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Generalized derivatives of spherical associated homogeneous distributions on $\mathbf{R}^n$

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ABSTRACT

Full proofs are given of general expressions for the generalized partial derivatives of spherically symmetric associated homogeneous distributions (SAHDs) based on $\mathbf{R}^n$. This work complements earlier work began by Estrada and Kanwal. Special attention is given to the cases when the derivative of the distribution is a singular distribution, being either an analytic continuation or a distributional extension (or regularization). The presented results are useful for the distributional treatment of potential (i.e., time-invariant) problems in n dimensions.

1. Introduction

Homogeneous distributions (HDs) generalize the concept of homogeneous functions, such as $|\mathbf{x}|^z : \mathbf{R}^n \to \mathbf{C}$, which is a homogeneous function of complex degree z. Associated to homogeneous functions are power-log functions, which arise when taking the derivative with respect to the degree of homogeneity z. The set of associated homogeneous distributions (AHDs) with support in (or based on) $\mathbf{R}^n$, denoted by $\mathcal{H}'(\mathbf{R}^n)$, generalizes these power-log functions [5] (or [6]), [11,13,14]. Distributions in $\mathcal{H}'(\mathbf{R}^n)$ are important for solving distributional potential problems in n dimensions, arising as static (i.e., time-invariant) problems in various branches of physics such as electrostatics, magnetostatics, stationary gravity, molecular theory, etc.

An important subset of $\mathcal{H}'(\mathbf{R}^n)$ are the $O(n)$-invariant AHDs on $\mathbf{R}^n$, called spherical associated homogeneous distributions (SAHDs), and which we denote by $\mathcal{SH}'(\mathbf{R}^n)$. A prominent example is r^z, having degree of homogeneity $z \in \mathbf{C}$ and order of association $m = 0$, see e.g., [11, pp. 71, 98 and 192]. For a more detailed study of SAHDs, including the singular distributions $((D_z^m |\mathbf{x}|^z)_e)_{z=-n-2p}$, $\forall m, p \in \mathbf{N}$, and how they can be generated as pullbacks of AHDs on $\mathbf{R}$, see [9], [10, Chapter 7]. For an introduction to the more modern concept of extension of a partial distribution, which generalizes the classical concept of regularization of a singular distribution, see [5,6,9]. SAHDs on $\mathbf{R}^n$ arise in spherically symmetric potential problems, such as the construction of a fundamental solution g (i.e., Green's distribution) for Poisson's equation $\Delta g = \delta$ and its complex degree generalizations (i.e., involving "complex powers of the Laplacian Δ"). Also, proper associated distributions (having order of association $m > 0$) do arise in this context, e.g., g in $\Delta^{m+p} g = \delta$, in case $n = 2m$, $\forall m \in \mathbf{Z}_+$ and $\forall p \in \mathbf{N}$, is proportional to the distribution $r^{2p} \ln r$.

A distributional treatment of potential problems imposes itself when the problem involves point sources and one needs to calculate derivatives of potentials at the source point. A function description fails in this case since a potential proportional to r^{-p} for $p > 0$, is not differentiable at the origin. To give a proper distributional formulation of these type of problems, expressions for generalized partial derivatives of distributions in $\mathcal{SH}'(\mathbf{R}^n)$ are required. In this paper, we give gen-

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eral expressions for generalized partial derivatives of SAHDs on $\mathbf{R}^n$. Notice that any monomial combination of generalized partial derivatives of any $f \in \mathcal{SH}'(\mathbf{R}^n)$ is in $\mathcal{H}'(\mathbf{R}^n)\backslash\mathcal{SH}'(\mathbf{R}^n)$.

Generalized partial derivatives of the distribution r^{-p}, $\forall p \in \mathbf{Z}_+$, were derived in [3] and in [4] (see also [12]). More general expressions, valid for all complex degrees of homogeneity and all orders of association appear not to be available yet. The here presented study fills this gap and so completes (and corrects minor errors in) the work began in [3,4,12]. Our main results are (i) expressions for the derivatives of the singular distributions $((D_z^m|\mathbf{x}|^z)_e)_{z=-n-2p}$, $\forall m, p \in \mathbf{N}$, of which only the case $m = 0$ was considered in [3,4], (ii) a careful investigation of the special cases when the distribution itself or its generalized partial derivative is either a regular distribution or a singular distribution that is either an analytic continuation or a distributional extension (in classical terms: a regularization) (cf. cases (a), (b) and (c) of Section 4.2), (iii) the (more demanding) proofs of all formulas and the identities (distributional and others) required for the main proofs. Although it is evident that all main proofs can be given by induction, the non-triviality of this procedure for the singular distributions was already noticed in [3, Section 1], and no proofs were given in [3,4,12].

In [9] it is shown that all SAHDs on $\mathbf{R}^n$ are pullbacks T^*, along the function $T : X = \mathbf{R}^n\backslash\{0\} \to Y = \mathbf{R}$ such that $\mathbf{x} \mapsto y = |\mathbf{x}|$, of AHDs on $\mathbf{R}$ [7,8]. A possible route to calculate generalized partial derivatives for SAHDs on $\mathbf{R}^n$ that then presents itself is to use the generalized chain rule, involving the ordinary derivatives $\partial T/\partial x^i$ of the function T and the generalized derivatives of AHDs on $\mathbf{R}$. However, this requires that $\partial T/\partial x^i$ are smooth functions in the whole of $\mathbf{R}^n$. Unfortunately, the functions $\partial T/\partial x^i = x^i/|\mathbf{x}|$ are not smooth at the origin. For this reason we must resort to a direct calculation, based on the definition of the generalized partial derivation operator D_i.

In Section 3, we start with distributions for which the generalized partial derivative is a regular distribution. In Section 4, we then consider distributions for which the generalized partial derivative is a singular distribution. In its first subsection, expressions for first degree derivatives are derived and in the second subsection, expressions for derivatives of arbitrary degree are proved.

2. Notation

We use the notation introduced in [5,6]. For convenience, some of these are repeated here.

1. Define $1_p \triangleq 1$ if p is true, else $1_p \triangleq 0$. Further, $e_m \triangleq 1_{m\in\mathbf{Z}_e}$, hence $e_m = 1$ if m is even, else $e_m = 0$ and similarly, $o_m \triangleq 1_{m\in\mathbf{Z}_o}$, hence $o_m = 1$ if m is odd, else $o_m = 0$.
2. The falling factorial polynomial and the rising factorial polynomial are, $\forall m \in \mathbf{N}$, respectively

$$u_{(m)} \triangleq 1_{m=0} + 1_{0<m} u(u-1)(u-2)\ldots\big(u-(m-1)\big) = \frac{\Gamma(u+1)}{\Gamma(u+1-m)}, \tag{1}$$

$$u^{(m)} \triangleq 1_{m=0} + 1_{0<m} u(u+1)(u+2)\ldots\big(u+(m-1)\big) = \frac{\Gamma(u+m)}{\Gamma(z)}. \tag{2}$$

3. Let $\mathbf{D} \triangleq (D_i \in \mathbf{N}, \forall i \in \mathbf{Z}_{[1,n]})$ denote the generalized partial derivation operators with respect to the coordinates $\mathbf{x} \triangleq (x^i, \forall i \in \mathbf{Z}_{[1,n]})$. Let $\mathbf{k} \triangleq (k_i \in \mathbf{N}, \forall i \in \mathbf{Z}_{[1,n]})$ be a multi-index and $K \triangleq \sum_{i=1}^n k_i$. We will use the following implicit multiplication notation

$$\mathbf{x}^{\mathbf{k}} \triangleq \prod_{i=1}^n \big(x^i\big)^{k_i}, \tag{3}$$

$$\frac{\mathbf{D}^{\mathbf{k}}}{\mathbf{k}!} \triangleq \prod_{i=1}^n \frac{D_i^{k_i}}{k_i!}, \tag{4}$$

$$\frac{(2\mathbf{x})^{\mathbf{l}}}{\mathbf{l}!} \triangleq \prod_{i=1}^n \frac{(2x^i)^{l_i}}{l_i!}, \tag{5}$$

$$\frac{e_{\mathbf{k}-\mathbf{l}}}{((\mathbf{k}-\mathbf{l})/2)!} \triangleq \prod_{i=1}^n \frac{e_{k_i-l_i}}{((k_i-l_i)/2)!}. \tag{6}$$

Further define

$$\sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{k}} \triangleq \sum_{l_1=0}^{k_1} \cdots \sum_{l_n=0}^{k_n}. \tag{7}$$

In any dummy multi-index $\mathbf{l} \triangleq (l_i, \forall i \in \mathbf{Z}_{[1,n]})$, L will be a shorthand for $\sum_{i=1}^n l_i$.

4. The surface area A_{n-1} of the $(n-1)$-dimensional unit sphere S^{n-1} and the volume V_n of the n-dimensional unit ball B^n it bounds, are given by

$$A_{n-1} = 2\frac{\pi^{n/2}}{\Gamma(n/2)}, \tag{8}$$

$$V_n = \frac{A_{n-1}}{n} = \frac{\pi^{n/2}}{\Gamma(n/2+1)}. \tag{9}$$

Some identities that are used throughout the paper are, $\forall n, p, m \in \mathbf{N}$,

$$A_{2p+2} = \frac{(4\pi)^{p+1} p!}{(2p+1)!}, \tag{10}$$

$$4^m(-n/2-p)_{(m)} \frac{A_{n+2(p+m)-1}}{(4\pi)^{p+m}} = (-1)^m \frac{A_{n+2p-1}}{(4\pi)^p}, \tag{11}$$

$$\frac{2}{(4\pi)^{p-q}} \frac{A_{n+2p+1}}{A_{2q+2}} = \frac{V_{n+2p}}{(4\pi)^p} \frac{(2q+1)!}{q!}. \tag{12}$$

3. Partial derivatives being regular distributions

Proposition 1. *There holds,* $\forall i \in \mathbf{Z}_{[1,n]}$, $\forall k \in \mathbf{Z}_+$ *and* $-(n-2k) < \mathrm{Re}(z)$,

$$\frac{D_i^k}{k!}|\mathbf{x}|^z = \sum_{l=0}^{k} (z/2)_{((k+l)/2)} \frac{e_{k-l}}{((k-l)/2)!} \left(\frac{(2x^i)^l}{l!} |\mathbf{x}|^{z-(k+l)} \right). \tag{13}$$

Proof. For each $k \in \mathbf{Z}_+$ and $-(n-2k) < \mathrm{Re}(z)$, all distributions in (13) are regular. From the definition of D_i and by using partial integration it follows that the generalized partial derivative in this case is given by the same rule as the ordinary partial derivative. The proof then immediately follows by induction over k. $\square$

Corollary 2. *Eq.* (13) *generalizes to, for* $-(n-2K) < \mathrm{Re}(z)$,

$$\frac{\mathbf{D}^\mathbf{k}}{\mathbf{k}!}|\mathbf{x}|^z = \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{k}} (z/2)_{((K+L)/2)} \frac{e_{\mathbf{k}-\mathbf{l}}}{((\mathbf{k}-\mathbf{l})/2)!} \left(\frac{(2\mathbf{x})^\mathbf{l}}{\mathbf{l}!} |\mathbf{x}|^{z-(K+L)} \right). \tag{14}$$

Proof. Follows trivially from (13). $\square$

Proposition 3. *There holds,* $\forall i \in \mathbf{Z}_{[1,n]}$, $\forall k, m \in \mathbf{Z}_+$ *and* $-(n-2k) < \mathrm{Re}(z)$,

$$\frac{D_i^k}{k!}\left(|\mathbf{x}|^z \ln^m |\mathbf{x}|\right) = \sum_{j=0}^{m} \frac{\binom{m}{j}}{2^{m-j}} \sum_{l=0}^{k} P^{m-j}_{(k+l)/2-(m-j)}(z/2) \frac{e_{k-l}}{((k-l)/2)!} \left(\frac{(2x^i)^l}{l!} |\mathbf{x}|^{z-(k+l)} \ln^j |\mathbf{x}| \right), \tag{15}$$

with $P_q^m(u)$, $\forall m \in \mathbf{N}$, *the following polynomials in* u *of degree* q,

$$P_q^m(u) \triangleq \frac{d^m}{du^m} u_{(m+q)} = 1_{0 \leqslant q} \sum_{r=0}^{q} \frac{(m+r)!}{r!} s(m+q, m+r) u^r, \tag{16}$$

with $s(k,r)$ *Stirling numbers of the first kind* [1, p. 824].

