends to $\wedge : \mathbf{M}^* \times \mathbf{M}^* \rightarrow \mathbf{M}^*$ by distributivity over the direct sum. The exterior product is independent of any choice of cobases.

Define a (left) contraction product $\lrcorner : \mathbf{V}^{*p,q} \times (\mathbf{V}^{*p,q})^{\wedge k} \rightarrow (\mathbf{V}^{*p,q})^{\wedge k-1}$ such that (i) for $\forall k \in \mathbf{Z}_{[1,m]}$,

$$(\mathbf{v}^*, \mathbf{w}_k^*) \mapsto \mathbf{v}^* \lrcorner \mathbf{w}_k^* \triangleq \frac{1}{k!} \sum_{l=1}^k (-1)^{l-1} v_l w_{j_1 \dots j_k} h^{-1}(\boldsymbol{\theta}^i, \boldsymbol{\theta}^{j_l}) \boldsymbol{\theta}^{j_1 \dots \widehat{j_l} \dots j_k} \quad (11)$$

and (ii) for $k = 0$, $\mathbf{v}^* \lrcorner \mathbf{w}_k^* \triangleq 0$. The contraction product extends to $\lrcorner : \mathbf{V}^{*p,q} \times \mathbf{M}^* \rightarrow \mathbf{M}^*$ by distributivity over the direct sum and reduces for $k = 1$ to the sesquilinear product of $\mathbf{V}^{*p,q}$. The contraction product is independent of any choice of cobases.

Define a Clifford product, denoted by juxtaposition, first between a covector \mathbf{v}^* and a k -covector \mathbf{w}_k^* as

$$\mathbf{v}^* \mathbf{w}_k^* \triangleq +\mathbf{v}^* \lrcorner \mathbf{w}_k^* + \mathbf{v}^* \wedge \mathbf{w}_k^*, \quad (12)$$

$$(-1)^k \mathbf{w}_k^* \mathbf{v}^* \triangleq -\mathbf{v}^* \lrcorner \mathbf{w}_k^* + \mathbf{v}^* \wedge \mathbf{w}_k^*, \quad (13)$$

and then extend it, by distributivity over the direct sum and associativity, to $\mathbf{M}^* \times \mathbf{M}^* \rightarrow \mathbf{M}^*$. The linear space M^* together with the Clifford product defined on M^* is the covariant universal real Clifford Algebra $Cl(\mathbf{V}^{*p,q})$ generated by $\mathbf{V}^{*p,q}$.

Practical calculations are considerably simplified by choosing orthonormal cobases $B_k^* \triangleq \{\mathbf{t}^{j_1 \dots j_k}, \forall j_1 < \dots < j_k\}$, with respect to h^{-1} , for each $(\mathbf{V}^{*p,q})^{\wedge k}$, as then $\mathbf{t}^{j_1 \dots j_k} = \mathbf{t}^{j_1} \dots \mathbf{t}^{j_k} = \mathbf{t}^{j_1} \wedge \dots \wedge \mathbf{t}^{j_k}$ and $\mathbf{t}^j \mathbf{t}^j = \pm 1$.

2.4 Real pseudo-Euclidean space E^n

Recall that the differential manifold of real n -dimensional pseudo-Euclidean space E^n is an affine space over \mathbb{R} , i.e., a principal homogeneous space with automorphism group the affine group, given by the semi-direct product $T_n(\mathbb{R}) \rtimes GL_n(\mathbb{R})$. At any point $x \in E^n$, the tangent space $V_x \triangleq T_x E$ and the dual cotangent space $V_x^* \triangleq T_x^* E$ are isomorphic (as linear spaces) to \mathbb{R}^n . Further, E^n comes with a real sesquilinear product structure, given by a nondegenerate symmetric bilinear function $g : V_x \times V_x \rightarrow \mathbb{R}, \forall x \in E^n$, such that $(\mathbf{v}, \mathbf{w}) \mapsto g(\mathbf{v}, \mathbf{w})$, with g of signature (r, s) and independent of x . The space E^n is in addition tacitly endowed with a trivial flat connection, i.e., the standard rule for parallel transport. The latter property and the independence of g of x make that all $V_x, \forall x \in E^n$, can be identified with a common single sesquilinear product space $\mathbf{E}^{r,s} \triangleq (\mathbb{R}^n, g)$. Similarly, all $V_x^*, \forall x \in E^n$, can be identified with $\mathbf{E}^{*r,s} \triangleq (\mathbb{R}^n, g^{-1})$.

Although it is possible to also impose an independent Clifford structure on $\mathbf{E}^{*r,s}$, we will refrain from doing so at this point. We will see further how $\mathbf{E}^{*r,s}$ may inherit a Clifford structure from $\mathbf{V}^{*p,q}$ in a natural way (then requiring $(r, s) = (p, q)$). One could also impose an independent Clifford structure on $\mathbf{E}^{r,s}$, but such structure appears to be uninteresting. It will be shown in the next section that the algebra $Cl(\mathbf{E}^{r,s})$ does not generate a natural $Cl(\mathbf{E}^{r,s})$ -valued Clifford Analysis.

2.5 Rings

We will make use of the following rings.

(i) $\mathcal{F}^\infty \triangleq (C^\infty(\Omega, \mathbb{R}), +, \cdot)$: the unital ring of smooth real-valued functions from $\Omega \rightarrow \mathbb{R}$, together with pointwise addition $+$ and pointwise multiplication (denoted by juxtaposition).

(ii) $\mathcal{D} \triangleq (C_c^\infty(E^n, \mathbb{R}), +, \cdot)$: the ring of smooth real-valued functions with compact support from $E^n \rightarrow \mathbb{R}$. Denote by \mathcal{D}' the linear space of distributions as the continuous dual of \mathcal{D} .

We will also use the multiplication between a smooth function $\psi \in \mathcal{F}^\infty$ and a distribution $f \in \mathcal{D}'$, which is defined as the distribution $\psi f = f\psi$ given by, $\forall \varphi \in \mathcal{D}$,

$$\langle \psi f, \varphi \rangle \triangleq \langle f, \psi \varphi \rangle, \quad (14)$$

which is legitimate since $\psi \varphi \in \mathcal{D}$.

3 CLIFFORD ANALYSIS OVER E^N

3.1 Definition

Contravariant Clifford Analysis (CA) over pseudo-Euclidean space E^n is the study of functions from $\Omega \rightarrow Cl(\mathbf{V}^{p,q})$, together with a first order vector differential operator ∂ . Covariant Clifford Analysis (CA^{*}) over pseudo-Euclidean space E^n is the study of functions from $\Omega \rightarrow Cl(\mathbf{V}^{*p,q})$, together with a first order covector differential operator ∂^* .

