Delta distributions supported on quadratic $O(p,q)$-invariant surfaces

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An important subset of distributions encountered in physics is the set of multiplet delta distributions $\delta_{c_y}^{(k)}$, with support a quadratic $O(p,q)$-invariant surface $c_y \triangleq \{\mathbf{x} \in \mathbf{R}^n : P(\mathbf{x})=y\}$, $n=p+q$. The evaluation of these distributions for a general test function is not always sufficiently detailed in the classical literature, especially for those distributions that are defined as a regularization and/or when one needs their causal or anticausal version (for $p=1$ or $q=1$). This work intends to improve this situation by deriving explicit expressions for $\langle \delta_{c_y}^{(k)}, \varphi \rangle$, $\forall p,q \in \mathbf{Z}_+$, $\forall k \in \mathbf{N}$, $\forall y \in \mathbf{R}$, and $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$, in a form suitable for practical applications. In addition, we also apply to these distributions a new approach to regularization. Distributions that need to be regularized are (equivalently) defined as extensions of a partial distribution. This extension process reveals in a natural way that regularized multiplet delta distributions are in general, uncountably multivalued. In the work of Gel'fand and Shilov [*Generalized Functions* (Academic, New York, 1964), Vol. 1] four types of multiplet delta distributions with support the null space c_0 were introduced: $\delta_1^{(k)}(P)$, $\delta_2^{(k)}(P)$ and $\delta^{(k)}(P_+)$, $\delta^{(k)}(P_-)$. Our regularization method explains why this particular nonuniqueness was observed and further discloses the full extent of this nonuniqueness. © *2009 American Institute of Physics.*
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I. INTRODUCTION

Let $p,q \in \mathbf{Z}_+$ and $n=p+q$. We revisit the definition of multiplet delta distributions $\delta_{c_y}^{(k)}$, $\forall k \in \mathbf{N}$, having as support a quadratic $O(p,q)$-invariant surface $c_y \triangleq \{\mathbf{x} \in \mathbf{R}^n : P(\mathbf{x})=y\}$, $\forall y \in \mathbf{R}$, wherein P has the form (5). This includes c_0, the null space of the ultrahyperbolic metric (6), which reduces to the (full) light cone in the case of Minkowski space for $p=1$ and $q=3$.

Multiplet delta distributions are of fundamental importance in physics. For instance, the delta distribution δ_{c_y} allows to calculate the integral of a (compactly supported) function φ [actually of $\varphi/(2|\mathbf{x}|)$, see (13)] over a mass hyperboloid, with mass $m=y^{1/2}$, as $\langle \delta_{c_y}, \varphi \rangle$. Further, the distribution $\delta_{c_0}^{((n-2)/2-l)}$ is (up to a multiplicative constant) a fundamental solution of the "iterated" ultrahyperbolic wave equation $\square^l u=\delta$, for $l \in \mathbf{Z}_{[1,(n-2)/2]}$, in a higher-dimensional Minkowski space $M^{p,q}$, with odd number of time dimensions p and odd number of space dimensions q, and when $4 \leqslant p+q$ (Ref. 4, p. 281), (Huygens' principle). More generally, the multiplet delta distribution $\delta_{c_y}^{(k)}$ has as support de Sitter space $dS^{p-1,q}$ if $y>0$ or anti-de Sitter space $AdS^{p,q-1}$ if $y<0$.

The multiplet delta distributions concentrated at c_0, $\delta_{c_0}^{(k)}$, $\forall k \in \mathbf{N}$, were introduced and discussed in Ref. 4 (Chap. III, Sec. II A, p. 247) and there denoted $\delta^{(k)}(P)$. However, the expressions given there are not always of a sufficient explicitness to readily evaluate $\langle \delta_{c_y}^{(k)}, \varphi \rangle$ in practical applications. For instance, it is not immediately obvious from the material in Ref. 4 how to evaluate these functionals in case they are defined as regularizations and/or one needs temporal

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causal and anticausal ($p=1$, $q>1$) or spatial causal and anticausal ($p>1$, $q=1$) distributions. One of the purposes of this paper is to supply the missing details. The scope of the here presented treatment is also somewhat more general than Ref. 4 (Chapter III, Sec. II A) as we also consider multiplet delta distributions having as support single and double hyperboloid sheets (i.e., the cases $y\neq 0$). We derive explicit formulas to evaluate $\langle \delta^{(k)}_{c_y},\varphi\rangle$, $\forall p,q\in \mathbf{Z}_+$, $\forall k\in \mathbf{N}$, $\forall y\in \mathbf{R}$ and $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$, including the temporal (if $p=1$) and spatial (if $q=1$) causal (or retarded, future, forward) and anticausal (or advanced, past, backward) versions.