Proof. For $-n < \mathrm{Re}(z)$, $D_z^m|\mathbf{x}|^z$ is a regular distribution, so $D_z^m|\mathbf{x}|^z = |\mathbf{x}|^z \ln^m |\mathbf{x}|$. By combining the definition of the derivative of a distribution with respect to a parameter, D_z [11, pp. 147–151], with the definition of D_i, it easily follows that $D_i^k D_z^m = D_z^m D_i^k$, $\forall k, m \in \mathbf{Z}_+$. Then, from Eq. (13), the formula for the mth derivative of a product and the identity $u_{(p)} = \sum_{r=0}^{p} s(p,r) u^r$ for Stirling numbers of the first kind $s(p,r)$ [1, 24.1.3, p. 824], the proposition follows. $\square$

Corollary 4. *Eq.* (15) *generalizes to,* $\forall m \in \mathbf{Z}_+$ *and* $-(n-2K) < \mathrm{Re}(z)$,

$$\frac{\mathbf{D}^\mathbf{k}}{\mathbf{k}!}\left(|\mathbf{x}|^z \ln^m |\mathbf{x}|\right) = \sum_{j=0}^{m} \frac{\binom{m}{j}}{2^{m-j}} \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{k}} P^{m-j}_{(K+L)/2-(m-j)}(z/2) \frac{e_{\mathbf{k}-\mathbf{l}}}{((\mathbf{k}-\mathbf{l})/2)!} \left(\frac{(2\mathbf{x})^\mathbf{l}}{\mathbf{l}!} |\mathbf{x}|^{z-(K+L)} \ln^j |\mathbf{x}| \right). \tag{17}$$

Proof. Using $D_i^k D_z^m = D_z^m D_i^k$, this immediately follows from (14). $\square$

By analytic continuation, Eqs. (13), (15) and (14), (17) continue to hold $\forall (z+n) \in \mathbf{C} \backslash (\mathbf{Z}_- \cup \mathbf{Z}_{[0,2K]})$.

4. Partial derivatives being singular distributions

The evaluation of distributions that are analytic continuations or extensions require the following projection operator, $\forall n \in \mathbf{Z}_+$ and $\forall p, q \in \mathbf{N}$, $T^n_{p,q} : \mathcal{D}(\mathbf{R}^n) \to \mathcal{D}(\mathbf{R}^n)$ such that $\varphi \mapsto \psi = T^n_{p,q}\varphi$, with

$$\big(T^n_{p,q}\varphi\big)(\mathbf{x}) \triangleq \varphi(\mathbf{x}) - \sum_{l=0}^{p+q} \left(\sum_{\mathbf{l}=\mathbf{0}}^{(l\ldots l)} 1_{L=l} \left(\frac{\partial^L \varphi}{(\partial \mathbf{x})^{\mathbf{l}}} \right)_{\mathbf{x}=\mathbf{0}} \frac{\mathbf{x}^{\mathbf{l}}}{\mathbf{l}!} \right) \big(1_{l<p} + 1_{p\leqslant l} 1_{[+}\big(1 - |\mathbf{x}|^2\big)\big), \tag{18}$$

wherein $(\partial \mathbf{x})^{\mathbf{l}} \triangleq (\partial x^1)^{l_1} \ldots (\partial x^n)^{l_n}$ and the step function $1_{[+}(x) = 1$ iff $x \geqslant 0$. We further define $T^n_{p,0}$, $\forall p \in \mathbf{Z}_-$, as the identity operator on $\mathcal{D}(\mathbf{R}^n)$. We will need the following properties of this operator, $\forall i \in \mathbf{Z}_{[1,n]}$ and $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$,

$$T^n_{p,q}\left(\frac{\partial \varphi}{\partial x^i}\right) = \frac{\partial}{\partial x^i}\big(T^n_{p+1,q}\varphi\big), \tag{19}$$

$$T^n_{p,q}\big(x^i \varphi\big) = x^i\big(T^n_{p-1,q}\varphi\big). \tag{20}$$

Eqs. (19)–(20) are easily verified by direct substitution.

We will need the following distributions, $\forall p, m \in \mathbf{N}$ and $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$.

(i) At $z = -n - (2p+1)$,

$$\big\langle \big(\mathbf{x}^{\mathbf{k}} |\mathbf{x}|^{-n-(2p+1)}\big) \ln^m |\mathbf{x}|, \varphi \big\rangle \triangleq \int_{\mathbf{R}^n} \big(\mathbf{x}^{\mathbf{k}} |\mathbf{x}|^{-n-(2p+1)} \ln^m |\mathbf{x}|\big)\big(T^n_{2p+1-K,0}\varphi\big)\omega_{\mathbf{R}^n}. \tag{21}$$

Definition (21) coincides for $K = 0$ with the definition for the distributions $|\mathbf{x}|^{-n-(2p+1)} \ln^m |\mathbf{x}|$ given in [9, Eq. (43)]. It is easily verified, by invoking (20), that

$$\big(\mathbf{x}^{\mathbf{k}} |\mathbf{x}|^{-n-(2p+1)}\big) \ln^m |\mathbf{x}| = \mathbf{x}^{\mathbf{k}}.\big(|\mathbf{x}|^{-n-(2p+1)} \ln^m |\mathbf{x}|\big) \triangleq \mathbf{x}^{\mathbf{k}}\big(|\mathbf{x}|^{-n-(2p+1)} \ln^m |\mathbf{x}|\big), \tag{22}$$

with the dot in the middle expression denoting multiplication of a smooth function with a distribution. Eq. (21) shows that the distributions $(\mathbf{x}^{\mathbf{k}} |\mathbf{x}|^{-n-(2p+1)}) \ln^m |\mathbf{x}|$ are analytic continuations for $K \leqslant 2p+1$ and regular distributions for $2p+1 < K$.

(ii) At $z = -n - 2p$,

$$\big\langle \big(\big(D^m_z\big(\mathbf{x}^{\mathbf{k}} |\mathbf{x}|^z\big)\big)_0\big)_{z=-n-2p}, \varphi \big\rangle \triangleq \int_{\mathbf{R}^n} \big(\mathbf{x}^{\mathbf{k}} |\mathbf{x}|^{-n-2p} \ln^m |\mathbf{x}|\big)\big(T^n_{2p-K,0}\varphi\big)\omega_{\mathbf{R}^n}. \tag{23}$$

Definition (23) coincides for $K = 0$ with the definition for the distributions $((D^m_z |\mathbf{x}|^z)_0)_{z=-n-2p}$ given in [9, Eqs. (51) and (18)]. The general associated homogeneous extension is given by

$$\begin{aligned} \left(\left(D^m_z\left(\frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}|\mathbf{x}|^z\right)\right)_e\right)_{z=-n-2p} &= \left(\left(D^m_z\left(\frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}|\mathbf{x}|^z\right)\right)_0\right)_{z=-n-2p} \\ &\quad + c(-1)^K \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{k}} 1_{L\leqslant 2p-K} \frac{e_{\mathbf{k}-\mathbf{l}}}{((\mathbf{k}-\mathbf{l})/2)!} \frac{(2\mathbf{D})^{\mathbf{l}}}{\mathbf{l}!} \frac{\Delta^{p-(K+L)/2}}{(p-(K+L)/2)!}\delta, \end{aligned} \tag{24}$$

wherein $c \in \mathbf{C}$ is arbitrary and

$$\big\langle \mathbf{D}^{\mathbf{l}} \Delta^q \delta, \varphi \big\rangle = \big\langle \Delta^q \mathbf{D}^{\mathbf{l}} \delta, \varphi \big\rangle \triangleq \left(\Delta^q \frac{\partial^L}{(\partial \mathbf{x})^{\mathbf{l}}} \varphi \right)_{\mathbf{x}=\mathbf{0}}, \tag{25}$$

with Δ the generalized and ordinary Laplacian on $\mathbf{R}^n$, respectively. It is easily verified, by invoking (20) and (78), that also

$$\big(\big(D^m_z\big(\mathbf{x}^{\mathbf{k}} |\mathbf{x}|^z\big)\big)_e\big)_{z=-n-2p} = \mathbf{x}^{\mathbf{k}}.\big(\big(\big(D^m_z |\mathbf{x}|^z\big)_e\big)_{z=-n-2p}\big) \triangleq \mathbf{x}^{\mathbf{k}}\big(\big(\big(D^m_z |\mathbf{x}|^z\big)_e\big)_{z=-n-2p}\big), \tag{26}$$

wherein

$$\big(\big(D^m_z |\mathbf{x}|^z\big)_e\big)_{z=-n-2p} = \big(\big(D^m_z |\mathbf{x}|^z\big)_0\big)_{z=-n-2p} + c\frac{\Delta^p}{p!}\delta, \tag{27}$$

and c in (27) is the same constant as in (24). Eq. (23) shows that $((D^m_z(\mathbf{x}^{\mathbf{k}}|\mathbf{x}|^z))_0)_{z=-n-2p}$ are particular extensions for $K \leqslant 2p$ and regular distributions for $2p < K$. This is also clear from the residue of $D^m_z(\mathbf{x}^{\mathbf{k}}|\mathbf{x}|^z)$ at $z = -n-2p$, which is easily obtained from the residue of $D^m_z |\mathbf{x}|^z$ [9, Eq. (48)] and by applying (78), as

$$\operatorname*{Res}_{z=-n-2p}\left(D^m_z\left(\frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}|\mathbf{x}|^z\right)\right) = (-1)^m \frac{A_{n+2p-1}}{(4\pi)^p}(-1)^K \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{k}} 1_{L\leqslant 2p-K} \frac{e_{\mathbf{k}-\mathbf{l}}}{((\mathbf{k}-\mathbf{l})/2)!} \frac{(2\mathbf{D})^{\mathbf{l}}}{\mathbf{l}!} \frac{\Delta^{p-(K+L)/2}}{(p-(K+L)/2)!}\delta. \tag{28}$$

4.1. First degree derivatives

We now consider the following three remaining cases.

Case (a) refers to the regular distributions $|\mathbf{x}|^{-n+1}\ln^m|\mathbf{x}|$, $\forall m\in\mathbf{N}$. By the definition of a regular distribution and of D_i, in terms of the operator $T^n_{p,q}$ and after singling out the integration over x^i, we obtain, $\forall\varphi\in\mathcal{D}(\mathbf{R}^n)$,

$$\langle D_i(|\mathbf{x}|^{-n+1}\ln^m|\mathbf{x}|),\varphi\rangle=-\int_{\mathbf{R}^{n-1}}\int_{-\infty}^{+\infty}(|\mathbf{x}|^{-n+1}\ln^m|\mathbf{x}|)\left(\frac{\partial}{\partial x^i}(T^n_{0,0}\varphi)\right)^i\omega_{\mathbf{R}^{n-1}}. \tag{29}$$

Case (b) refers to singular distributions of the form (21) with $K=0$, which are analytic continuations. From (21), the definition of D_i, the property (19) and after singling out the integration over x^i, we obtain, $\forall\varphi\in\mathcal{D}(\mathbf{R}^n)$,

$$\langle D_i(|\mathbf{x}|^{-n-(2p+1)}\ln^m|\mathbf{x}|),\varphi\rangle=-\int_{\mathbf{R}^{n-1}}\int_{-\infty}^{+\infty}(|\mathbf{x}|^{-n-(2p+1)}\ln^m|\mathbf{x}|)\left(\frac{\partial}{\partial x^i}(T^n_{2(p+1),0}\varphi)\right)dx^i\omega_{\mathbf{R}^{n-1}}. \tag{30}$$

Case (c) refers to singular distributions of the form (24) with $K=0$, which are extensions. From (23), the definition of D_i, the property (19) and after singling out the integration over x^i, we obtain, $\forall\varphi\in\mathcal{D}(\mathbf{R}^n)$,

$$\langle D_i(((D^m_z|\mathbf{x}|^z)_0)_{z=-n-2p}),\varphi\rangle=-\int_{\mathbf{R}^{n-1}}\int_{-\infty}^{+\infty}(|\mathbf{x}|^{-n-2p}\ln^m|\mathbf{x}|)\left(\frac{\partial}{\partial x^i}(T^n_{2p+1,0}\varphi)\right)dx^i\omega_{\mathbf{R}^{n-1}}. \tag{31}$$

In each of Eqs. (29), (30) and (31), the inner integral is of the form

$$\int_{-\infty}^{+\infty}(|(x^i,\boldsymbol{\rho})|^{-n-(k-1)}\ln^m|(x^i,\boldsymbol{\rho})|)\left(\frac{\partial}{\partial x^i}(T^n_{k,0}\varphi)\right)dx^i, \tag{32}$$

with $k\in\{-1\}\cup\mathbf{N}$. In order to perform the partial integration of the integral (32), we subdivide the integration region $\mathbf{R}^{n-1}$ in (31) in the closed ball with radius $0\leqslant\rho\leqslant 1$ and its exterior $1<\rho$. Inside the closed ball, the integration interval of the integral over x^i is subdivided in the three subintervals $]-\infty,-\sqrt{1-\rho^2}[$, $[-\sqrt{1-\rho^2},+\sqrt{1-\rho^2}]$ and $]+\sqrt{1-\rho^2},+\infty[$, in each of which $T^n_{k,0}\varphi$ is continuous in x^i. The integral (32) is calculated in Appendix A.1 and the result is given by (58). In Appendix A.2 we further apply the integration $-\int_{\mathbf{R}^{n-1}}\omega_{\mathbf{R}^{n-1}}$ to the boundary term $B_{k,m}(\boldsymbol{\rho})$ occurring in (58). The result of this integration is given by (60).