Within CA over E^n , we want in particular derive an integral representation for functions $F \in C^\infty(\Omega, Cl(\mathbf{V}^{p,q}))$ satisfying

$$\partial F = -S, \quad (15)$$

with $S \in C_c^\infty(E^n, Cl(\mathbf{V}^{p,q}))$. Similarly, within CA^{*} over E^n , we want to derive an integral representation for functions $F^* \in C^\infty(\Omega, Cl(\mathbf{V}^{*p,q}))$ satisfying

$$\partial^* F^* = -S^*, \quad (16)$$

with $S^* \in C_c^\infty(E^n, Cl(\mathbf{V}^{*p,q}))$.

From a mathematical point of view, it is not necessary to relate the space $\mathbf{V}^{p,q}$ to $\mathbf{E}^{r,s}$ (or $\mathbf{V}^{*p,q}$ to $\mathbf{E}^{*r,s}$) any further. Their only connection so far is through the functions that we study, which: (i) have as domain a subset of the affine space E^n underlying $\mathbf{E}^{r,s}$ and $\mathbf{E}^{*r,s}$ and (ii) have as codomain the Clifford algebra generated by either $\mathbf{V}^{p,q}$ or $\mathbf{V}^{*p,q}$.

For further convenience, we denote by $\{\epsilon_i\}$ a general basis for $\mathbf{V}^{p,q}$ and by $\{\theta^j\}$ the canonically associated cobasis for $\mathbf{V}^{*p,q}$ with respect to h . Similarly, let $\{\epsilon_\mu\}$ be a general basis for $\mathbf{E}^{r,s}$ and $\{\vartheta^\nu\}$ the canonically associated cobasis for $\mathbf{E}^{*r,s}$ with respect to g .

3.2 Solderings

Sometimes a physics application might require that an additional relation between the spaces $\mathbf{V}^{p,q}$ and $\mathbf{E}^{r,s}$ is to be imposed and that $m = n$. Such a relation is commonly defined in the literature by a smooth bijective map $\mathbf{E}^{r,s} \rightarrow \mathbf{V}^{p,q}$ such that $u \mapsto v \triangleq \langle \chi, u \rangle$ with either (a) $\chi \in \mathbf{V}^{p,q} \otimes \mathbf{E}^{*r,s}$ or (b) $\chi \in \mathbf{E}^{*r,s} \otimes \mathbf{V}^{p,q}$. With respect to the above bases, $\langle \chi, u \rangle$ is defined in terms of the bilinear binary function (5), relative to g , as for (a) and (b) respectively,

$$\langle \chi, u \rangle = \langle \chi^i{}_\nu (\epsilon_i \otimes \vartheta^\nu), u^\mu \epsilon_\mu \rangle \triangleq \chi^i{}_\nu u^\mu \epsilon_i \otimes \langle \vartheta^\nu, \epsilon_\mu \rangle = \chi^i{}_\kappa u^\kappa \epsilon_i. \quad (17)$$

$$\langle \chi, u \rangle = \langle \chi_\nu{}^j (\vartheta^\nu \otimes \epsilon_j), u^\mu \epsilon_\mu \rangle \triangleq u^\mu \chi_\nu{}^j \langle \vartheta^\nu, \epsilon_\mu \rangle \otimes \epsilon_j = u^\kappa \chi_\kappa{}^j \epsilon_j. \quad (18)$$

In the physics literature, this map is usually called a *soldering* as it “solders” $\mathbf{V}^{p,q}$ to E^n by establishing a bijective relation between the linear space $\mathbf{V}^{p,q}$ and the common tangent space $\mathbf{E}^{r,s}$. We assume here that χ is independent of $x \in E^n$.

	soldering	type (a)	type (b)
1	$\mathbf{E}^{p,q} \rightarrow \mathbf{V}^{p,q}$	$\chi \in \mathbf{V}^{p,q} \otimes \mathbf{E}^{*p,q}$	$\chi \in \mathbf{E}^{*p,q} \otimes \mathbf{V}^{p,q}$
2	$\mathbf{E}^{p,q} \rightarrow \mathbf{V}^{*p,q}$	$\chi \in \mathbf{V}^{*p,q} \otimes \mathbf{E}^{*p,q}$	$\chi \in \mathbf{E}^{*p,q} \otimes \mathbf{V}^{*p,q}$
3	$\mathbf{E}^{*p,q} \rightarrow \mathbf{V}^{p,q}$	$\chi \in \mathbf{V}^{p,q} \otimes \mathbf{E}^{p,q}$	$\chi \in \mathbf{E}^{p,q} \otimes \mathbf{V}^{p,q}$
4	$\mathbf{E}^{*p,q} \rightarrow \mathbf{V}^{*p,q}$	$\chi \in \mathbf{V}^{*p,q} \otimes \mathbf{E}^{p,q}$	$\chi \in \mathbf{E}^{p,q} \otimes \mathbf{V}^{*p,q}$

Table 1: All possible solderings relating $\mathbf{V}^{p,q}$ to E^n .

The inverse soldering is the map $\mathbf{V}^{p,q} \rightarrow \mathbf{E}^{r,s}$ such that $v \mapsto u \triangleq \langle \chi^{-1}, v \rangle$ with (a) $\chi^{-1} \in \mathbf{V}^{*p,q} \otimes \mathbf{E}^{r,s}$ or (b) $\chi^{-1} \in \mathbf{E}^{r,s} \otimes \mathbf{V}^{*p,q}$. With respect to the above bases, $\langle \chi^{-1}, v \rangle$ is defined in terms of the bilinear binary function (5) now relative to h , as for (a) and (b) respectively,

$$\langle \chi^{-1}, v \rangle = \left\langle (\chi^{-1})_j^\nu (\boldsymbol{\theta}^j \otimes \boldsymbol{\varepsilon}_\nu), v^i \boldsymbol{\varepsilon}_i \right\rangle \triangleq v^i (\chi^{-1})_j^\nu \langle \boldsymbol{\theta}^j, \boldsymbol{\varepsilon}_i \rangle \otimes \boldsymbol{\varepsilon}_\nu = v^k (\chi^{-1})_k^\nu \boldsymbol{\varepsilon}_\nu, \quad (19)$$

$$\langle \chi^{-1}, v \rangle = \left\langle (\chi^{-1})_j^\mu (\boldsymbol{\varepsilon}_\mu \otimes \boldsymbol{\theta}^j), v^i \boldsymbol{\varepsilon}_i \right\rangle \triangleq (\chi^{-1})_j^\mu v^i \boldsymbol{\varepsilon}_\mu \otimes \langle \boldsymbol{\theta}^j, \boldsymbol{\varepsilon}_i \rangle = (\chi^{-1})_k^\mu v^k \boldsymbol{\varepsilon}_\mu, \quad (20)$$

wherein

$$\chi_{\kappa}^i (\chi^{-1})_j^\kappa = \delta_j^i \text{ and } \chi_{\mu}^k (\chi^{-1})_k^\nu = \delta_{\mu}^\nu, \quad (21)$$

$$\chi_{\kappa}^i (\chi^{-1})_j^\kappa = \delta_j^i \text{ and } \chi_{\mu}^k (\chi^{-1})_k^\nu = \delta_{\mu}^\nu. \quad (22)$$

A soldering χ also relates higher tensor spaces in a natural way. In particular, the sesquilinear product structures g and h become related by

$$g_{\mu\nu} = h_{ij} \chi_{\mu}^i \chi_{\nu}^j. \quad (23)$$

Hence, if a soldering χ is present, then only two members of the triple (g, χ, h) can be chosen freely. The earlier requirement that $m = n$ avoids that either g or h becomes degenerated. Further, by Sylvester's law of inertia, both sesquilinear product structures must have the same signature. Hence, once a soldering is present, we will write $(r, s) = (p, q)$.