The distributions $\delta^{(k)}_{c_y}$ are most naturally defined as the pullback P^* of the one-dimensional delta distributions $\delta^{(k)}$ along the quadratic submersion $P:\mathbf{R}^n\backslash\{\mathbf{0}\}\to\mathbf{R}$ such that $\mathbf{x}\mapsto y=P(\mathbf{x})$. The fact that the domain of the submersion P must necessarily exclude the origin in $\mathbf{R}^n$ emerges as a technical difficulty if one wants to define $\delta^{(k)}_{c_y}$ for all test functions in $\mathcal{D}(\mathbf{R}^n)$. This is the reason that, for certain values of p,q and k, one finds that the defining integrals for $\delta^{(k)}_{c_0}$ require a regularization. In Ref. 4 (p. 249), an intriguing fact is observed in this respect. For even n and when $(n-2)/2\leq k$, two different distributions are obtained with support c_0, there labeled $\delta^{(k)}_1(P)$ and $\delta^{(k)}_2(P)$. Moreover in Ref. 4 (p. 278) two more distinct delta distributions are introduced, $\delta^{(k)}(P_+)$ and $\delta^{(k)}(P_-)$, also having support c_0. Any two of these four distributions differ from each other by a distribution having as support the origin $\{\mathbf{0}\in\mathbf{R}^n\}$ (Ref. 4, p. 279). The second aim of this paper is to clarify this apparent nonuniqueness of certain multiplet delta distributions. We will show that, for n even and when $(n-2)/2\leq k$, the distributions $\delta^{(k)}_{c_y}$ are, in general, uncountably multivalued. The key that fully unravels this nonuniqueness property is the more modern concept of the extension of a partial distribution, introduced by the author in Ref. 2 and here recalled in Sec. II C. The four distributions $\delta^{(k)}_1(P)$, $\delta^{(k)}_2(P)$, $\delta^{(k)}(P_+)$ and $\delta^{(k)}(P_-)$ are then found to be just four particular extensions of a common unique partial distribution $\delta^{(k)}_{c_0}$.

We also find that if $p=1$ ($q=1$) and $(n-2)/2\leq k$, the temporal (spatial) causal and anticausal distributions are uncountably multivalued extensions, irrespective the parity of n. This behavior of the causal and anticausal distributions differs from that of their time-symmetric (space-symmetric) counterparts, which are unique analytic continuations for n odd and uncountably multivalued extensions for n even. This appears to be a new result.

The outline of the paper is as follows.

(a) We first recall in Sec. II the definition of the pullback of a distribution along a scalar submersion and apply this to the quadratic function P.
(b) We consider in Sec. III the distribution δ_{c_y} for general $y\in\mathbf{R}$.
(c) The distribution δ_{c_y} is fundamental to define the pullback of any distribution along P. We use δ_{c_y} to calculate the particular distributions $\delta^{(k)}_{c_y}$, $\forall k\in\mathbf{N}$, in Sec. IV.
(d) We discuss in Sec. V, as an interesting and practical example, the temporal causal ($+$) and anticausal ($-$) distributions $(\delta^{(1)}_{c_0^{t\pm}})_e$ for $p=1$ and $q=3$.

Finally a note on terminology. We say that a distribution $f\in\mathcal{D}'(X)$, having support $U\subseteq X$, is *defined* for test functions in $\mathcal{D}(X)$ [with $\mathcal{D}'(X)$ the continuous dual space of $\mathcal{D}(X)$], is *based* on X (called the base space of f) and is *concentrated* at U. Further, we use the notation and definitions as stated in Ref. 2. For convenience we recall here the 1-symbol, $1_p\triangleq 1$ if p is true, else $1_p\triangleq 0$, the parity symbols, $e_m\triangleq 1_{m\in\mathbf{Z}_e}$ and $o_m\triangleq 1_{m\in\mathbf{Z}_o}$ ($\mathbf{Z}_{e,o}$ even or odd integers), and the step functions $1_\pm$ (characteristic functions of $\mathbf{R}_\pm$).

II. PULLBACK TO R^n OF A DISTRIBUTION BASED ON R

A. Pullback along a scalar submersion

Denote by $\langle,\rangle_{\mathrm{dR}}$ the de Rham pairing, of a chain and a form, and by $\langle,\rangle$ the Schwartz pairing, of a generalized function and a test function.