Substituting expression (58) for the inner integral (32) and using (60) and (20) yields for the right-hand sides of (29), (30) and (31) the common expression

$$1_{1\leqslant k}o_k 1_{m=0}\frac{V_{n+k-1}}{(4\pi)^{(k-1)/2}}\left(\frac{\Delta^{(k-1)/2}}{((k-1)/2)!}\frac{\partial}{\partial x^i}\varphi\right)_{\mathbf{x}=\mathbf{0}}+\int_{\mathbf{R}^n}\begin{pmatrix}(-n-(k-1))x^i|\mathbf{x}|^{-n-(k+1)}\ln^m|\mathbf{x}|\\ +mx^i|\mathbf{x}|^{-n-(k+1)}\ln^{m-1}|\mathbf{x}|\end{pmatrix}(T^n_{k,0}\varphi)\omega_{\mathbf{R}^n}. \tag{33}$$

Case (a). We get for (29), from (33) with $k=0$,

$$\langle D_i(|\mathbf{x}|^{-n+1}\ln^m|\mathbf{x}|),\varphi\rangle=\int_{\mathbf{R}^n}\begin{pmatrix}c(-n+1)x^i|\mathbf{x}|^{-n-1}\ln^m|\mathbf{x}|\\ +mx^i|\mathbf{x}|^{-n-1}\ln^{m-1}|\mathbf{x}|\end{pmatrix}(T^n_{0,0}\varphi)\omega_{\mathbf{R}^n}.$$

By (21), with $k_i=1$, $K=1$ and $p=0$, this implies that

$$D_i(|\mathbf{x}|^{-n+1}\ln^m|\mathbf{x}|)=(-n+1)(x^i|\mathbf{x}|^{-n-1})\ln^m|\mathbf{x}|+m(x^i|\mathbf{x}|^{-n-1})\ln^{m-1}|\mathbf{x}|. \tag{34}$$

This shows that the generalized partial derivative $D_i(|\mathbf{x}|^{-n+1}\ln^m|\mathbf{x}|)$ is no longer a regular distribution, but an analytic continuation.

Case (b). We get for (30), from (33) with $k=2(p+1)$,

$$\langle D_i(|\mathbf{x}|^{-n-(2p+1)}\ln^m|\mathbf{x}|),\varphi\rangle=\int_{\mathbf{R}^n}\begin{pmatrix}(-n-(2p+1))x^i|\mathbf{x}|^{-n-(2(p+1)+1)}\ln^m|\mathbf{x}|\\ +mx^i|\mathbf{x}|^{-n-(2(p+1)+1)}\ln^{m-1}|\mathbf{x}|\end{pmatrix}(T^n_{2(p+1),0}\varphi)\omega_{\mathbf{R}^n}.$$

By (21), with $k_i=1$, $K=1$ and p replaced by $p+1$, this implies that

$$D_i(|\mathbf{x}|^{-n-(2p+1)}\ln^m|\mathbf{x}|)=-(n+(2p+1))(x^i|\mathbf{x}|^{-n-(2p+3)})\ln^m|\mathbf{x}|+m(x^i|\mathbf{x}|^{-n-(2p+3)})\ln^{m-1}|\mathbf{x}|. \tag{35}$$

This shows that the generalized partial derivative $D_i(|\mathbf{x}|^{-n-(2p+1)}\ln^m|\mathbf{x}|)$ is also an analytic continuation.

Case (c). We get for (31), from (33) with $k = 2p + 1$,

$$\left\langle D_i\left(\left(\left(D_z^m|\mathbf{x}|^z\right)_0\right)_{z=-n-2p}\right), \varphi\right\rangle = 1_{m=0}\frac{V_{n+2p}}{(4\pi)^p}\left(\frac{\Delta^p}{p!}\frac{\partial}{\partial x^i}\varphi\right)_{\mathbf{x}=\mathbf{0}} + \int_{\mathbf{R}^n}\left(\begin{matrix}(-n-2p)x^i|\mathbf{x}|^{-n-2(p+1)}\ln^m|\mathbf{x}| \\ +mx^i|\mathbf{x}|^{-n-2(p+1)}\ln^{m-1}|\mathbf{x}|\end{matrix}\right)\left(T^n_{2p+1,0}\varphi\right)\omega_{\mathbf{R}^n}.$$

By (23), with $k_i = 1$, $K = 1$ and p replaced by $p + 1$, this implies that

$$D_i\left(\left(\left(D_z^m|\mathbf{x}|^z\right)_0\right)_{z=-n-2p}\right) = 1_{m=0}\frac{V_{n+2p}}{(4\pi)^p}\frac{\Delta^p}{p!}D_i\delta - (n+2p)\left(\left(\left(D_z^m x^i|\mathbf{x}|^z\right)_0\right)_{z=-n-2(p+1)}\right) + m\left(\left(\left(D_z^{m-1}x^i|\mathbf{x}|^z\right)_0\right)_{z=-n-2(p+1)}\right). \tag{36}$$

Eq. (36) reveals that the generalized partial derivative of $((D_z^m|\mathbf{x}|^z)_0)_{z=-n-2p}$ contains delta terms if $m = 0$, but not if $m > 0$.

Example 1. For $n = 1$ and $x^i = x$, $D_i = D$, (34) yields

$$D\left(\ln^m|x|\right) = mx^{-1}\ln^{m-1}|x|, \tag{37}$$

wherein the analytic continuation x^{-1} is Cauchy's principal value (also written as $\operatorname{Pf}\frac{1}{x}$). Eq. (37) agrees with [5, Eqs. (143) and (171)], [6, Eqs. (142) and (170)].

Example 2. For $m = 0$ follows from (36) that

$$D_i|\mathbf{x}|_0^{-n-2p} = -(n+2p)x^i|\mathbf{x}|_0^{-n-2(p+1)} - \frac{V_{n+2p}}{(4\pi)^p}\frac{\Delta^p}{p!}D_i\delta, \tag{38}$$

and in particular for $p = 0$,

$$D_i|\mathbf{x}|_0^{-n} = -nx^i|\mathbf{x}|_0^{-n-2} - V_n D_i\delta. \tag{39}$$

Let $n = 3$, put $-n - 2p = -(k+2)$ or $2p = k - 1$ (with $k \in \mathbf{Z}_{o,+}$) in (38) and use the following identity (which is a direct consequence of (10) for $k = 2p + 1$),

$$\frac{V_{k+2}}{(4\pi)^{(k-1)/2}((k-1)/2)!} = \frac{4\pi}{k+2}\frac{1}{k!}, \tag{40}$$

to get

$$D_i|\mathbf{x}|_0^{-(k+2)} = -(k+2)x^i|\mathbf{x}|_0^{-(k+4)} - o_k\frac{4\pi}{k+2}\frac{1}{k!}\Delta^{(k-1)/2}D_i\delta. \tag{41}$$

This shows that [12, p. 136, Eq. (16)] is wrong. Further, taking $k = 1$ in (41) we get

$$D_i|\mathbf{x}|_0^{-3} = -3x^i|\mathbf{x}|_0^{-5} - \frac{4\pi}{3}D_i\delta, \tag{42}$$

instead of [12, p. 135, Eq. (14)].

4.2. Higher degree derivatives

4.2.1. The cases (a) and (b)

Proposition 5. *There holds, $\forall i \in \mathbf{Z}_{[1,n]}$, $\forall k \in \mathbf{Z}_+$, $\forall m \in \mathbf{N}$ and $\forall z \in \{-1\} \cup \mathbf{Z}_{o,+}$,*

$$\frac{D_i^k}{k!}\left(|\mathbf{x}|^{-n-z}\ln^m|\mathbf{x}|\right) = \sum_{j=0}^{m}\frac{\binom{m}{j}}{2^{m-j}}\sum_{l=0}^{k}P^{m-j}_{(k+l)/2-(m-j)}(z/2)\frac{e_{k-l}}{((k-l)/2)!}\left(\frac{(2x^i)^l}{l!}|\mathbf{x}|^{-n-z-(k+l)}\right)\ln^j|\mathbf{x}|, \tag{43}$$

wherein the distributions in the right-hand side are analytic continuations.

Proof. It follows from (34) and (35) that the generalized derivative of $|\mathbf{x}|^{-n-z}\ln^m|\mathbf{x}|$, $\forall m \in \mathbf{N}$ and $z \in \{-1\} \cup \mathbf{Z}_{o,+}$, is given by Leibniz' rule. Hence the proof of (43) is identical to the one leading to (15). □

Proposition 6. *There holds,* $\forall m \in \mathbf{N}$ *and* $\forall z \in \{-1\} \cup \mathbf{Z}_{o,+}$,

$$\frac{\mathbf{D^k}}{\mathbf{k}!}\left(|\mathbf{x}|^{-n-z}\ln^m|\mathbf{x}|\right) = \sum_{j=0}^{m}\frac{\binom{m}{j}}{2^{m-j}}\sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{k}} P^{m-j}_{(K+L)/2-(m-j)}\big((-n-z)/2\big)\frac{e_{\mathbf{k}-\mathbf{l}}}{((\mathbf{k}-\mathbf{l})/2)!}\left(\frac{(2\mathbf{x})^{\mathbf{l}}}{\mathbf{l}!}|\mathbf{x}|^{-n-z-(K+L)}\right)\ln^j|\mathbf{x}|, \tag{44}$$

wherein the distributions in the right-hand side are analytic continuations.