More generally, a soldering can be specified by any of the eight smooth bijective maps given in Table 1. We will use this observation in the next subsection.

3.3 Dirac operators

We want here introduce a Dirac operator as a geometrical invariant object, rather than as an ad hoc operator.

Let $d_{VE} \in \mathbf{V}^{p,q} \otimes \mathbf{E}^{p,q}$ and $d_{V^*E} \in \mathbf{V}^{*p,q} \otimes \mathbf{E}^{p,q}$ be nondegenerate (i.e., maximal rank) tensors. The tensors d_{VE} and d_{V^*E} have representatives,

$$d_{VE} \triangleq d^{i\kappa} (\boldsymbol{\varepsilon}_i \otimes \boldsymbol{\varepsilon}_\kappa), \quad (24)$$

$$d_{V^*E} \triangleq d_j{}^\kappa (\boldsymbol{\theta}^j \otimes \boldsymbol{\varepsilon}_\kappa). \quad (25)$$

Since tangent vectors at a point of a manifold, such as $\boldsymbol{\varepsilon}_\kappa$, are defined as derivations on \mathcal{F}^∞ , they are partial differential operators, [6, p. 117]. Hence the tensors (24)–(25) are also first order partial derivative operators, acting on \mathcal{F}^∞ through the Pfaff derivatives $\boldsymbol{\varepsilon}_\kappa$, [6, p.138]. This becomes even more apparent if we choose a natural (i.e., a coordinate) basis $\{\partial/\partial x^\kappa\}$ in $\mathbf{E}^{p,q}$. Therefore, the tensors d_{VE} and d_{V^*E} can be identified as anisotropic Dirac operators, taking values in $\mathbf{V}^{p,q}$ and $\mathbf{V}^{*p,q}$, respectively.

In the presence of a soldering χ of type 1.a (Table 1), the tensors d_{VE} and d_{V^*E} are mapped to the respective tensors $d_{EE}^S \in \mathbf{E}^{p,q} \otimes \mathbf{E}^{p,q}$ and $d_{E^*E}^S \in \mathbf{E}^{*p,q} \otimes \mathbf{E}^{p,q}$, having representatives

$$d_{EE}^S = (\chi^{-1})_i^\lambda d^{i\kappa} (\boldsymbol{\varepsilon}_\lambda \otimes \boldsymbol{\varepsilon}_\kappa), \quad (26)$$

$$d_{E^*E}^S = \chi^j_\nu d_j^\kappa (\boldsymbol{\vartheta}^\nu \otimes \boldsymbol{\varepsilon}_\kappa). \quad (27)$$

Eq. (26) shows that d_{EE}^S is a second order differential operator acting on \mathcal{F}^∞ and hence does not qualify as a Dirac operator, taking values in $\mathbf{E}^{p,q}$. Moreover, the tensor d_{VE} itself defines a soldering of type 3.a, which corresponds to a type 1.a soldering $\chi = b_g d_{VE}$, having representative $\chi^i_\nu = d^{i\kappa} g_{\kappa\nu}$. Under its own soldering, d_{VE} is mapped to $d_{EE}^S = (g^{-1})_i^{\lambda\kappa} (\boldsymbol{\varepsilon}_\lambda \otimes \boldsymbol{\varepsilon}_\kappa)$, which is the Laplacian for scalar functions (since the connection coefficients are zero on E^n).

Eq. (27) shows that $d_{E^*E}^S$ is a first order differential operator acting on \mathcal{F}^∞ . Hence, the covector operator $d_{E^*E}^S$ qualifies as a natural anisotropic Dirac operator, taking values in $\mathbf{E}^{*p,q}$. Also, any covector Dirac operator d_{V^*E} defines itself a soldering of type 4.a, which corresponds to a type 1.a inverse soldering $\chi^{-1} = d_{V^*E}$, having representative $(\chi^{-1})_i^\nu = d_i^\nu$. Under its own soldering, d_{V^*E} is thus mapped to $d_{E^*E}^S = \boldsymbol{\vartheta}^\kappa \otimes \boldsymbol{\varepsilon}_\kappa$, which is the standard isotropic covector Dirac operator. In this special case,

$$d_{E^*E}^S = \delta, \quad (28)$$

which shows that the isotropic covector Dirac operator, taking values in $\mathbf{E}^{*p,q}$, is just the 1-covariant, 1-contravariant Kronecker tensor $\delta \in (\mathbf{E}^{p,q} \otimes \mathbf{E}^{*p,q}) \cap (\mathbf{E}^{*p,q} \otimes \mathbf{E}^{p,q})$.

I.b Let $d_{EV} \in \mathbf{E}^{p,q} \otimes \mathbf{V}^{p,q}$ and $d_{EV^*} \in \mathbf{E}^{p,q} \otimes \mathbf{V}^{*p,q}$ be nondegenerate tensors, with representatives

$$d_{EV} \triangleq d^{\kappa i} (\boldsymbol{\varepsilon}_\kappa \otimes \boldsymbol{\varepsilon}_i), \quad (29)$$

$$d_{EV^*} \triangleq d^\kappa_j (\boldsymbol{\varepsilon}_\kappa \otimes \boldsymbol{\theta}^j). \quad (30)$$

Similarly, the tensors d_{EV} and d_{EV^*} can be identified as anisotropic Dirac type operators, taking values in $\mathbf{V}^{p,q}$ and $\mathbf{V}^{*p,q}$, respectively.

In the presence of a soldering χ of type 1.b, the tensors d_{EV} and d_{EV^*} are mapped to the respective tensors $d_{EE}^S \in \mathbf{E}^{p,q} \otimes \mathbf{E}^{p,q}$ and $d_{E^*E}^S \in \mathbf{E}^{p,q} \otimes \mathbf{E}^{*p,q}$, having representatives

$$d_{EE}^S = d^{\kappa i} (\chi^{-1})_i^\lambda (\boldsymbol{\varepsilon}_\kappa \otimes \boldsymbol{\varepsilon}_\lambda), \quad (31)$$

$$d_{E^*E}^S = d^\kappa_j \chi_\nu^j (\boldsymbol{\varepsilon}_\kappa \otimes \boldsymbol{\vartheta}^\nu). \quad (32)$$

Again, the vector operator d_{EE}^S is not a Dirac operator, while the covariant operator $d_{E^*E}^S$ qualifies as an anisotropic Dirac operator. Under its own soldering, d_{EV^*} is also mapped to $\boldsymbol{\varepsilon}_\kappa \otimes \boldsymbol{\vartheta}^\kappa = \delta$.