The de Rham pairing $\langle , \rangle_{\mathrm{dR}}$ of an m-chain $c_y \subseteq X \subseteq \mathbf{R}^n$ and an m-form $\varphi\omega_T$ is defined as the integral of $\varphi\omega_T$ over c_y, (Ref. 1, p. 217). Herein is ω_T the Leray form of the chain c_y, such that $\omega_X = dT \wedge \omega_T$, with ω_X the volume form on X.

For any regular distribution $f \in \mathcal{D}'(X)$, the Schwartz pairing is defined as

$$\langle f, \varphi \rangle \triangleq \langle X, f\varphi\omega_X \rangle_{\mathrm{dR}}. \tag{1}$$

Definition 1: *Let $n \in \mathbf{N}: 2 \leq n$, $X \subseteq \mathbf{R}^n$, $Y = \mathbf{R}$ and $\delta_y \in \mathcal{D}'(Y)$ with $\langle \delta_y, \psi \rangle \triangleq \psi(y)$, $\forall \psi \in \mathcal{D}(Y)$. Let $f \in \mathcal{D}'(Y)$ and $T: X \to Y$ such that $\mathbf{x} \mapsto y = T(\mathbf{x})$ be a C^∞ function with $(dT)(\mathbf{x}) \neq 0$, $\forall \mathbf{x} \in c_y \triangleq \{\mathbf{x} \in X : T(\mathbf{x}) = y\}$, and $\forall y \in \operatorname{supp} f$. The pullback $T^* f$ of f along T is defined $\forall \varphi \in \mathcal{D}(X)$ as*

$$\langle T^* f, \varphi \rangle \triangleq \langle f, \Sigma_T \varphi \rangle, \tag{2}$$

with

$$(\Sigma_T \varphi)(y) \triangleq \langle T^* \delta_y, \varphi \rangle. \tag{3}$$

The distribution $\delta_{c_y} \triangleq T^* \delta_y \in \mathcal{D}'(X)$ represents a delta distribution having as support the level set surface c_y of T, with level parameter y, and is defined as (Ref. 3, p. 85, and Ref. 1, p. 438)

$$\langle \delta_{c_y}, \varphi \rangle \triangleq \langle c_y, \varphi\omega_T \rangle_{\mathrm{dR}}. \tag{4}$$

It is clear from (2) and (3) that the pullback δ_{c_y} of the one-dimensional distribution δ_y, defined by (4), is fundamental to calculate the pullback $T^* f$ of any distribution f along T.

The condition on dT in Definition 1 is necessary and sufficient for the Leray form ω_T to exist on c_y. Moreover, although $\omega_X = dT \wedge \omega_T$ does not specify ω_T uniquely in a neighborhood of c_y, ω_T is unique on c_y (Ref. 4, pp. 220 and 221). The Leray form ω_T is the natural surface form on c_y.

We can not speak of the delta distribution with support c_y since the pullback $T^* \delta_y$, as defined above, depends on the equation used to represent the surface c_y, through the Leray form ω_T (Ref. 4, p. 222, and Ref. 1, p. 439). For this reason, it might sometimes be appropriate to use a more explicit notation for δ_{c_y}, that displays the form of the equation used to describe c_y, such as $\delta_{(T(\mathbf{x})-y)}$.

It is shown in, e.g., Ref. 3, p. 82, Theorem 7.2.1, that, under the conditions given in Definition 1, $\Sigma_T \varphi \in \mathcal{D}(Y)$, $T^* f \in \mathcal{D}'(X)$, and T^* is a sequentially continuous linear operator.

B. The quadratic submersion *P*

Let $p, q \in \mathbf{Z}_+$ with $n = p + q$, $(x^\mu) \triangleq \mathbf{x} \triangleq (\mathbf{x}_t \in \mathbf{R}^p, \mathbf{x}_s \in \mathbf{R}^q) \in \mathbf{R}^n$ and $|\,|$ the Euclidean norm on $\mathbf{R}^n$. Define a function $P: X = \mathbf{R}^n \backslash \{\mathbf{0}\} \to Y = \mathbf{R}$ such that $\mathbf{x} \mapsto y = P(\mathbf{x})$ with (implicit summation over 1 to n)

$$P(\mathbf{x}) \triangleq \eta_{\mu\nu} x^\mu x^\nu = |\mathbf{x}_t|^2 - |\mathbf{x}_s|^2, \tag{5}$$

wherein

$$[\eta_{\mu\nu}] \triangleq \operatorname{diag}[\underbrace{+1, \ldots, +1}_{p \text{ times}}, \underbrace{-1, \ldots, -1}_{q \text{ times}}]. \tag{6}$$