Proof. It follows from (34) and (35) that the generalized derivative of $|\mathbf{x}|^{-n-z}\ln^m|\mathbf{x}|$, $\forall m \in \mathbf{N}$ and $z \in \{-1\} \cup \mathbf{Z}_{o,+}$, is given by Leibniz' rule. Hence the proof of (44) is identical to the one leading to (17). □

4.2.2. The case (c)

Proposition 7. *There holds,* $\forall i \in \mathbf{Z}_{[1,n]}$, $\forall k \in \mathbf{Z}_+$ *and* $\forall p \in \mathbf{N}$,

$$\begin{aligned}\frac{D_i^k}{k!}|\mathbf{x}|_0^{-n-2p} &= \sum_{l=0}^{k}(-n/2-p)_{((k+l)/2)}\frac{e_{k-l}}{((k-l)/2)!}\frac{(2x^i)^l}{l!}|\mathbf{x}|_0^{-n-2(p+(k+l)/2)}\\ &\quad-\frac{A_{n+2p-1}}{2^k(4\pi)^p}\sum_{l=0}^{k}\beta\big(n+2p;k,(k-l)/2\big)\frac{e_{k-l}}{((k-l)/2)!}\frac{(2D_i)^l}{l!}\frac{\triangle^{p+(k-l)/2}}{(p+(k-l)/2)!}\delta,\end{aligned} \tag{45}$$

with $\triangle$ *the generalized Laplacian in* $\mathbf{R}^n$ *and the functions* $\beta(u;k,m)$ *satisfying,* $\forall k \in \mathbf{Z}_+$ *and* $\forall m \in \mathbf{N}$,

$$\begin{aligned}&\beta(u;0,0) = 0,\\ &\beta(u;k,m) = \frac{k-2m}{k}\beta(u;k-1,m) + \frac{1}{k}\sum_{q=0}^{m}\binom{m}{q}\frac{2q+k-2m}{2q+u+2(k-1-m)}(-1)^{m-q},\end{aligned} \tag{46}$$

and given by

$$\beta(u;k,m) = 1_{k>0}1_{m=0}\sum_{i=0}^{k-1}\frac{1}{u+2i} + 1_{m>0}\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^j}{u+2(k-1)-2j}. \tag{47}$$

(i) The proposition is true for $k=1$ since it reduces to (38).

(ii) Assume that the proposition holds for some $k>1$. Applying D_i to (45) and using Leibniz' rule, which holds since $(x^i)^l$ is a smooth function, we get

$$\begin{aligned}\frac{D_i^{k+1}}{k!}|\mathbf{x}|_0^{-n-2p} &= \sum_{l=1}^{k}(-n/2-p)_{((k+l)/2)}\frac{e_{k-l}}{((k-l)/2)!}2\frac{(2x^i)^{l-1}}{(l-1)!}|\mathbf{x}|_0^{-n-2(p+(k+l)/2)}\\ &\quad+\sum_{l=0}^{k}(-n/2-p)_{((k+l)/2)}\frac{e_{k-l}}{((k-l)/2)!}\frac{(2x^i)^l}{l!}D_i|\mathbf{x}|_0^{-n-2(p+(k+l)/2)}\\ &\quad-\frac{A_{n+2p-1}}{2^{k+1}(4\pi)^p}\sum_{l=0}^{k}\beta\big(n+2p;k,(k-l)/2\big)\frac{e_{k-l}}{((k-l)/2)!}\frac{(2D_i)^{l+1}}{l!}\frac{\triangle^{p+(k-l)/2}}{(p+(k-l)/2)!}\delta.\end{aligned}$$

Substitute herein (38), with p replaced by $p+(k+l)/2$, to get

$$\begin{aligned}\frac{D_i^{k+1}}{k!}|\mathbf{x}|_0^{-n-2p} &= \sum_{l=1}^{k}(-n/2-p)_{((k+l)/2)}\frac{e_{k-l}}{((k-l)/2)!}2\frac{(2x^i)^{l-1}}{(l-1)!}|\mathbf{x}|_0^{-n-2(p+(k+l)/2)}\\ &\quad-\sum_{l=0}^{k}(-n/2-p)_{((k+l)/2)}\frac{e_{k-l}}{((k-l)/2)!}\frac{(2x^i)^l}{l!}\big(n+2\big(p+(k+l)/2\big)\big)x^i|\mathbf{x}|_0^{-n-2(p+(k+l)/2+1)}\\ &\quad-\sum_{l=0}^{k}e_{k-l}\frac{(-n/2-p)_{((k+l)/2)}}{((k-l)/2)!}\frac{(2x^i)^l}{l!}\frac{A_{n+2(p+(k+l)/2)-1}}{(4\pi)^{p+(k+l)/2}(n+2(p+(k+l)/2))}\frac{\triangle^{p+(k+l)/2}D_i\delta}{(p+(k+l)/2)!}\\ &\quad-\frac{A_{n+2p-1}}{2^{k+1}(4\pi)^p}\sum_{l=0}^{k}\beta\big(n+2p;k,(k-l)/2\big)\frac{e_{k-l}}{((k-l)/2)!}\frac{(2D_i)^{l+1}}{l!}\frac{\triangle^{p+(k-l)/2}}{(p+(k-l)/2)!}\delta.\end{aligned} \tag{48}$$

(a) The first two sums in the right-hand side of (48) can be combined in a single sum, denoted S_{1+2}, and is obtained (exactly as in the proof of Proposition 1) as

$$S_{1+2}=(k+1)\sum_{l=0}^{k+1}(-n/2-p)_{((k+1+l)/2)}\frac{e_{k+1-l}}{((k+1-l)/2)!}\frac{(2x^i)^l}{l!}|\mathbf{x}|_0^{-n-2(p+(k+1+l)/2)}. \tag{49}$$

(b) With the use of identity (11) we write the third sum in the right-hand side of (48), denoted S_3, as

$$S_3=-\frac{A_{n+2p-1}}{2^k(4\pi)^p}\sum_{l=0}^{k}\frac{e_{k-l}}{((k-l)/2)!}\frac{(-1)^{(k+l)/2}}{n+2p+(k+l)}\left(\frac{(x^i)^l}{l!}\frac{\Delta^{p+(k+l)/2}}{(p+(k+l)/2)!}D_i\delta\right).$$

Substituting herein identity (77) gives

$$S_3=-\frac{A_{n+2p-1}}{2^k(4\pi)^p}\sum_{l=0}^{k}\frac{e_{k-l}}{((k-l)/2)!}\frac{(-1)^{(k+l)/2}}{n+2p+(k+l)}(-1)^l(l+1)\sum_{r=-1}^{l}e_{k-r}2^r\frac{e_{l-r}}{((l-r)/2)!}\frac{\Delta^{p+(k-r)/2}}{(p+(k-r)/2)!}\frac{D_i^{r+1}\delta}{(r+1)!},$$

or

$$S_3=-\frac{A_{n+2p-1}}{2^k(4\pi)^p}\left(\sum_{l=0}^{k}\frac{(-1)^{(k-l)/2}e_{k+1}}{((k-l)/2)!}\frac{l+1}{n+2p+(k+l)}\frac{2^{-1}e_{l+1}}{((l+1)/2)!}\right)\frac{\Delta^{p+(k+1)/2}\delta}{(p+(k+1)/2)!}$$
$$-\frac{A_{n+2p-1}}{2^k(4\pi)^p}\sum_{l=0}^{k}\sum_{r=0}^{l}\frac{(-1)^{(k-l)/2}e_{k-l}}{((k-l)/2)!}\frac{l+1}{n+2p+(k+l)}\frac{2^re_{l-r}}{((l-r)/2)!}\frac{e_{k-r}\Delta^{p+(k-r)/2}}{(p+(k-r)/2)!}\frac{D_i^{r+1}\delta}{(r+1)!}.$$

Exchanging the summation order gives

$$S_3=-\frac{A_{n+2p-1}}{2^k(4\pi)^p}\left(\begin{array}{l}(\sum_{l=0}^{k}\frac{(-1)^{(k-l)/2}e_{k+1}}{((k-l)/2)!}\frac{l+1}{n+2p+(k+l)}\frac{2^{-1}e_{l+1}}{((l+1)/2)!})\frac{\Delta^{p+(k+1)/2}\delta}{(p+(k+1)/2)!}\\ \quad+\sum_{r=0}^{k}(\sum_{l=r}^{k}\frac{(-1)^{(k-l)/2}e_{k-r}}{((k-l)/2)!}\frac{l+1}{n+2p+(k+l)}\frac{2^re_{l-r}}{((l-r)/2)!})\frac{\Delta^{p+(k-r)/2}}{(p+(k-r)/2)!}\frac{D_i^{r+1}\delta}{(r+1)!}\end{array}\right).$$

We can absorb the first single sum in the second double sum by letting r running from -1 and so bring S_3 in the form

$$S_3=-\frac{A_{n+2p-1}}{2^{k+1}(4\pi)^p}\sum_{j=0}^{k+1}S\big(-1;(k+1-j)/2,j,n+2(p-1)+k+1+j\big)\frac{e_{k+1-j}}{((k+1-j)/2)!}$$
$$\times\frac{(2D_i)^j}{j!}\frac{\Delta^{p+(k+1-j)/2}}{(p+(k+1-j)/2)!}\delta, \tag{50}$$

wherein we defined, $\forall a,b\in\mathbf{N}$,

$$S(x;m,a,b)\triangleq\sum_{q=0}^{m}\binom{m}{q}\frac{2q+a}{2q+b}x^{m-q}. \tag{51}$$

(c) We write the fourth sum in the right-hand side of (48), denoted S_4, as

$$S_4=-\frac{A_{n+2p-1}}{2^{k+1}(4\pi)^p}\sum_{j=0}^{k+1}e_{k+1-j}\frac{j\beta(n+2p;k,(k+1-j)/2)}{((k+1-j)/2)!}\frac{(2D_i)^j}{j!}\frac{\Delta^{p+(k+1-j)/2}\delta}{(p+(k+1-j)/2)!}. \tag{52}$$

Now, adding (49), (50) and (52) gives for (48)

$$\frac{D_i^{k+1}}{k!}|\mathbf{x}|_0^{-n-2p}=(k+1)\sum_{j=0}^{k+1}e_{k+1-j}(-1)^{(k+1+j)/2}\frac{(2x^i)^j}{j!}|\mathbf{x}|_0^{-n-2(p+(k+1+j)/2)}$$
$$-\frac{A_{n+2p-1}}{2^{k+1}(4\pi)^p}\sum_{l=0}^{k+1}\frac{e_{k+1-l}}{((k+1-l)/2)!}\left(\begin{array}{l}S(-1;(k+1-l)/2,l,n+2(p-1)+k+1+l)\\ \quad+l\beta(n+2p;k,(k+1-l)/2)\end{array}\right)$$
$$\times\frac{(2D_i)^l}{l!}\frac{\Delta^{p+(k+1-l)/2}\delta}{(p+(k+1-l)/2)!}. \tag{53}$$

The given recurrence relation for the functions β, (46), can be restated in the following form, with $m=(k+1-l)/2$,

$$(k+1)\beta\big(n+2p;k+1,(k+1-l)/2\big)=l\beta\big(n+2p;k,(k+1-l)/2\big)+S\big(-1;(k+1-l)/2,l,n+2(p-1)+k+1+l\big).$$

Substituting this in (53) yields

$$\frac{D_i^{k+1}}{(k+1)!}|\mathbf{x}|_0^{-n-2p} = \sum_{l=0}^{k+1}(-n/2-p)_{((k+1+l)/2)}\frac{e_{k+1-l}}{((k+1-l)/2)!}\frac{(2x^i)^l}{l!}|\mathbf{x}|_0^{-n-2(p+(k+1+l)/2)}$$
$$-\frac{A_{n+2p-1}}{2^{k+1}(4\pi)^p}\sum_{l=0}^{k+1}\beta\big(n+2p;k+1,(k+1-l)/2\big)\frac{e_{k+1-l}}{((k+1-l)/2)!}$$
$$\times\frac{(2D_i)^l}{l!}\frac{\Delta^{p+(k+1-l)/2}}{(p+(k+1-l)/2)!}\delta,$$

which equals the proposed form for $k+1$. Hence, the proposition is true for $k+1$.

(iii) By induction the proposition holds $\forall k \in \mathbf{Z}_+$.

In Appendix A.3 it is shown that expression (47) for the functions β satisfies (46). In (45) only functions β with parameters $k, m \in \mathbf{N}$ such that $k-2m \in \mathbf{N}$ enter.

Expression (45) is equivalent to [4, Eq. (3.25)] (or [12, p. 135, Eq. (9)]), except for a printing error in the power of the Laplacian. Also, the sum in parentheses in [3, Eq. (5.26)] is wrong as it reduces the second sum to just one term: $j=0$. This was apparently corrected in [4, Eq. (3.25)].