II.a Let $d_{VE^*} \in \mathbf{V}^{p,q} \otimes \mathbf{E}^{*p,q}$ and $d_{V^*E^*} \in \mathbf{V}^{*p,q} \otimes \mathbf{E}^{*p,q}$ be nondegenerate tensors, with representatives

$$d_{VE^*} \triangleq d^i_\kappa (\boldsymbol{\varepsilon}_i \otimes \boldsymbol{\vartheta}^\kappa), \quad (33)$$

$$d_{V^*E^*} \triangleq d_{j\kappa} (\boldsymbol{\theta}^j \otimes \boldsymbol{\vartheta}^\kappa). \quad (34)$$

Since cotangent vectors at a point of a manifold, such as $\boldsymbol{\vartheta}^\kappa$, are not derivations on \mathcal{F}^∞ , they are not partial differential operators [6, p. 135]. Hence the tensors d_{VE^*} and $d_{V^*E^*}$ can not be identified as Dirac operators.

In the presence of a soldering χ of type 1.a, the tensors d_{VE^*} and $d_{V^*E^*}$ are mapped to the respective tensors $d_{EE^*}^S \in \mathbf{E}^{p,q} \otimes \mathbf{E}^{*p,q}$ and $d_{E^*E^*}^S \in \mathbf{E}^{*p,q} \otimes \mathbf{E}^{*p,q}$, having representatives

$$d_{EE^*}^S = (\chi^{-1})_i^\lambda d_{\kappa}^i (\boldsymbol{\varepsilon}_\lambda \otimes \boldsymbol{\vartheta}^\kappa), \quad (35)$$

$$d_{E^*E^*}^S = \chi^j_\nu d_{j\kappa} (\boldsymbol{\vartheta}^\nu \otimes \boldsymbol{\vartheta}^\kappa). \quad (36)$$

Eq. (35) shows that $d_{EE^*}^S$ is a first order differential operator acting on \mathcal{F}^∞ . Hence, the covector operator $d_{EE^*}^S$ qualifies as an anisotropic Dirac operator, taking values in $\mathbf{E}^{*p,q}$. Under its own soldering, d_{VE^*} is also mapped to $\boldsymbol{\varepsilon}_\kappa \otimes \boldsymbol{\vartheta}^\kappa = \delta$.

Eqs. (34) and (36) show that $d_{E^*E^*}^S$ is not a differential operator. Any tensor $d_{V^*E^*}$ defines a soldering of type 2.a, which corresponds to a type 1.a soldering $\chi = \sharp_h d_{V^*E^*}$, having representative $\chi^i_\nu = h^{ik} d_{k\nu}$. Under its own soldering, $d_{V^*E^*}$ is mapped to $d_{E^*E^*}^S = g_{\lambda\kappa} (\boldsymbol{\vartheta}^\lambda \otimes \boldsymbol{\vartheta}^\kappa)$, which is the ‘‘dual Laplacian’’ (with respect to a natural basis, $d_{E^*E^*}^S = ds^2$, the square of the infinitesimal line element in E^n).

II.b Let $d_{E^*V} \in \mathbf{E}^{*p,q} \otimes \mathbf{V}^{p,q}$ and $d_{E^*V^*} \in \mathbf{E}^{*p,q} \otimes \mathbf{V}^{*p,q}$ be nondegenerate tensors, with representatives

$$d_{E^*V} \triangleq d_{\kappa}^i (\boldsymbol{\vartheta}^\kappa \otimes \boldsymbol{\varepsilon}_i), \quad (37)$$

$$d_{E^*V^*} \triangleq d_{\kappa j} (\boldsymbol{\vartheta}^\kappa \otimes \boldsymbol{\theta}^j). \quad (38)$$

In the presence of a soldering χ of type 1.b, the tensors d_{E^*V} and $d_{E^*V^*}$ are mapped to the respective tensors $d_{E^*E}^S \in \mathbf{E}^{*p,q} \otimes \mathbf{E}^{p,q}$ and $d_{E^*E^*}^S \in \mathbf{E}^{*p,q} \otimes \mathbf{E}^{*p,q}$, having representatives

$$d_{E^*E}^S = d_{\kappa}^i (\chi^{-1})^\mu_i (\boldsymbol{\vartheta}^\kappa \otimes \boldsymbol{\varepsilon}_\mu), \quad (39)$$

$$d_{E^*E^*}^S = d_{\kappa j} \chi_\nu^j (\boldsymbol{\vartheta}^\kappa \otimes \boldsymbol{\vartheta}^\nu). \quad (40)$$

Again, the covector operator $d_{E^*E}^S$ qualifies as an anisotropic Dirac operator, taking values in $\mathbf{E}^{*p,q}$. The tensor $d_{E^*E^*}^S$ is not a Dirac operator.

In summary, we have the following geometrical Dirac operators:

(i) taking values in $\mathbf{V}^{*p,q}$, and irrespective of the presence of a soldering χ ,

$$\partial_{V^*E} \triangleq d_{V^*E} = d_j^\kappa (\boldsymbol{\theta}^j \otimes \boldsymbol{\varepsilon}_\kappa), \quad (41)$$

$$\partial_{EV^*} \triangleq d_{EV^*} = d_j^\kappa (\boldsymbol{\varepsilon}_\kappa \otimes \boldsymbol{\theta}^j); \quad (42)$$

(ii) taking values in $\mathbf{E}^{*p,q}$, and in the presence of an inverse soldering χ^{-1} ,

(a) of type 1.a,

$$\begin{aligned} \partial_{EE^*}^{(1a)} &\triangleq \langle \chi^{-1}, d_{E^*V} \rangle = \langle (\chi^{-1})_i^\nu (\boldsymbol{\theta}^i \otimes \boldsymbol{\varepsilon}_\nu), d_\mu^j (\boldsymbol{\vartheta}^\mu \otimes \boldsymbol{\varepsilon}_j) \rangle \\ &\triangleq (\chi^{-1})_i^\nu d_\mu^j \langle \boldsymbol{\theta}^i, \boldsymbol{\varepsilon}_j \rangle (\boldsymbol{\varepsilon}_\nu \otimes \boldsymbol{\vartheta}^\mu) = (\chi^{-1})_k^\nu d_\mu^k (\boldsymbol{\varepsilon}_\nu \otimes \boldsymbol{\vartheta}^\mu), \end{aligned} \quad (43)$$

$$\begin{aligned} \partial_{EE^*}^{(2a)} &\triangleq \langle \chi^{-1}, d_{VE^*} \rangle = \langle (\chi^{-1})_i^\nu (\boldsymbol{\theta}^i \otimes \boldsymbol{\varepsilon}_\nu), (\boldsymbol{\varepsilon}_j \otimes \boldsymbol{\vartheta}^\mu) d_\mu^j \rangle \\ &\triangleq (\chi^{-1})_i^\nu d_\mu^j \langle \boldsymbol{\theta}^i, \boldsymbol{\varepsilon}_j \rangle (\boldsymbol{\varepsilon}_\nu \otimes \boldsymbol{\vartheta}^\mu) = (\chi^{-1})_k^\nu d_\mu^k (\boldsymbol{\varepsilon}_\nu \otimes \boldsymbol{\vartheta}^\mu); \end{aligned} \quad (44)$$