We have $dP = 2\eta_{\mu\nu} x^\mu dx^\nu \neq 0$, $\forall \mathbf{x} \in X$, hence dP is everywhere surjective and P is a submersion. Define the level sets $c_y \triangleq \{\mathbf{x} \in \mathbf{R}^n : P(\mathbf{x}) = y\}$, $\forall y \in \mathbf{R}$. In particular, c_0 is the null space (or null cone if $p=1$ or $q=1$) with respect to the origin. Notice that our definition of c_0 includes the origin, so $c_0 \subsetneq X$ and c_0 is not a manifold, (Ref. 1, p. 113).

1. Volume form

The natural volume form on X most easily follows from Hodge's star operator $*$ as $\omega_X = *1$ or

$$\omega_X = \epsilon_{1\cdots n} dx^1 \wedge \cdots \wedge dx^n, \tag{7}$$

with ϵ the n-covariant Levi–Civita pseudotensor field, involving the metric on X. The volume form ω_X is an antisymmetric tensor density field of order n and weight 1. A tensor density field of weight 1 is also called a pseudotensor field. An antisymmetric tensor density field of weight 1 is usually called an odd (or twisted or pseudo-) form.

The classical volume element dV is the infinitesimal Euclidean measure corresponding to the volume form ω_X on X according to, $\forall \varphi \in \mathcal{D}(X)$,

$$\langle X, \varphi\omega_X\rangle_{\mathrm{dR}} \triangleq \int_X \varphi(\mathbf{x}) dV(\mathbf{x}). \tag{8}$$

2. Leray form

Introduce the interior product $\vee : \Gamma(T^*X) \times \Gamma(T^*X) \to \mathbf{R}$ such that $(\alpha,\beta) \mapsto \alpha \vee \beta = \eta^{\mu\nu}\alpha_\mu\beta_\nu$, with $[\eta^{\mu\nu}] \triangleq [\eta_{\mu\nu}]^{-1}$. The Leray form ω_P follows from the following property of Hodge's star operator, $\forall \alpha, \beta \in \Gamma(T^*X)$:

$$\alpha \wedge (*\beta) = (*\alpha) \wedge \beta = (\alpha \vee \beta)\omega_X, \tag{9}$$

with $\wedge$ the exterior product. Let $\alpha = dP$, $*\beta = \omega_P$, and impose that $dP \vee (*^{-1}\omega_P) = 1$. Since $dP \vee dP = 4P$, we get $\omega_P = *(dP/4P)$ or

$$\omega_P = \frac{1}{1!(n-1)!}\epsilon_{\mu_1\mu_2\cdots\mu_n}\frac{x^{\mu_1}}{2P}(dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}). \tag{10}$$

The Leray form ω_P is also an odd form.

Let $\cdot : \Gamma(T^*X) \times \Gamma(T^*X) \to \mathbf{R}$ such that $(\alpha,\beta) \mapsto \alpha\cdot\beta = \delta^{\kappa\lambda}\alpha_\kappa\beta_\lambda$ stand for the Euclidean inner product on $\Gamma(T^*X)$, so $|\alpha| = \sqrt{\alpha\cdot\alpha}$. The outward unit normal 1-form $n \in \Gamma(T^*X)$ such that $|n| = 1$ is defined as

$$n \triangleq \frac{dP}{|dP|} = \frac{2\eta_{\mu\nu}x^\mu dx^\nu}{2|\mathbf{x}|}. \tag{11}$$

The classical surface element dS is the infinitesimal Euclidean measure corresponding to the Leray form ω_P on c_y according to (Ref. 1, p. 439), $\forall \varphi \in \mathcal{D}(X)$,

$$\left\langle c_y, \varphi \frac{1}{1!(n-1)!}\epsilon_{\mu_1\mu_2\cdots\mu_n}(dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n})\right\rangle_{\mathrm{dR}} \triangleq \int_{c_y} n_{\mu_1}(\mathbf{x})\varphi(\mathbf{x}) dS(\mathbf{x}). \tag{12}$$

Substituting in (12) the expressions for the normal components n_{μ_1}, read off from (11), and using expression (10) for ω_P, we obtain for (4) the integral form, $\forall y \in \mathbf{R}$,

$$(\Sigma_P\varphi)(y) = \frac{1}{2}\int_{c_y}\frac{\varphi(\mathbf{x})}{|\mathbf{x}|} dS(\mathbf{x}). \tag{13}$$

3. Pullback along *P*

The following two cases can occur.