Proposition 8. *Let* $\mathbf{D}_m \triangleq (D_i, \forall i \in \mathbf{Z}_{[1,m]})$, $\mathbf{x}_m \triangleq (x^i, \forall i \in \mathbf{Z}_{[1,m]})$, $\mathbf{k}_m \triangleq (k_i \in \mathbf{N}, \forall i \in \mathbf{Z}_{[1,m]})$ *and* $K_m \triangleq \sum_{i=1}^m k_i$. *There holds,* $\forall p \in \mathbf{N}$ *and* $\forall m \in \mathbf{Z}_{[1,n]}$,

$$\frac{\mathbf{D}_m^{\mathbf{k}_m}}{\mathbf{k}_m!}|\mathbf{x}|_0^{-n-2p} = \sum_{\mathbf{l}_m=\mathbf{0}}^{\mathbf{k}_m}(-n/2-p)_{((K_m+L_m)/2)}\frac{e_{\mathbf{k}_m-\mathbf{l}_m}}{((\mathbf{k}_m-\mathbf{l}_m)/2)!}\frac{(2\mathbf{x}_m)^{\mathbf{l}_m}}{\mathbf{l}_m!}|\mathbf{x}|_0^{-n-2p-(K_m+L_m)}$$
$$-\frac{A_{n+2p-1}}{2^{K_m}(4\pi)^p}\sum_{\mathbf{l}_m=\mathbf{0}}^{\mathbf{k}_m}\beta\big(n+2p;K_m,(K_m-L_m)/2\big)\frac{e_{\mathbf{k}_m-\mathbf{l}_m}}{((\mathbf{k}_m-\mathbf{l}_m)/2)!}$$
$$\times\frac{(2\mathbf{D}_m)^{\mathbf{l}_m}}{\mathbf{l}_m!}\frac{\Delta^{p+(K_m-L_m)/2}}{(p+(K_m-L_m)/2)!}\delta, \tag{54}$$

with L_m *a shorthand for* $\sum_{i=1}^m l_i$ *and the other quantities are as in Proposition* 7.

(i) For $m=1$ the proposition is true as it reduces to (45).

(ii) Assume the proposition is true for $1<m<n$. Calculate

$$\frac{(2D_{m+1})^{k_{m+1}}}{k_{m+1}!}\frac{(2\mathbf{D}_m)^{\mathbf{k}_m}}{\mathbf{k}_m!}|\mathbf{x}|_0^{-n-2p} = 2^{K_m}\sum_{\mathbf{l}_m=\mathbf{0}}^{\mathbf{k}_m}\left(-\frac{n}{2}-p\right)_{((K_m+L_m)/2)}\frac{e_{\mathbf{k}_m-\mathbf{l}_m}}{((\mathbf{k}_m-\mathbf{l}_m)/2)!}\frac{(2\mathbf{x}_m)^{\mathbf{l}_m}}{\mathbf{l}_m!}\frac{(2D_{m+1})^{k_{m+1}}}{k_{m+1}!}|\mathbf{x}|_0^{-n-2p-(K_m+L_m)}$$
$$-\frac{A_{n+2p-1}}{(4\pi)^p}\sum_{\mathbf{l}_m=\mathbf{0}}^{\mathbf{k}_m}\beta\big(n+2p;K_m,(K_m-L_m)/2\big)\frac{(2D_{m+1})^{k_{m+1}}}{k_{m+1}!}$$
$$\times\frac{e_{\mathbf{k}_m-\mathbf{l}_m}}{((\mathbf{k}_m-\mathbf{l}_m)/2)!}\frac{(2\mathbf{D}_m)^{\mathbf{l}_m}}{\mathbf{l}_m!}\frac{\Delta^{p+(K_m-L_m)/2}}{(p+(K_m-L_m)/2)!}\delta.$$

Substituting herein expression (45) for $\frac{(2D_{m+1})^{k_{m+1}}}{k_{m+1}!}|\mathbf{x}|_0^{-n-2p-(K_m+L_m)}$ gives

$$\frac{(2\mathbf{D}_{m+1})^{\mathbf{k}_{m+1}}}{\mathbf{k}_{m+1}!}|\mathbf{x}|_0^{-n-2p} = 2^{K_{m+1}}\sum_{\mathbf{l}_{m+1}=\mathbf{0}}^{\mathbf{k}_{m+1}}\left(-\frac{n}{2}-p\right)_{((K_{m+1}+L_{m+1})/2)}\frac{e_{\mathbf{k}_{m+1}-\mathbf{l}_{m+1}}}{((\mathbf{k}_{m+1}-\mathbf{l}_{m+1})/2)!}$$
$$\times\frac{(2\mathbf{x}_{m+1})^{\mathbf{l}_{m+1}}}{\mathbf{l}_{m+1}!}|\mathbf{x}|_0^{-n-2p-(K_{m+1}+L_{m+1})}$$
$$-\sum_{\mathbf{l}_{m+1}=\mathbf{0}}^{\mathbf{k}_{m+1}}\left(2^{K_m}\left(-\frac{n}{2}-p\right)_{((K_m+L_m)/2)}\frac{A_{n+2p+(K_m+L_m)-1}}{(4\pi)^{p+(K_m+L_m)/2}}\right)$$
$$\times\beta\big(n+2p+(K_m+L_m);k_{m+1},(k_{m+1}-l_{m+1})/2\big)\frac{e_{\mathbf{k}_{m+1}-\mathbf{l}_{m+1}}}{((\mathbf{k}_{m+1}-\mathbf{l}_{m+1})/2)!}$$
$$\times\frac{(2D_{m+1})^{l_{m+1}}}{l_{m+1}!}\left(\frac{(2\mathbf{x}_m)^{\mathbf{l}_m}}{\mathbf{l}_m!}\frac{\Delta^{p+(K_m+L_m)/2+(k_{m+1}-l_{m+1})/2}\delta}{(p+(K_m+L_m)/2+(k_{m+1}-l_{m+1})/2)!}\right)$$

$$
-\frac{A_{n+2p-1}}{(4\pi)^p}\sum_{\mathbf{l}_m=\mathbf{0}}^{\mathbf{k}_m}\beta\big(n+2p;K_m,(K_m-L_m)/2\big)
\times\frac{e_{\mathbf{k}_m-\mathbf{l}_m}}{((\mathbf{k}_m-\mathbf{l}_m)/2)!}\frac{(2\mathbf{D}_m)^{\mathbf{l}_m}}{\mathbf{l}_m!}\frac{(2D_{m+1})^{k_{m+1}}}{k_{m+1}!}\frac{\triangle^{p+(K_m-L_m)/2}}{(p+(K_m-L_m)/2)!}\delta. \tag{55}
$$

We now work on the second multi-sum in the right-hand side of (55). With the identity (11) this term becomes

$$
-\frac{A_{n+2p-1}}{(4\pi)^p}\sum_{\mathbf{l}_{m+1}=\mathbf{0}}^{\mathbf{k}_{m+1}}(-1)^{(K_m+L_m)/2}\beta\big(n+2p+(K_m+L_m);k_{m+1},(k_{m+1}-l_{m+1})/2\big)
\times\frac{e_{\mathbf{k}_{m+1}-\mathbf{l}_{m+1}}}{((\mathbf{k}_{m+1}-\mathbf{l}_{m+1})/2)!}\frac{(2D_{m+1})^{l_{m+1}}}{l_{m+1}!}\left(\frac{(\mathbf{x}_m)^{\mathbf{l}_m}}{\mathbf{l}_m!}\frac{\triangle^{p+(K_m+L_m)/2+(k_{m+1}-l_{m+1})/2}}{(p+(K_m+L_m)/2+(k_{m+1}-l_{m+1})/2)!}\delta\right).
$$

Substitute herein identity (78), with p replaced by $p+(K_m+L_m)/2+(k_{m+1}-l_{m+1})/2$, so that we obtain

$$
-\frac{A_{n+2p-1}}{(4\pi)^p}\sum_{\mathbf{l}_{m+1}=\mathbf{0}}^{\mathbf{k}_{m+1}}(-1)^{(K_m-L_m)/2}\beta\big(n+2p+(K_m+L_m);k_{m+1},(k_{m+1}-l_{m+1})/2\big)
\times\frac{e_{\mathbf{k}_{m+1}-\mathbf{l}_{m+1}}}{((\mathbf{k}_{m+1}-\mathbf{l}_{m+1})/2)!}\frac{(2D_{m+1})^{l_{m+1}}}{l_{m+1}!}\sum_{\mathbf{r}_m=\mathbf{0}}^{\mathbf{l}_m}1_{R_m\leqslant K_m+2p+k_{m+1}-l_{m+1}}
\times\frac{e_{\mathbf{l}_m-\mathbf{r}_m}}{((\mathbf{l}_m-\mathbf{r}_m)/2)!}\frac{(2\mathbf{D}_m)^{\mathbf{r}_m}}{\mathbf{r}_m!}\frac{\triangle^{p+(K_m-R_m)/2+(k_{m+1}-l_{m+1})/2}}{(p+(K_m-R_m)/2+(k_{m+1}-l_{m+1})/2)!}\delta.
$$

Singling out in this expression the sum over l_{m+1} and noticing that $R_m\leqslant K_m+2p+k_{m+1}-l_{m+1}$ is always true since $R_m\leqslant K_m$ and $0\leqslant 2p+k_{m+1}-l_{m+1}$, our second multi-sum can be further written as

$$
-\frac{A_{n+2p-1}}{(4\pi)^p}\sum_{l_{m+1}=0}^{k_{m+1}}\frac{e_{k_{m+1}-l_{m+1}}}{((k_{m+1}-l_{m+1})/2)!}
\times\sum_{\mathbf{l}_m=\mathbf{0}}^{\mathbf{k}_m}\sum_{\mathbf{r}_m=\mathbf{0}}^{\mathbf{l}_m}(-1)^{(K_m-L_m)/2}\beta\big(n+2p+(K_m+L_m);k_{m+1},(k_{m+1}-l_{m+1})/2\big)\frac{(2D_{m+1})^{l_{m+1}}}{l_{m+1}!}
\times\frac{e_{\mathbf{k}_m-\mathbf{l}_m}}{((\mathbf{k}_m-\mathbf{l}_m)/2)!}\frac{e_{\mathbf{l}_m-\mathbf{r}_m}}{((\mathbf{l}_m-\mathbf{r}_m)/2)!}\frac{(2\mathbf{D}_m)^{\mathbf{r}_m}}{\mathbf{r}_m!}\frac{\triangle^{p+(K_m-R_m)/2+(k_{m+1}-l_{m+1})/2}}{(p+(K_m-R_m)/2+(k_{m+1}-l_{m+1})/2)!}\delta.
$$

Exchanging the summations and invoking the symmetry of the binomial coefficients allows us to bring it in the form

$$
-\frac{A_{n+2p-1}}{(4\pi)^p}\sum_{l_{m+1}=0}^{k_{m+1}}\frac{e_{k_{m+1}-l_{m+1}}}{((k_{m+1}-l_{m+1})/2)!}\sum_{\mathbf{r}_m=\mathbf{0}}^{\mathbf{k}_m}\begin{pmatrix}\sum_{q_1=0}^{(k_1-r_1)/2}\cdots\sum_{q_m=0}^{(k_m-r_m)/2}(\prod_{i=1}^m\binom{(k_i-r_i)/2}{q_i})\\ \beta(n+2(p+K_m)-2Q_m;k_{m+1},(k_{m+1}-l_{m+1})/2)(-1)^{Q_m}\end{pmatrix}\frac{(2D_{m+1})^{l_{m+1}}}{l_{m+1}!}
\times\frac{e_{\mathbf{k}_m-\mathbf{r}_m}}{((\mathbf{k}_m-\mathbf{r}_m)/2)!}\frac{(2\mathbf{D}_m)^{\mathbf{r}_m}}{\mathbf{r}_m!}\frac{\triangle^{p+(K_m-R_m)/2+(k_{m+1}-l_{m+1})/2}}{(p+(K_m-R_m)/2+(k_{m+1}-l_{m+1})/2)!}\delta,
$$

with Q_m a shorthand for $\sum_{i=1}^m q_i$.