(b) of type 1.b,

$$\begin{aligned} \partial_{EE^*}^{(1b)} &\triangleq \langle \chi^{-1}, d_{E^*V} \rangle = \langle (\chi^{-1})_i^\nu (\boldsymbol{\varepsilon}_\nu \otimes \boldsymbol{\theta}^i), d_\mu^j (\boldsymbol{\vartheta}^\mu \otimes \boldsymbol{\varepsilon}_j) \rangle \\ &\triangleq (\chi^{-1})_i^\nu d_\mu^j \langle \boldsymbol{\theta}^i, \boldsymbol{\varepsilon}_j \rangle (\boldsymbol{\varepsilon}_\nu \otimes \boldsymbol{\vartheta}^\mu) = (\chi^{-1})_k^\nu d_\mu^k (\boldsymbol{\varepsilon}_\nu \otimes \boldsymbol{\vartheta}^\mu), \end{aligned} \quad (45)$$

$$\begin{aligned}
\partial_{EE^*}^{(2b)} &\triangleq \langle \chi^{-1}, d_{VE^*} \rangle = \langle (\chi^{-1})^\nu_i (\epsilon_\nu \otimes \theta^i), (\epsilon_j \otimes \vartheta^\mu) d^j_\mu \rangle \\
&\triangleq (\chi^{-1})^\nu_i d^j_\mu \langle \theta^i, \epsilon_j \rangle (\epsilon_\nu \otimes \vartheta^\mu) = (\chi^{-1})^\nu_k d^k_\mu (\epsilon_\nu \otimes \vartheta^\mu). \tag{46}
\end{aligned}$$

Remarks.

(i) Since geometrical Dirac operators themselves define a soldering, solderings cannot be avoided in a geometrical Clifford Analysis.

(ii) In contrast to tangent vectors at a point of a manifold, cotangent vectors at a point of a manifold are not differential operators. This fundamental difference in the analytic nature of vectors and covectors is the origin of an essential asymmetry between contravariant and covariant Clifford Analysis.

(iii) Contravariant Clifford Analysis over E^n together with a first order vector Dirac operator, taking values in $Cl(\mathbf{V}^{p,q})$, can always be defined. Also, covariant Clifford Analysis over E^n together with a first order covector Dirac operator, taking values in $Cl(\mathbf{V}^{*p,q})$, can always be defined.

(iv) Contravariant Clifford Analysis over E^n together with a first order vector Dirac operator, taking values in $Cl(\mathbf{E}^{p,q})$, can not be defined. Covariant Clifford Analysis over E^n with a first order covector Dirac operator, taking values in $Cl(\mathbf{E}^{*p,q})$, can naturally be defined based on the image of $Cl(\mathbf{V}^{*p,q})$ under a soldering.

3.4 Reproducing kernels

Let $x_0 \in E^n$ and ∂^* an anisotropic covariant Dirac operator, given by (41). We use a superscript $*$ to indicate the covector character of the Dirac operator and the functions and distributions involved.

Let $C_{x_0}^* \in \mathcal{D}' \otimes Cl(\mathbf{V}^{*p,q})$ be of grade 1, i.e., $C_{x_0}^*$ is a $Cl(\mathbf{V}^{*p,q})$ -valued distribution such that $\langle C_{x_0}^*, \varphi \rangle \in \mathbf{V}^{*p,q}$, $\forall \varphi \in \mathcal{D}$, satisfying (with ∂^* acting on the left)

$$C_{x_0}^* \partial^* = \delta_{x_0}, \tag{47}$$

and wherein δ_{x_0} is the delta distribution with support the calculation point $\{x_0\}$. The distribution $C_{x_0}^*$ is called a reproducing or Cauchy kernel, [1, p. 50].

The grade 2 component of (47) together with the fact that δ_{x_0} is scalar yields that $C_{x_0}^* \wedge \partial^* = 0$. From Poincaré's lemma follows that $C_{x_0}^*$ can be generated from a scalar distribution $f_{x_0} \in \mathcal{D}'$ as

$$C_{x_0}^* = f_{x_0} \partial^*. \tag{48}$$

Substituting (48) in (47) shows that f_{x_0} must satisfy

$$(f_{x_0} \partial^*) \partial^* = \delta_{x_0}. \tag{49}$$

The left-hand side of (49) is a triple convolution product of distributions of which only one (f_{x_0}) is of non-compact support. Therefore, associativity holds for the convolution product. Moreover, associativity also holds for the Clifford product. Since (i) $\partial^* \partial^*$ is scalar, due to the commutativity of the composition $\epsilon_\kappa \circ \epsilon_\lambda$, and (ii) after a suitable change of coordinates, (49) is equivalent to

$$\square_{p,q} f_{x_0} = \delta_{x_0}, \tag{50}$$

with $\square_{p,q} \triangleq \Delta_p - \Delta_q$, the canonical d'Alembertian of signature (p, q) . Eq. (50) is known as the ultrahyperbolic wave equation of E^n .

A (real) fundamental solution of (50) is, with $P \triangleq g(x - x_0, x - x_0)$ and $A_{n-1} \triangleq \frac{2\pi^{n/2}}{\Gamma(n/2)}$, [7, p. 280],

(i) for $n > 2$,

$$f_{x_0} = -\frac{1}{(n-2)A_{n-1}} \frac{1}{2} \left(e^{iq\frac{\pi}{2}} (P + i0)^{1-\frac{n}{2}} + e^{-iq\frac{\pi}{2}} (P - i0)^{1-\frac{n}{2}} \right), \quad (51)$$

(ii) for $n = 2$,

$$f_{x_0} = \frac{\cos\left(q\frac{\pi}{2}\right)}{4} \frac{1}{\pi} \ln|P| - \frac{\sin\left(q\frac{\pi}{2}\right)}{4} 1_-(P). \quad (52)$$

The Cauchy kernel $C_{x_0}^*$ then follows from (48). Its explicit calculation however, involves distributional technicalities, on the one hand related to the non-applicability of the generalized chain rule, [8, p. 83], and on the other hand related to the singularities of the distributions $(P \pm i0)^z$ at $z = -n/2$, both of which are outside the scope of this article.

3.5 Solution

We now derive an integral representation for functions satisfying (16) within the framework of covariant Clifford Analysis with an anisotropic covariant Dirac operator ∂^* , given by (41).