(i) A $f \in \mathcal{D}'(\mathbf{R})$ for which $0 \in \operatorname{supp} f$ has a pullback $P^*f \in \mathcal{D}'(\mathbf{R}^n \backslash \{\mathbf{0}\})$.

(ii) A $f \in \mathcal{D}'(\mathbf{R})$ for which $0 \notin \operatorname{supp} f$ has a pullback $P^*f \in \mathcal{D}'(\mathbf{R}^n)$, since it is sufficient that $dP \neq 0$ on all level sets of P corresponding to the support of f.

C. Extension of a partial distribution

In distribution theory, one is often forced to impose restrictions on the test functions to which certain functionals can be applied. An example of this is case (i) above. In this section we describe a general procedure which can remove such restrictions. We will use this procedure in the subsequent sections to construct multiplet delta distributions, which are defined for all test functions in $\mathcal{D}(\mathbf{R}^n)$. This means that our results will be applicable to test functions having a support that may intersect with the origin $\{\mathbf{0}\in\mathbf{R}^n\}$. If one only wants to apply a multiplet delta distribution to test functions, vanishing in a neighborhood of the origin, then extensions are obviously unnecessary.

Definition 2: *Let $\mathcal{D}_r\subset\mathcal{D}$ and denote by $\mathcal{D}'_r\supset\mathcal{D}'$ the continuous dual space of $\mathcal{D}_r$. Elements of $\mathcal{D}'_r$ that are only defined on $\mathcal{D}_r$ are called* partial distributions defined on $\mathcal{D}_r$.

Example: P^*f in case (i) above is a partial distribution defined on $\mathcal{D}(\mathbf{R}^n\backslash\{\mathbf{0}\})\subset\mathcal{D}(\mathbf{R}^n)$.

A distribution, by definition, is a linear (sequentially) continuous functional defined on the whole of $\mathcal{D}$.[8,9] We want to extend a partial distribution, defined on $\mathcal{D}_r$, so that it becomes a distribution. A natural way to do this is as follows.

Since the space of test functions $\mathcal{D}$ is locally convex (Ref. 7, p. 9), the Hahn–Banach theorem guarantees that a continuous linear functional f, defined on a $\mathcal{D}_r\subset\mathcal{D}$ (i.e., a partial distribution), can be extended to a continuous linear functional f_e defined on the whole of $\mathcal{D}$ (i.e., a distribution), and that both coincide on $\mathcal{D}_r$. The subset of $\mathcal{D}'_r$ that maps $\mathcal{D}_r$ to zero is called the annihilator of $\mathcal{D}_r$, denoted $\mathcal{D}'^{\perp}_r$. Clearly, any two extensions $f_{e,1}$ and $f_{e,2}$ of f can differ by an element in $\mathcal{D}'^{\perp}_r$. If $\mathcal{D}_r$ is dense in $\mathcal{D}$, then the extension is unique.

Notice that this method, when applied to $\mathcal{D}_r=\mathcal{D}\subset\mathcal{S}$, shows that any tempered distribution in $\mathcal{S}'$ (Ref. 8, Vol. II, pp. 89–104) can be regarded as the extension of a distribution in $\mathcal{D}'\supset\mathcal{S}'$. Since $\mathcal{D}$ is a dense subspace of $\mathcal{S}$ (Ref. 9, p. 101), any tempered distribution, regarded as an extension, is unique.

The Hahn–Banach theorem does not say how an extension can be constructed. A natural extension process, however, is the one used in the distribution literature for the regularization of divergent integrals (Ref. 4, pp. 10 and 45). This procedure basically consists in the construction of a projection operator $T:\mathcal{D}\to\mathcal{D}_r$ and to replace an integral, convergent $\forall\psi\in\mathcal{D}_r$ but divergent for test functions $\varphi\in\mathcal{D}\backslash\mathcal{D}_r$ (e.g., in one dimension),

$$\int_{-\infty}^{+\infty} f(x)\varphi(x)dx, \tag{14}$$

by the integral

$$\int_{-\infty}^{+\infty} f(x)(T\varphi)(x)dx, \tag{15}$$

which now by construction is convergent $\forall\varphi\in\mathcal{D}$. The integral (15) is then said to be a regularization of the integral (14). Equivalently, we could say that (15) defines a distribution which is an extension of the partial distribution defined by (14). It is important to notice, however, that this method is by no means unique, since infinitely many projection operators T can be constructed, mapping $\mathcal{D}$ to $\mathcal{D}_r$.