Now, invoking definition (69) and by applying identity (70) we see that (after putting $u=n+2(p+K_m)$, $m_i=(k_i-r_i)/2$ and $m_{m+1}=(k_{m+1}-l_{m+1})/2$)

$$
\sum_{q_1=0}^{m_1}\cdots\sum_{q_m=0}^{m_m}\left(\prod_{i=1}^m\binom{m_i}{q_i}\right)\beta(u-2Q_m;k_{m+1},m_{m+1})(-1)^{Q_m}=\sigma\big(u;(K_m-R_m)/2,k_{m+1},m_{m+1}\big).
$$

Substituting this simplification and renaming the indices r_i back to l_i, our second multi-sum finally becomes

$$
-\frac{A_{n+2p-1}}{(4\pi)^p}\sum_{\mathbf{l}_{m+1}=\mathbf{0}}^{\mathbf{k}_{m+1}}\sigma\big(n+2(p+K_m);(K_m-L_m)/2,k_{m+1},(k_{m+1}-l_{m+1})/2\big)
\times\frac{e_{\mathbf{k}_{m+1}-\mathbf{l}_{m+1}}}{((\mathbf{k}_{m+1}-\mathbf{l}_{m+1})/2)!}\frac{(2\mathbf{D}_{m+1})^{\mathbf{l}_{m+1}}}{\mathbf{l}_{m+1}!}\frac{\triangle^{p+(K_{m+1}-L_{m+1})/2}}{(p+(K_{m+1}-L_{m+1})/2)!}\delta.
$$

Substituting this expression for the second multi-sum in the right-hand side of (55) now gives for the whole equation

$$\frac{(2\mathbf{D}_{m+1})^{\mathbf{k}_{m+1}}}{\mathbf{k}_{m+1}!}|\mathbf{x}|_0^{-n-2p} = 2^{K_{m+1}} \sum_{\mathbf{l}_{m+1}=\mathbf{0}}^{\mathbf{k}_{m+1}} (-n/2-p)_{((K_{m+1}+L_{m+1})/2)} \frac{e_{\mathbf{k}_{m+1}-\mathbf{l}_{m+1}}}{((\mathbf{k}_{m+1}-\mathbf{l}_{m+1})/2)!}$$
$$\times \frac{(2\mathbf{x}_{m+1})^{\mathbf{l}_{m+1}}}{\mathbf{l}_{m+1}!}|\mathbf{x}|_0^{-n-2p-(K_{m+1}+L_{m+1})}$$
$$- \frac{A_{n+2p-1}}{(4\pi)^p} \sum_{\mathbf{l}_{m+1}=\mathbf{0}}^{\mathbf{k}_{m+1}} \begin{pmatrix} \sigma(n+2(p+K_m);(K_m-L_m)/2,k_{m+1},(k_{m+1}-l_{m+1})/2) \\ +1_{l_{m+1}=k_{m+1}}\beta(n+2p;K_m,(K_m-L_m)/2) \end{pmatrix}$$
$$\times \frac{e_{\mathbf{k}_{m+1}-\mathbf{l}_{m+1}}}{((\mathbf{k}_{m+1}-\mathbf{l}_{m+1})/2)!} \frac{(2\mathbf{D}_{m+1})^{\mathbf{l}_{m+1}}}{\mathbf{l}_{m+1}!} \frac{\Delta^{p+(K_{m+1}-L_{m+1})/2}}{(p+(K_{m+1}-L_{m+1})/2)!}\delta.$$

Using herein identity (74) in the form

$$\sigma\big(n+2(p+K_m);(K_m-L_m)/2,k_{m+1},(k_{m+1}-l_{m+1})/2\big)+1_{l_{m+1}=k_{m+1}}\beta\big(n+2p;K_m,(K_m-L_m)/2\big)$$
$$=\beta\big(n+2p;K_{m+1},(K_{m+1}-L_{m+1})/2\big),$$

finally yields for (55),

$$\frac{\mathbf{D}_{m+1}^{\mathbf{k}_{m+1}}}{\mathbf{k}_{m+1}!}|\mathbf{x}|_0^{-n-2p} = \sum_{\mathbf{l}_{m+1}=\mathbf{0}}^{\mathbf{k}_{m+1}} (-n/2-p)_{((K_{m+1}+L_{m+1})/2)} \frac{e_{\mathbf{k}_{m+1}-\mathbf{l}_{m+1}}}{((\mathbf{k}_{m+1}-\mathbf{l}_{m+1})/2)!} \frac{(2\mathbf{x}_{m+1})^{\mathbf{l}_{m+1}}}{\mathbf{l}_{m+1}!}|\mathbf{x}|_0^{-n-2p-(K_{m+1}+L_{m+1})}$$
$$- \frac{A_{n+2p-1}}{2^{K_{m+1}}(4\pi)^p} \sum_{\mathbf{l}_{m+1}=\mathbf{0}}^{\mathbf{k}_{m+1}} \beta\big(n+2p;K_{m+1},(K_{m+1}-L_{m+1})/2\big)$$
$$\times \frac{e_{\mathbf{k}_{m+1}-\mathbf{l}_{m+1}}}{((\mathbf{k}_{m+1}-\mathbf{l}_{m+1})/2)!} \frac{(2\mathbf{D}_{m+1})^{\mathbf{l}_{m+1}}}{\mathbf{l}_{m+1}!} \frac{\Delta^{p+(K_{m+1}-L_{m+1})/2}}{(p+(K_{m+1}-L_{m+1})/2)!}\delta.$$

This shows that the proposition is also true for $m+1$.

(iii) By induction it holds $\forall m \in \mathbf{Z}_{[1,n]}$.

Proposition 9. *There holds, $\forall m \in \mathbf{Z}_+$ and $\forall p \in \mathbf{N}$,*

$$\frac{\mathbf{D}^{\mathbf{k}}}{\mathbf{k}!}\big(\big((D_z^m|\mathbf{x}|^z)_0\big)_{z=-n-2p}\big) = \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{k}} (-n/2-p)_{((K+L)/2)} \frac{e_{\mathbf{k}-\mathbf{l}}}{((\mathbf{k}-\mathbf{l})/2)!} \frac{(2\mathbf{x})^{\mathbf{l}}}{\mathbf{l}!}\big(\big((D_z^m|\mathbf{x}|^z)_0\big)_{z=-n-2p-(K+L)}\big). \tag{56}$$

Proof. It follows from (36) that the generalized derivative of $((D_z^m|\mathbf{x}|^z)_0)_{z=-n-2p}$ for $m>0$ does not contain delta terms. Hence the proof of (56) is identical to the one leading to (14). □

The higher degree generalized partial derivatives of any extension $((D_z^m|\mathbf{x}|^z)_e)_{z=-n-2p}$ are finally found from (27) together with (54) (for $m=0$) or (56) (for $m>0$).

Appendix A

A.1. Partial integration of the integral (32)

Put $\mathbf{x} = (x^i, \boldsymbol{\rho}) \in \mathbf{R}^n$ and $\rho \triangleq |\boldsymbol{\rho}|$. Define

$$I_{k,m}(\boldsymbol{\rho}) \triangleq \int_{-\infty}^{+\infty} \big(\big|(x^i,\boldsymbol{\rho})\big|^{-n-(k-1)} \ln^m\big|(x^i,\boldsymbol{\rho})\big|\big)\bigg(\frac{\partial}{\partial x^i}\big(T_{k,0}^n\varphi\big)\bigg)dx^i,$$
$$J_{k,m}(\boldsymbol{\rho}) \triangleq \int_{-\infty}^{+\infty} \begin{pmatrix} (-n-(k-1))x^i|(x^i,\boldsymbol{\rho})|^{-n-(k+1)}\ln^m|(x^i,\boldsymbol{\rho})| \\ +mx^i|(x^i,\boldsymbol{\rho})|^{-n-(k+1)}\ln^{m-1}|(x^i,\boldsymbol{\rho})| \end{pmatrix}\big(T_{k,0}^n\varphi\big)\,dx^i.$$

(i) For $0 \leqslant \rho \leqslant 1$, the integration interval in the integral $I_{k,m}(\boldsymbol{\rho})$ is divided in the subintervals $]-\infty,-\sqrt{1-\rho^2}[$, $[-\sqrt{1-\rho^2},+\sqrt{1-\rho^2}]$ and $]+\sqrt{1-\rho^2},+\infty[$. Partial integration of each of the corresponding subintegrals gives

$$I_{k,m}(\boldsymbol{\rho}) = B_{k,m}(\boldsymbol{\rho}) - J_{k,m}(\boldsymbol{\rho}),$$

with, since $1^{-(n-1)-k}\ln^m 1 = 1_{m=0}$,

$$B_{k,m}(\boldsymbol{\rho}) \triangleq 1_{m=0}\big(T^n_{k,0}\varphi\big)\big(-\sqrt{1-\rho^2},\boldsymbol{\rho}\big) - 0 + 1_{m=0}\big(T^n_{k,0}\varphi\big)\big(+\sqrt{1-\rho^2},\boldsymbol{\rho}\big) - 1_{m=0}\big(T^n_{k,0}\varphi\big)\big(-\sqrt{1-\rho^2},\boldsymbol{\rho}\big) + 0 - 1_{m=0}\big(T^n_{k,0}\varphi\big)\big(+\sqrt{1-\rho^2},\boldsymbol{\rho}\big).$$

The operator $T^n_{k,0}$, in the boundary terms produced by the middle subintegral, has a last term which is proportional to $1_{[+}(1-|\mathbf{x}|^2)$, while in the first and last boundary term this last term is absent (due to the chosen ranges of integration and the definition of the step function $1_{[+}$). Therefore, the four boundary terms in the above expression do not cancel completely and we get

$$B_{k,m}(\boldsymbol{\rho}) = 1_{0\leqslant k}1_{m=0}\sum_{\mathbf{l}=\mathbf{0}}^{(k\ldots k)} 1_{L=k}(-2o_{l_i})\frac{(\sqrt{1-\rho^2})^{l_i}}{l_i!}\left(\frac{\partial^L\varphi}{(\partial\mathbf{x})^{\mathbf{l}}}\right)_{\mathbf{x}=\mathbf{0}}\left(\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{(x^j)^{l_j}}{l_j!}\right)_{(x^j)=\boldsymbol{\rho}}.$$

Put $\mathbf{x} = (x^i,\boldsymbol{\rho})\in\mathbf{R}^n$, $\boldsymbol{\rho} = (\rho^j,\forall j\in\mathbf{Z}_{[1,n-1]})\in\mathbf{R}^{n-1}$, $\rho\triangleq|\boldsymbol{\rho}|$, $\omega^j\triangleq\rho^j/\rho$ with $0<\rho\leqslant 1$. Then,

$$B_{k,m}(\boldsymbol{\rho}) = 1_{0\leqslant k}1_{m=0}\sum_{l=0}^{k}(-2o_l)\frac{\rho^{k-l}}{(k-l)!}\frac{(\sqrt{1-\rho^2})^l}{l!}\left(\sum_{l_1=0}^{k}\cdots\sum_{l_{n-1}=0}^{k}\right)1_{L'=k-l}\left(L'!\left(\prod_{j=1}^{n-1}\frac{(\omega^j)^{l_j}}{l_j!}\frac{\partial^{l_j}}{(\partial\rho^j)^{l_j}}\right)\frac{\partial^l}{(\partial x^i)^l}\varphi\right)_{x^i=0,\,\boldsymbol{\rho}=\mathbf{0}},$$

wherein L' is a shorthand for $\sum_{j=1}^{n-1}l_j$, so $L = L'+l$. We can further bring this in the form

$$B_{k,m}(\boldsymbol{\rho}) = 1_{0\leqslant k}1_{m=0}\sum_{l=0}^{k}(-2o_l)\frac{\rho^{k-l}}{(k-l)!}\frac{(\sqrt{1-\rho^2})^l}{l!}\left(\frac{\partial^k}{(\partial\rho)^{k-l}(\partial x^i)^l}\varphi\right)_{x^i=0,\,\rho=0}. \tag{57}$$

(ii) For $1<\rho$, we can apply partial integration directly to the complete integral (32) and the two boundary terms are zero due to the finite support of φ.