3.5.1 Local reciprocity

Start from

$$\underline{\partial}^* F^* = -S^*, \quad (53)$$

$$C_{x_0}^* \underline{\partial}^* = \delta_{x_0}. \quad (54)$$

Multiply (53) on the left by $C_{x_0}^*$ and (54) on the right by F^* and add, giving

$$\left(C_{x_0}^* \underline{\partial}^* \right) F^* + C_{x_0}^* \left(\underline{\partial}^* F^* \right) = \delta_{x_0} F^* - C_{x_0}^* S^*. \quad (55)$$

The under arrow indicates the acting direction of the Dirac operator. The products in (55) exist since they are based on the multiplication between a distribution and a smooth function, as defined in (14). Eq. (55) relates $Cl(\mathbf{V}^{*p,q})$ -valued distributions with support in the base space E^n . Using Leibniz' rule and preserving the order of the cobasis elements, eq. (55) can be written as

$$\varepsilon_\kappa \left(C_{x_0}^* \theta^j d_j^\kappa F^* \right) = \delta_{x_0} F^* - C_{x_0}^* S^*. \quad (56)$$

This is a local reciprocity relation between the $Cl(\mathbf{V}^{*p,q})$ -valued function F^* and distribution $C_{x_0}^*$.

Tensor multiply (56) by the volume form ω_V on E^n (with respect to a general cobasis),

$$\omega_V = \sqrt{|\det g|} \left(\vartheta^1 \wedge \dots \wedge \vartheta^n \right), \quad (57)$$

to turn it into an equation of n -covectors over $Cl(\mathbf{V}^{*p,q})$ -valued distributions,

$$\left(\varepsilon_\kappa \left(C_{x_0}^* \theta^j d_j^\kappa F^* \right) \right) \otimes \omega_V = \left(\delta_{x_0} F^* \right) \otimes \omega_V - \left(C_{x_0}^* S^* \right) \otimes \omega_V. \quad (58)$$

The objects appearing in (58) are of quite an advanced nature, being elements of

$$\left(\mathcal{D}' \otimes Cl(\mathbf{V}^{*p,q}) \right) \otimes \left(\mathcal{F}^\infty \otimes \Lambda(\mathbf{E}^{*p,q}) \right), \quad (59)$$

with $\Lambda(\mathbf{E}^{*p,q})$ the exterior Grassmann algebra generated by $\mathbf{E}^{*p,q}$. They can be regarded as generalized (in the distributional sense) Clifford-algebra-valued differential forms, with the latter sometimes called cliffords, [9]. We need form (58) of the reciprocity relation in order to be able to integrate it later on.

Rewrite (58) as

$$\begin{aligned} & (\varepsilon_\kappa (C_{x_0}^* \boldsymbol{\theta}^j d_j^\lambda F^*)) \otimes \sqrt{|\det g|} \delta_\lambda^\kappa (\boldsymbol{\vartheta}^1 \wedge \dots \wedge \boldsymbol{\vartheta}^n) \\ &= (\delta_{x_0} F^*) \otimes \omega_V - (C_{x_0}^* S^*) \otimes \omega_V. \end{aligned} \quad (60)$$

We now need the identity

$$\delta_\lambda^\kappa \omega_V = \boldsymbol{\vartheta}^\kappa \wedge \omega_\lambda, \quad (61)$$

holding for ω_V and the $(n-1)$ -covectors

$$\omega_\lambda \triangleq \sqrt{|\det g|} \frac{1}{(n-1)!} \delta_{\lambda \mu_2 \dots \mu_n}^{1 \dots n} (\boldsymbol{\vartheta}^{\mu_2} \wedge \dots \wedge \boldsymbol{\vartheta}^{\mu_n}).$$

Also, introduce the following definition, $\forall \mathbf{v}^* \in \mathcal{D}' \otimes Cl(\mathbf{V}^{*p,q})$ and $\forall \omega \in \mathcal{F}^\infty \otimes \Lambda(\mathbf{E}^{*p,q})$,

$$d(\mathbf{v}^* \otimes \omega) \triangleq (\varepsilon_\kappa \mathbf{v}^*) \otimes (\boldsymbol{\vartheta}^\kappa \wedge \omega) \quad (62)$$

for the (left) covariant exterior derivation d acting on the objects in (60). In (62), the action of ε_κ involves generalized derivation. Now, eq. (60) can be brought in the form

$$(\delta_{x_0} F^*) \otimes \omega_V = (C_{x_0}^* S^*) \otimes \omega_V + d((C_{x_0}^* \boldsymbol{\theta}^j d_j^\lambda F^*) \otimes \omega_\lambda). \quad (63)$$

3.5.2 Smoothing

We want to convert (63) to integral form. Eq. (63) is a distributional equation, so we can not (definite) integrate it right away. A possible way to proceed is to convert it to a smoothed version. Smoothing of a distribution is usually done by convolving it with a test function $\lambda \in \mathcal{D}$, called a mollifier. Such an operation, applied to a distribution $f \in \mathcal{D}'$, is called a regularization of f and the resulting smooth function, $f * \lambda$, is regarded as an approximation to f , [10, p. 132]. Convolution $*$ on E^n is defined as, $\forall f \in \mathcal{D}'$ and $\forall \varphi \in \mathcal{D}$,

$$(f * \varphi)(x) \triangleq \langle f_{(y)}, \varphi(x-y) \rangle. \quad (64)$$

Herein, the subscript (y) denotes that y is a dummy variable with scope the Schwartz pairing \langle, \rangle . An essential property, enjoyed by convolution on E^n , is, $\forall f \in \mathcal{D}'$ and $\forall \varphi \in \mathcal{D}$,

$$(df) * \varphi = d(f * \varphi), \quad (65)$$

wherein d denotes generalized exterior derivation in the left-hand side and ordinary exterior derivation in the right-hand side. We further extend the definition of convolution to our objects as

$$\begin{aligned} & ((\mathcal{D}' \otimes Cl(\mathbf{V}^{*p,q})) \otimes (\mathcal{F}^\infty \otimes \Lambda(\mathbf{E}^{*p,q}))) * \varphi \\ & \triangleq ((\mathcal{D}' * \varphi) \otimes Cl(\mathbf{V}^{*p,q})) \otimes (\mathcal{F}^\infty \otimes \Lambda(\mathbf{E}^{*p,q})). \end{aligned} \quad (66)$$