(iii) Hence, $\forall\rho\in\mathbf{R}_{[+}$,

$$I_{k,m}(\boldsymbol{\rho}) = 1_{0\leqslant\rho\leqslant 1}B_{k,m}(\boldsymbol{\rho}) - J_{k,m}(\boldsymbol{\rho}). \tag{58}$$

A.2. Integration of the boundary term (57)

Here we integrate the boundary term $B_{k,m}(\boldsymbol{\rho})$, given in (57), over $\mathbf{R}^{n-1}$ in spherical coordinates. We get

$$-\int_{\mathbf{R}^{n-1}} B_{k,m}(\boldsymbol{\rho})\omega_{\mathbf{R}^{n-1}} = -1_{0\leqslant k}1_{m=0}\sum_{l=0}^{k}(-2o_l)C_{k,l}\int_0^1\rho^{k-l}\big(\sqrt{1-\rho^2}\big)^l\rho^{n-2}\,d\rho, \tag{59}$$

with

$$C_{k,l}\triangleq\frac{A_{n-2}}{(k-l)!l!}\left(\left(\frac{d}{d\rho}\right)^{k-l}S\left(\frac{\partial^l\varphi}{(\partial x^i)^l}\right)\right)_{x^i=0,\,\rho=0},$$

and wherein we denoted the spherical mean operator in $\mathbf{R}^{n-1}$ by S. We only need the constants $C_{k,2q+1}$, $\forall q\in\mathbf{N}$.

(i) The constants $C_{k,2q+1}$ will only be non-zero if k is also odd, since all odd derivatives of a spherical mean function are zero at $\rho=0$ ([11, pp. 72–73], [9, Eq. (13)]). Let $k = 2p+1$, $\forall p\in\mathbf{N}$. Now use Pizetti's formula ([11, p. 73, Eq. (6)], [2, p. 287], [9, Eq. (19)]), which becomes here

$$\frac{1}{(2(p-q))!}\left(\left(\frac{d}{d\rho}\right)^{2(p-q)}S\left(\frac{\partial^{2q+1}}{(\partial x^i)^{2q+1}}\varphi\right)\right)_{x^i=0,\,\rho=0} = \frac{A_{n-1+2(p-q)-1}}{(4\pi)^{p-q}A_{n-1-1}}\left(\frac{\Delta^{p-q}_{\mathbf{R}^{n-1}}}{(p-q)!}\frac{\partial^{2q+1}}{(\partial x^i)^{2q+1}}\varphi\right)_{x^i=0,\,\boldsymbol{\rho}=\mathbf{0}},$$

with $\Delta_{\mathbf{R}^{n-1}}$ the Laplacian on $\mathbf{R}^{n-1}$. Then,

$$C_{2p+1,2q+1} = \frac{A_{n+2(p-q-1)}}{(4\pi)^{p-q}(2q+1)!}\left(\frac{\Delta^{p-q}_{\mathbf{R}^{n-1}}}{(p-q)!}\frac{\partial^{2q+1}}{(\partial x^i)^{2q+1}}\varphi\right)_{x^i=0,\,\boldsymbol{\rho}=\mathbf{0}}.$$

(ii) By definition of the beta function and its relation to the gamma function, we get

$$\int_0^1 \rho^{n+2(p-q-1)}\left(\sqrt{1-\rho^2}\right)^{2q+1} d\rho = \frac{1}{2}\frac{\Gamma((n-1)/2+p-q)\Gamma(q+\frac{3}{2})}{\Gamma((n-1)/2+p+\frac{3}{2})}.$$

Substitution of these two expressions in (59) and using (8) yields, $\forall p \in \mathbf{N}$,

$$-\int_{\mathbf{R}^{n-1}} B_{2p+1,m}(\boldsymbol{\rho})\omega_{\mathbf{R}^{n-1}} = 1_{m=0}\sum_{q=0}^{p}\frac{2A_{n+2p+1}}{(4\pi)^{p-q}A_{2q+2}(p-q)!(2q+1)!}\left(\Delta_{\mathbf{R}^{n-1}}^{p-q}\frac{\partial^{2q+1}}{(\partial x^i)^{2q+1}}\varphi\right)_{x^i=0,\,\boldsymbol{\rho}=\mathbf{0}}.$$

With the identity (12) this reduces to

$$-\int_{\mathbf{R}^{n-1}} B_{2p+1,m}(\boldsymbol{\rho})\omega_{\mathbf{R}^{n-1}} = 1_{m=0}\frac{V_{n+2p}}{(4\pi)^p p!}\left(\left(\Delta_{\mathbf{R}^{n-1}}+\frac{\partial^2}{(\partial x^i)^2}\right)^p\frac{\partial}{\partial x^i}\varphi\right)_{x^i=0,\,\boldsymbol{\rho}=\mathbf{0}}.$$

Hence, $\forall k, m \in \mathbf{N}$,

$$-\int_{\mathbf{R}^{n-1}} B_{k,m}(\boldsymbol{\rho})\omega_{\mathbf{R}^{n-1}} = o_k 1_{1\leqslant k}1_{m=0}\frac{V_{n+k-1}}{(4\pi)^{(k-1)/2}}\left(\frac{\Delta^{(k-1)/2}}{((k-1)/2)!}\frac{\partial}{\partial x^i}\varphi\right)_{\mathbf{x}=\mathbf{0}}, \tag{60}$$

with Δ the Laplacian on $\mathbf{R}^n$.

A.3. The functions β

Consider functions, $\forall k, m \in \mathbf{N}$, $\beta : \mathbf{R}\backslash(-\mathbf{N}) \to \mathbf{R}$ such that $u \mapsto \beta(u; k, m)$ defined by the recurrence relation (46). Define further $\beta(u; k, m) \triangleq 0$, $\forall k, m \in \mathbf{Z}_-$.

Proposition 10. *The functions β are given by,* $\forall k, m \in \mathbf{N}$,

$$\beta(u; k, m) = 1_{k>0}1_{m=0}\sum_{j=0}^{k-1}\frac{1}{u+2j} + 1_{m>0}\sum_{q=0}^{m-1}\binom{m-1}{q}\frac{(-1)^q}{u+k+(k-2m)+2(m-1-q)}. \tag{61}$$

Proof. (i) Let $m = 0$. The recurrence relation (46) readily shows that $\beta(u; k, 0)$ is given by the first sum in (61).

(ii) Let $m > 0$. Write the sum in (46) as

$$\begin{aligned}
\frac{1}{k}\sum_{q=0}^{m}\binom{m}{q}\frac{2q+k-2m}{2q+u+2(k-1-m)}(-1)^{m-q} &= \frac{k-2m}{k}\sum_{q=0}^{m}\binom{m}{q}\frac{(-1)^{m-q}}{2q+u+2(k-1-m)} \\
&\quad + \frac{1}{k}\sum_{q=1}^{m}\binom{m}{q}\frac{2q}{2q+u+2(k-1-m)}(-1)^{m-q} \\
&= \frac{k-2m}{k}\sum_{j=-1}^{m-1}\binom{m}{j+1}\frac{(-1)^{m-1-j}}{u+2(k-m)+2j} \\
&\quad + \frac{2m}{k}\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{m-1-j}}{u+2(k-m)+2j} \\
&= \frac{k-2m}{k}\sum_{j=-1}^{m-1}\left(\binom{m}{j+1} - 1_{0\leqslant j}\binom{m-1}{j}\right)\frac{(-1)^{m-1-j}}{u+2(k-m)+2j} \\
&\quad + \sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{m-1-j}}{u+2(k-m)+2j}.
\end{aligned}$$

With the binomial identity, $\forall j \in \mathbf{Z}_{[-1,m-1]}$,

$$\binom{m}{j+1} - 1_{0\leqslant j}\binom{m-1}{j} = 1_{j\leqslant m-2}\binom{m-1}{j+1},$$

we get

$$\frac{1}{k}\sum_{q=0}^{m}\binom{m}{q}\frac{2q+k-2m}{2q+u+2(k-1-m)}(-1)^{m-q}$$
$$=-\frac{k-2m}{k}\sum_{i=0}^{m-1}\binom{m-1}{i}\frac{(-1)^{m-1-i}}{u+2(k-1-m)+2i}+\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{m-1-j}}{u+2(k-m)+2j}. \tag{62}$$

Substitution of (62) in (46) yields

$$\beta(u;k,m)-\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{m-1-j}}{u+2(k-m)+2j}=\frac{k-2m}{k}\left(\beta(u;k-1,m)-\sum_{i=0}^{m-1}\binom{m-1}{i}\frac{(-1)^{m-1-i}}{u+2(k-1-m)+2i}\right).$$

Expanding this recurrence relation gives

$$\beta(u;k,m)-\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{m-1-j}}{u+2(k-m)+2j}$$
$$=\frac{k-2m}{k}\frac{k-1-2m}{k-1}\cdots\frac{1}{2m+1}\left(\beta(u;2m,m)-\sum_{i=0}^{m-1}\binom{m-1}{i}\frac{(-1)^{m-1-i}}{u+2m+2i}\right). \tag{63}$$

In particular for $k=2m$, the recurrence relation (46) yields, $\forall m\in\mathbf{Z}_+$,

$$\beta(u;2m,m)=\sum_{i=0}^{m-1}\binom{m-1}{i}\frac{(-1)^{m-1-i}}{u+2(m+i)}. \tag{64}$$

Substituting this result in (63) shows that its right-hand side is zero and hence that

$$\beta(u;k,m)=\sum_{q=0}^{m-1}\binom{m-1}{q}\frac{(-1)^{q}}{u+2(k-1)-2q}. \qquad \square \tag{65}$$

Proposition 11. *The functions β given by* (61) *also satisfy the recurrence relation, $\forall k,m\in\mathbf{N}$,*

$$\beta(u;k,m)=\beta(u;k,m-1)-\beta(u;k-1,m-1). \tag{66}$$

Proof. (i) For all $k,m\in\mathbf{Z}_+$. Substituting (61) for $\beta(u;k,m-1)$ and $\beta(u;k-1,m-1)$ gives

$$\begin{aligned}
\beta(u;k,m-1)-\beta(u;k-1,m-1)&=1_{k>0}1_{m=1}\sum_{j=0}^{k-1}\frac{1}{u+2j}+1_{m>1}\sum_{q=0}^{m-2}\binom{m-2}{q}\frac{(-1)^q}{u+2(k-1)-2q}\\
&\quad-1_{k>1}1_{m=1}\sum_{j=0}^{k-2}\frac{1}{u+2j}-1_{m>1}\sum_{q=0}^{m-2}\binom{m-2}{q}\frac{(-1)^q}{u+2(k-2)-2q}\\
&=1_{k=1}1_{m=1}\sum_{j=0}^{k-1}\frac{1}{u+2j}+1_{k>1}1_{m=1}\sum_{j=0}^{k-1}\frac{1}{u+2j}-1_{k>1}1_{m=1}\sum_{j=0}^{k-2}\frac{1}{u+2j}\\
&\quad+1_{m>1}\sum_{q=0}^{m-2}\binom{m-2}{q}\frac{(-1)^q}{u+2(k-1)-2q}+1_{m>1}\\
&\quad\times\sum_{q+1=1}^{m-1}\binom{m-2}{q+1-1}\frac{(-1)^{q+1}}{u+2(k-1)-2(q+1)}\\
&=1_{k>0}1_{m=1}\frac{1}{u+2(k-1)}+1_{m>1}\sum_{j=0}^{m-1}\left(1_{j\leqslant m-2}\binom{m-2}{j}+1_{1\leqslant j}\binom{m-2}{j-1}\right)\\
&\quad\times\frac{(-1)^j}{u+2(k-1)-2j}.
\end{aligned}$$

Using the binomial identity, $\forall j\in\mathbf{Z}_{[-1,m-1]}$,

$$1_{j\leqslant m-2}\binom{m-2}{j}+1_{1\leqslant j}\binom{m-2}{j-1}=\binom{m-1}{j},$$

we get

$$\beta(u;k,m-1)-\beta(u;k-1,m-1)=1_{m>0}\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^j}{u+2(k-1)-2j},$$

which by identification yields (66) (since $m>0$).

(ii) Since we defined $\beta(u;k,m)=0$ for negative values of the parameters k and m, and since $\beta(u;0,0)=0$, the proposition is readily seen to also hold for $k=0$ and $m=0$. □

Proposition 12. *The β functions satisfy, $\forall k, m_i, m_j \in \mathbf{N}$,*

$$\beta(u;k,m_i+m_j)=\sum_{q_i=0}^{m_i}\binom{m_i}{q_i}\beta(u;k-q_i,m_j)(-1)^{q_i}, \tag{67}$$

and are also given by

$$\beta(u;k,m)=\sum_{q=0}^{m}\binom{m}{q}1_{k>q}\sum_{p=q}^{k-1}\frac{(-1)^q}{u+2p-2q}. \tag{68}$$

Proof. (i) Successive application of (66) gives, $\forall k, m_i, m_j \in \mathbf{N}$,

$$\beta(u;k,m_i+m_j)=\sum_{q_i=0}^{m_i}\binom{m_i}{q_i}\beta(u;k-q_i,m_j)(-1)^{q_i}.$$

(ii) In particular for $m_j=0$, this expression becomes, $\forall k, m \in \mathbf{N}$,

$$\beta(u;k,m)=\sum_{q=0}^{m}\binom{m}{q}\beta(u;k-q,0)(-1)^q=\sum_{q=0}^{m}\binom{m}{q}\left(1_{k>q}\sum_{j=0}^{k-q-1}\frac{1}{u+2j}\right)(-1)^q=\sum_{q=0}^{m}\binom{m}{q}1_{k>q}\sum_{p=q}^{k-1}\frac{(-1)^q}{u+2p-2q}. \qquad \square$$

Definition 1. For all $K, m_i, m_j \in \mathbf{N}$,

$$\sigma(u;m_i,K,m_j)\triangleq\sum_{q=0}^{m_i}\binom{m_i}{q}\beta(u-2q;K,m_j)(-1)^q. \tag{69}$$

Proposition 13. *There holds, $\forall m, k \in \mathbf{N}$ and $\forall m_i \in \mathbf{N}$, $\forall i \in \mathbf{Z}_{[1,m+1]}$,*

$$\sigma(u;M_m,k,m_{m+1})=\sum_{q_1=0}^{m_1}\cdots\sum_{q_m=0}^{m_m}\left(\prod_{i=1}^{m}\binom{m_i}{q_i}\right)\beta(u-2Q_m;k,m_{m+1})(-1)^{Q_m}, \tag{70}$$

with M_m a shorthand for $\sum_{i=1}^m m_i$ and Q_m a shorthand for $\sum_{i=1}^m q_i$.

Proof. Reordering summations gives

$$\begin{aligned}&\sum_{q_1=0}^{m_1}\cdots\sum_{q_m=0}^{m_m}\left(\prod_{i=1}^{m}\binom{m_i}{q_i}\right)\beta(u-2Q_m;k,m_{m+1})(-1)^{Q_m}\\&\quad=\sum_{s=0}^{M_m}\left(\sum_{q_1=0}^{m_1}\cdots\sum_{q_m=0}^{m_m}1_{s=Q_m}\left(\prod_{i=1}^{m}\binom{m_i}{q_i}\right)\right)\beta(u-2Q_m;k,m_{m+1})(-1)^{Q_m}.\end{aligned}$$

Due to a property of the binomial coefficients, we get

$$\begin{aligned}\sum_{q_1=0}^{m_1}\cdots\sum_{q_m=0}^{m_m}\left(\prod_{i=1}^{m}\binom{m_i}{q_i}\right)\beta(u-2Q_m;k,m_{m+1})(-1)^{Q_m}&=\sum_{s=0}^{M_m}\binom{M_m}{s}\beta(u-2s;k,m_{m+1})(-1)^s\\&=\sigma(u;M_m,k,m_{m+1}),\end{aligned}$$

by definition (69). □

Proposition 14. *There holds,* $\forall k_i, k_j, m_i \in \mathbf{N}$,

$$\sigma(u+2k_i; m_i, k_j, 0) = \beta(u; k_i + k_j, m_i) - \beta(u; k_i, m_i). \tag{71}$$

Proof. We have by (69) and (61),

$$\sigma(u+2k_i; m_i, k_j, 0) = \sum_{q=0}^{m_i} \binom{m_i}{q} \beta(u+2k_i-2q; k_j, 0)(-1)^q = \sum_{q=0}^{m_i} \binom{m_i}{q} \left(1_{k_j>0} \sum_{j=0}^{k_j-1} \frac{1}{u+2k_i-2q+2j} \right) (-1)^q$$
$$= 1_{k_j>0} \sum_{j=0}^{k_j-1} \left(\sum_{q=0}^{m_i} \binom{m_i}{q} \frac{(-1)^q}{u+2(k_i+j+1-1)-2q} \right),$$

or

$$\sigma(u+2k_i; m_i, k_j, 0) = 1_{k_j>0} \sum_{j=1}^{k_j-1} \beta(u; k_i+j+1, m_i+1).$$

Due to (66) this is equivalent to

$$\sigma(u+2k_i; m_i, k_j, 0) = 1_{k_j>0} \sum_{j=0}^{k_j-1} \beta(u; k_i+j+1, m_i) - 1_{k_j>0} \sum_{j=0}^{k_j-1} \beta(u; k_i+j, m_i),$$

or by telescoping

$$\sigma(u+2k_i; m_i, k_j, 0) = 1_{k_j>0} \beta(u; k_i+k_j, m_i) - 1_{k_j>0} \beta(u; k_i, m_i) = \beta(u; k_i+k_j, m_i) - \beta(u; k_i, m_i),$$

since $\sigma(u+2k_i; m_i, 0, 0) = 0$ due to $\beta(u+2k_i; 0, 0) = 0$. □

Proposition 15. *There holds,* $\forall k_i, k_j, m_i, m_j \in \mathbf{N}$,

$$\sigma(u+2k_i; m_i, k_j, m_j) + 1_{m_j=0} \beta(u; k_i, m_i)$$
$$= \sigma(u; m_i, k_i+k_j, m_j) + 1_{m_j=0} \big(\beta(u; k_i+k_j, m_i) - \sigma(u; m_i, k_i+k_j, 0) \big). \tag{72}$$

Proof. We have, $\forall k_i, k_j, m_i, m_j \in \mathbf{N}$,

$$\sigma(u+2k_i; m_i, k_j, m_j) + 1_{m_j=0} \beta(u; k_i, m_i) = 1_{m_j>0} \sigma(u+2k_i; m_i, k_j, m_j) + 1_{m_j=0} \big(\sigma(u+2k_i; m_i, k_j, 0) + \beta(u; k_i, m_i) \big)$$
$$= 1_{m_j>0} \sigma(u+2k_i; m_i, k_j, m_j) + 1_{m_j=0} \beta(u; k_i+k_j, m_i),$$

due to (71). From (69) easily follows that for $m_j > 0$,

$$\sigma(u+2k_i; m_i, k_j, m_j) = \sigma(u; m_i, k_i+k_j, m_j).$$

Hence,

$$\sigma(u+2k_i; m_i, k_j, m_j) + 1_{m_j=0} \beta(u; k_i, m_i) = 1_{m_j>0} \sigma(u; m_i, k_i+k_j, m_j) + 1_{m_j=0} \beta(u; k_i+k_j, m_i)$$
$$= \sigma(u; m_i, k_i+k_j, m_j) + 1_{m_j=0} \big(\beta(u; k_i+k_j, m_i) - \sigma(u; m_i, k_i+k_j, 0) \big). \quad \square$$

Proposition 16. *There holds,* $\forall K, m_i, m_j \in \mathbf{N}$,

$$\sigma(u; m_i, K, m_j) = \beta(u; K, m_i+m_j) - 1_{m_j=0} \beta(u; 0, m_i). \tag{73}$$

Proof. (i) From (61) readily follows, $\forall m_j \in \mathbf{Z}_+$ and $\forall K, m_i \in \mathbf{N}$,

$$\beta(u-2q_i; K, m_j) = \beta(u; K-q_i, m_j).$$

Substituting this in (69) and using (67) yields the proposition for $m_j > 0$.

(ii) The proposition follows for $m_j = 0$ and $\forall K, m_i \in \mathbf{N}$ from (71) with $k_i = 0$. □

Proposition 17. *There holds,* $\forall k_i, k_j, m_i, m_j \in \mathbf{N}$,

$$\sigma(u+2k_i; m_i, k_j, m_j) + 1_{m_j=0} \beta(u; k_i, m_i) = \beta(u; k_i+k_j, m_i+m_j). \tag{74}$$

Proof. Combining (72) with (73) and writing $K = k_i + k_j$ gives

$$\sigma(u+2k_i; m_i, k_j, m_j) + 1_{m_j=0}\beta(u; k_i, m_i)$$
$$= \beta(u; k_i+k_j, m_i+m_j) + 1_{m_j=0}\big(\beta(u; k_i+k_j, m_i) - \beta(u; 0, m_i) - \sigma(u; m_i, k_i+k_j, 0)\big).$$

Applying (73) with $m_j = 0$ to the right-hand side yields

$$\sigma(u+2k_i; m_i, k_j, m_j) + 1_{m_j=0}\beta(u; k_i, m_i) = \beta(u; k_i+k_j, m_i+m_j). \qquad \square$$

A.4. Distributional identities

Proposition 18. *There holds,* $\forall i \in \mathbf{Z}_{[1,n]}$ *and* $\forall k, p \in \mathbf{N}$,

$$x^i \frac{\Delta^p}{p!}\frac{D_i^k}{k!}\delta = -1_{1\leqslant p} 2(k+1)\frac{\Delta^{p-1}}{(p-1)!}\frac{D_i^{k+1}}{(k+1)!}\delta - 1_{1\leqslant k}\frac{\Delta^p}{p!}\frac{D_i^{k-1}}{(k-1)!}\delta. \tag{75}$$

Proof. Calculate, $\forall i \in \mathbf{Z}_{[1,n]}$ and $\forall k, p \in \mathbf{N}$,

$$\big\langle x^i \Delta^p\big(D_i^k\delta\big), \varphi\big\rangle = (-1)^k\big\langle \delta, d_i^k \Delta^p\big(x^i\varphi\big)\big\rangle,$$

by successive applications of $\Delta(x^i\varphi) = 2d_i\varphi + x^i\Delta\varphi$ and use $\langle x^i\delta, \varphi\rangle = 0$. $\square$

Proposition 19. *There holds,* $\forall i \in \mathbf{Z}_{[1,n]}$ *and* $\forall l, p \in \mathbf{N}$,

$$\frac{(x^i)^l}{l!}\frac{\Delta^p}{p!}\delta = (-1)^l \sum_{r=0}^{l} 1_{r\leqslant 2p-l}\frac{e_{l-r}}{((l-r)/2)!}\frac{\Delta^{p-(l+r)/2}}{(p-(l+r)/2)!}\frac{(2D_i)^r}{r!}\delta. \tag{76}$$

Proof. By induction over l and by using (75), $\forall p \in \mathbf{N}$. $\square$

Proposition 20. *There holds,* $\forall i \in \mathbf{Z}_{[1,n]}$ *and* $\forall l, p \in \mathbf{N}$,

$$\frac{(x^i)^l}{l!}\frac{\Delta^p}{p!}D_i\delta = (-1)^l(l+1)\frac{1}{2}\sum_{r=-1}^{l} 1_{r\leqslant 2p-l}\frac{e_{l-r}}{((l-r)/2)!}\frac{\Delta^{p-(l+r)/2}}{(p-(l+r)/2)!}\frac{(2D_i)^{r+1}}{(r+1)!}\delta. \tag{77}$$

Proof. By Leibniz' rule for a smooth function and a distribution and by using (76). $\square$

Proposition 21. *There holds,* $\forall p \in \mathbf{N}$ *and* $\forall m \in \mathbf{Z}_{[1,n]}$,

$$\frac{(\mathbf{x}_m)^{\mathbf{k}_m}}{\mathbf{k}_m!}\frac{\Delta^p}{p!}\delta = (-1)^{K_m}\sum_{\mathbf{l}_m=\mathbf{0}_m}^{\mathbf{k}_m} 1_{L_m\leqslant 2p-K_m}\frac{\Delta^{p-(K_m+L_m)/2}}{(p-(K_m+L_m)/2)!}\frac{e_{\mathbf{k}_m-\mathbf{l}_m}}{((\mathbf{k}_m-\mathbf{l}_m)/2)!}\frac{(2\mathbf{D}_m)^{\mathbf{l}_m}}{\mathbf{l}_m!}\delta. \tag{78}$$

Proof. Using (76) and induction over m. $\square$

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