The smoothed version of (63) becomes

$$((\delta_{x_0} F^*) \otimes \omega_V) * \lambda = ((C_{x_0}^* S^*) \otimes \omega_V) * \lambda + (d((C_{x_0}^* \boldsymbol{\theta}^k d_k^\kappa F^*) \otimes \omega_\kappa)) * \lambda,$$

or, with definition (66) and property (65),

$$((\delta_{x_0} F^*) * \lambda) \otimes \omega_V = ((C_{x_0}^* S^*) * \lambda) \otimes \omega_V + d \left(((C_{x_0}^* \theta^k d_k^\kappa F^*) * \lambda) \otimes \omega_\kappa \right),$$

or, with definition (64) and explicitly denoting the Schwartz pairing with a subscript S,

$$\begin{aligned} & \left\langle (\delta_{x_0} F^*)_{(y)}, \lambda(x-y) \right\rangle_S \otimes \omega_V \\ &= \left\langle (C_{x_0}^* S^*)_{(y)}, \lambda(x-y) \right\rangle_S \otimes \omega_V + d \left(\left\langle (C_{x_0}^* \theta^k d_k^\kappa F^*)_{(y)}, \lambda(x-y) \right\rangle_S \otimes \omega_\kappa \right). \end{aligned} \quad (67)$$

We will now choose our mollifier λ as follows. Define, $\forall y \in E^n$,

$$\mu(y) \triangleq \langle \bar{c}, \lambda(x-y) \otimes \omega_V(x) \rangle_{\text{dR}}, \quad (68)$$

so $\mu \in \mathcal{D}$. In (68) and further on, the subscript dR denotes de Rham pairing between a chain and a form. Let $U \subset c$ such that the calculation point $x_0 \in U$ and $\text{supp}(S) \subset U$. Choose λ such that $\mu(y) = 1, \forall y \in U$. In particular,

$$\mu(x_0) = 1, \quad (69)$$

$$\mu(y) = 1, \forall y \in \text{supp}(S). \quad (70)$$

For instance, we can use for λ a test function such that (i) $\text{supp}(\lambda) \subset c$ and (ii) λ integrates over \bar{c} to 1.

3.5.3 Integration

Integration of (67) over a chain \bar{c} , with boundary $\delta\bar{c}$ and interior c , such that $\text{supp}(S) \subset c \subset \bar{c}$ and with $x_0 \in c$, is actually a de Rham pairing, between a de Rham current (here in particular a chain) and a smooth form. To emphasize this, we denote each integral as $\langle \cdot, \cdot \rangle_{\text{dR}}$, and thus get for the integrated eq. (67),

$$\begin{aligned} & \left\langle \bar{c}, \left\langle (\delta_{x_0} F^*)_{(y)}, \lambda(x-y) \right\rangle_S \otimes \omega_V(x) \right\rangle_{\text{dR}} \\ &= \left\langle \bar{c}, \left\langle (C_{x_0}^* S^*)_{(y)}, \lambda(x-y) \right\rangle_S \otimes \omega_V(x) \right\rangle_{\text{dR}} \\ &+ \left\langle \bar{c}, d \left(\left\langle (C_{x_0}^* \theta^k d_k^\kappa F^*)_{(y)}, \lambda(x-y) \right\rangle_S \otimes \omega_\kappa(x) \right) \right\rangle_{\text{dR}}. \end{aligned} \quad (71)$$

(L₁) The left-hand side term Since $\lambda(x-y)$ is jointly continuous in x and y , we can exchange the order of pairings and get

$$\begin{aligned} L_1 &\triangleq \left\langle \bar{c}, \left\langle (\delta_{x_0} F^*)_{(y)}, \lambda(x-y) \right\rangle_S \otimes \omega_V(x) \right\rangle_{\text{dR}}, \\ &= \left\langle (\delta_{x_0} F^*)_{(y)}, \left\langle \bar{c}, \lambda(x-y) \otimes \omega_V(x) \right\rangle_{\text{dR}} \right\rangle_S. \end{aligned}$$

Using definition (68) this is

$$L_1 = \langle \delta_{x_0} F^*, \mu \rangle_S.$$

Since F^* is smooth, this equals

$$L_1 = \langle \delta_{x_0}, F^* \mu \rangle_S.$$

Using the sifting property of the delta distribution δ_{x_0} and invoking condition (69), we get

$$L_1 = F^*(x_0). \quad (72)$$

(R₁) The first right-hand side term Again exchanging the order of pairings and using definition (68) gives

$$\begin{aligned} R_1 &\triangleq \left\langle \bar{c}, \left\langle (C_{x_0}^* S^*)_{(y)}, \lambda(x-y) \right\rangle_S \otimes \omega_V(x) \right\rangle_{dR}, \\ &= \langle C_{x_0}^* S^*, \mu \rangle_S. \end{aligned}$$

Since S is assumed smooth and by condition (70), we get

$$R_1 = \langle C_{x_0}^*, S^* \rangle_S. \quad (73)$$

(R₂) The second right-hand side term The second term in the right-hand side of eq. (71) is, due to the pseudo-Euclidean version of the Clifford-valued Stokes' theorem for smooth forms, [1, p. 52],

$$\begin{aligned} R_2 &\triangleq \left\langle \bar{c}, d \left(\left\langle (C_{x_0}^* \boldsymbol{\theta}^k d_k{}^\kappa F^*)_{(y)}, \lambda(x-y) \right\rangle_S \otimes \omega_\kappa(x) \right) \right\rangle_{dR} \\ &= \left\langle \delta\bar{c}, \left\langle (C_{x_0}^* \boldsymbol{\theta}^k d_k{}^\kappa F^*)_{(y)}, \lambda(x-y) \right\rangle_S \otimes \omega_\kappa(x) \right\rangle_{dR}. \end{aligned} \quad (74)$$

We now make the following assumptions.

- (i) The region \bar{c} with boundary $\delta\bar{c}$ satisfies the conditions of [11, Lemma 4.2].
- (ii) The Cauchy kernel $C_{x_0}^*$ depends smoothly on the coordinate $y^1 \in I$ normal to $\delta\bar{c}$.
- (iii) The choice of our mollifier λ is further restricted so that it satisfies condition [11, Eq. (4.67)],

$$\langle f_b, \lambda(x^1 - b, x_\delta - y_\delta) \rangle_{S, \delta\bar{c}} = \langle f|_{\delta\bar{c}}, \lambda(x^1 - a, x_\delta - y_\delta) \rangle_{S, \delta\bar{c}}, \quad (75)$$

with $f = C_{x_0}^* \boldsymbol{\theta}^k d_k{}^\kappa F^*$, $f|_{\delta\bar{c}}$ the restriction of f to the boundary $\delta\bar{c}$ [12, p. 263], a and b defined as in [11, Eqs. (4.65) and (4.66)], and y_δ coordinates on $\delta\bar{c}$. We do not need to construct such a mollifier λ , all that is required is that it exists.

Applying [11, Lemma 4.2] to the Schwartz pairing in the right-hand side of (74), we get

$$R_2 = \left\langle \delta\bar{c}, \left\langle ((C_{x_0}^* \boldsymbol{\theta}^k d_k{}^\kappa F^*)|_{\delta\bar{c}})_{(z)}, (\lambda(x-y))_I(z) \right\rangle_{S, \delta\bar{c}} \otimes \omega_\kappa(x) \right\rangle_{dR}.$$

Exchanging the order of pairings gives

$$R_2 = \left\langle ((C_{x_0}^* \boldsymbol{\theta}^k d_k{}^\kappa F^*)|_{\delta\bar{c}})_{(z)}, \langle \delta\bar{c}, (\lambda(x-y))_I(z) \otimes \omega_\kappa(x) \rangle_{dR} \right\rangle_{S, \delta\bar{c}}.$$

Substitution of the result (83) obtained in the Appendix for the covector η_I^* , having as components the de Rham pairing in the right-hand side (see (81)), yields

$$R_2 = \langle (C_{x_0}^* \boldsymbol{\theta}^k d_k{}^\kappa F^*)|_{\delta\bar{c}}, n_\kappa \rangle_{S, \delta\bar{c}},$$

with n_κ^* the κ -component of the (outward to \bar{c}) unit normal covector $n^* \triangleq n_\kappa \boldsymbol{\theta}^\kappa \in \mathbf{E}^{*p,q}$, defined on $\delta\bar{c}$. Since F^* is smooth and n_κ are scalar quantities, this equals

$$R_2 = \langle C_{x_0}^*|_{\delta\bar{c}}, n^*(F^*|_{\delta\bar{c}}) \rangle_{S, \delta\bar{c}}. \quad (76)$$

This is the boundary term of our solution. It involves the restriction, to the boundary $\delta\bar{c}$, of the Cauchy kernel $C_{x_0}^*$, $C_{x_0}^*|_{\delta\bar{c}}$, and of the function F^* , $F^*|_{\delta\bar{c}}$.

Collecting results (72), (73) and (76), we have obtained the general solution of (16) as

$$F^*(x_0) = \langle C_{x_0}^*, S^* \rangle_S + \langle C_{x_0}^*|_{\delta\bar{c}}, n^*(F^*|_{\delta\bar{c}}) \rangle_{S, \delta\bar{c}}. \quad (77)$$

In order to evaluate this expression, a precise characterization of the distribution $C_{x_0}^*|_{\delta\bar{c}}$ must be given. This is currently a subject of further study.

4 APPENDIX

4.1 Calculation of the quantity η_I^*

A. First, define the covector, $\forall y \in E^n$,

$$\eta^*(y) \triangleq \langle \delta \bar{c}, \lambda(x-y) \boldsymbol{\vartheta}^\beta \otimes \omega_\beta(x) \rangle_{\text{dR}}. \quad (78)$$

Applying Stokes' theorem to (78) gives

$$\eta^*(y) = \langle \bar{c}, d_x (\lambda(x-y) \boldsymbol{\vartheta}^\beta \otimes \omega_\beta(x)) \rangle_{\text{dR}}.$$

After expanding the exterior derivation with respect to x , d_x , and using property (61), this becomes

$$\eta^*(\chi) = \langle \bar{c}, (\varepsilon_\alpha \lambda(x-y)) \boldsymbol{\vartheta}^\beta \otimes (\boldsymbol{\vartheta}^\alpha \wedge \omega_\beta(x)) \rangle_{\text{dR}},$$

or

$$\eta^*(\chi) = \langle \bar{c}, (\varepsilon_\beta \lambda(x-y)) \boldsymbol{\vartheta}^\beta \otimes \omega_V(x) \rangle_{\text{dR}}. \quad (79)$$

Since the Pfaff derivative ε_β with respect to x of $\lambda(x-y)$ equals minus the Pfaff derivative ε_β with respect to y , we get

$$\eta^*(y) = -\langle \bar{c}, (d_y \lambda(x-y)) \otimes \omega_V(x) \rangle_{\text{dR}}.$$

Using the continuity of de Rham currents, this becomes

$$\eta^*(y) = -d_y \langle \bar{c}, \lambda(x-y) \otimes \omega_V(x) \rangle_{\text{dR}}.$$

Substituting definition (68), we arrive at

$$\eta^* = -d\mu. \quad (80)$$

B. We now calculate the following covector, with $y = (y^1 \in I, y_\delta \in \delta \bar{c})$,

$$\eta_I^*(y_\delta) \triangleq \langle \delta \bar{c}, (\lambda(x-y))_I(x, y_\delta) \boldsymbol{\vartheta}^\beta \otimes \omega_\beta(x) \rangle_{\text{dR}}. \quad (81)$$

Using [11, Eq. (4.64)] in (81), we get

$$\eta_I^*(y_\delta) = \langle \delta \bar{c}, \langle I, \lambda(x^1 - y^1, x_\delta - y_\delta) \omega_I(y^1) \rangle_{\text{dR}} \boldsymbol{\vartheta}^\beta \otimes \omega_\beta(x^1, x_\delta) \rangle_{\text{dR}}.$$

Herein is I an interval, non tangential to $\delta \bar{c}$, starting in the region where μ is 0 and ending in the region where μ is 1. Since $\lambda(x-y)$ is jointly continuous in x and y , we can exchange the order of pairings and get

$$\begin{aligned} \eta_I^*(y_\delta) &= \langle I, \langle \delta \bar{c}, \lambda(x^1 - y^1, x_\delta - y_\delta) \boldsymbol{\vartheta}^\beta \otimes \omega_\beta(x^1, x_\delta) \rangle_{\text{dR}} \omega_I(y^1) \rangle_{\text{dR}}, \\ &= \langle I, \langle \delta \bar{c}, \lambda(x-y) \boldsymbol{\vartheta}^\beta \otimes \omega_\beta(x) \rangle_{\text{dR}} \omega_I(y^1) \rangle_{\text{dR}}, \end{aligned}$$

or, in terms of definition (78),

$$\eta_I^*(y_\delta) = \langle I, \eta^*(y^1, y_\delta) \omega_I(y^1) \rangle_{\text{dR}}. \quad (82)$$

Substituting (80) in (82) we finally get

$$\eta_I^*(y_\delta) = -\langle I, (d\mu)(y^1, y_\delta) \omega_I(y^1) \rangle_{\text{dR}},$$

or

$$\eta_I^*(y_\delta) = n^*(y_\delta), \quad (83)$$

with n^* the (outward to \bar{c}) unit normal covector, defined on $\delta\bar{c}$, due to the chosen normalization of μ .

Thus, η^* is the differential of μ and μ is the integral over \bar{c} of our shifted mollifier λ . If $\lambda(x - y)$ is a narrow smooth pulse concentrated at y that integrates to 1, then μ will be equal to 1 almost everywhere inside c and equal to 0 almost everywhere outside \bar{c} (since we integrate over \bar{c} in the definition of μ). Hence, μ will be a steep smooth function that rises from 0 to 1 across $\delta\bar{c}$. Thus, $\eta^* = -d\mu$ is a smoothed version of the (outward to \bar{c}) unit normal covector n^* , existing in a small neighborhood around the boundary $\delta\bar{c}$. The actual unit normal covector n^* , defined only on $\delta\bar{c}$, is obtained as η_I^* from η^* under the transformation (82).

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