For distributions that depend on a complex parameter z, such as $x^z_{\pm}$ (Ref. 4, p. 48, and Ref. 5, p. 84), this procedure can be applied to its Laurent series about a pole $z=-l\in\mathbf{Z}_-$. Any (associated homogeneous) extension $x^z_{\pm,e}$ is then obtained in the form

$$x^{-l}_{\pm,e}=x^{-l}_{\pm,0}+c\,\delta^{(l-1)}, \tag{16}$$

with $x^{-l}_{\pm,0}$ the analytic finite part (i.e., the zeroth term of the Laurent series) at the pole $z=-l$ and $c\in\mathbf{C}$ is arbitrary. The distribution $x^{-l}_{\pm,0}$ is given by [Ref. 2, Eq. (108)]

$$\langle x_{\pm,0}^{-l}, \varphi \rangle = \int_{-\infty}^{+\infty} (|x|^{-l} 1_{\pm}(x))(T_{l-1,0}\varphi)(x)dx, \tag{17}$$

wherein the projection operator is defined as

$$(T_{p,q}\varphi)(x) \triangleq \varphi(x) - \sum_{l=0}^{p+q} \varphi^{(l)}(0)\frac{x^l}{l!}(1_{l<p} + 1_{p\leqslant l}1_{+}(1-x^2)). \tag{18}$$

Any so constructed extension $x_{\pm,e}^{-l}$ is homogeneous of degree $-l$ and has first order of association. More details of this procedure can be found in Ref. 2.

An advantage of the concept "extension of a partial distribution" is that it clearly reveals why many singular distributions are generally nonunique. The classical definition of such a distribution is, almost always, given as a particular regularization of an integral, and this approach has largely obscured the intrinsic nonuniqueness of this process.

III. THE DELTA DISTRIBUTION δ_{c_y}

A. Summary of results

The distribution

$$\delta_{c_y} \triangleq P^* \delta_y \tag{19}$$

represents the delta distribution with $\mathrm{supp}(\delta_{c_y}) = c_y$, relative to the equation $P(\mathbf{x}) = y$ which defines c_y, and is given by (4).

In this section, we will obtain the following results.

(i) For $y \neq 0$, δ_{c_y} always exists as a distribution in $\mathcal{D}'(\mathbf{R}^n)$. For example, for $p=1$ and $y=m^2 > 0$, δ_{c_y} is the delta distribution concentrated at the (double sheeted) timelike mass m hyperboloid $P(\mathbf{x})=m^2$.

(ii) For $y=0$, we will find that δ_{c_0} is a *partial distribution* defined on $\mathcal{D}(\mathbf{R}^2 \backslash \{\mathbf{0}\})$ if $p=1=q$ and else is a distribution in $\mathcal{D}'(\mathbf{R}^n)$. In the former case, our extension process generates a (in general, nonunique) distribution $(\delta_{c_0})_e \in \mathcal{D}'(\mathbf{R}^2)$, called delta distribution concentrated at the (full) null cone $P(\mathbf{x})=0$, and which coincides with the partial delta distribution δ_{c_0} in the sense that $\langle \delta_{c_0}, \psi \rangle = \langle (\delta_{c_0})_e, \psi \rangle$, $\forall \psi \in \mathcal{D}(\mathbf{R}^2 \backslash \{\mathbf{0}\})$. For example, for $p=1$ and $q>1$, δ_{c_0} is the (unique) delta distribution concentrated at the null cone $P(\mathbf{x})=0$.

Deriving an explicit expression for the functional value $\langle \delta_{c_0}, \varphi \rangle$ requires distinguishing the following four cases.

B. Case p>1 and q>1

For distributions concentrated at an $O(p,q)$-invariant subspace of $\mathbf{R}^{p,q}$, one can typically derive two forms for $\Sigma_P \varphi$ in (2), hereafter labeled (a) and (b). Form (a) is obtained by integrating over the radius of the temporal section, ρ_t, and form (b) by integrating over the radius of the spatial section, ρ_s. This leads to two equivalent expressions as for $y \neq 0$ in the following.

Proposition 3: *The delta distribution δ_{c_y}, concentrated at $P(\mathbf{x})=y$, is for $p>1$, $q>1$ and $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$ given by the following expressions.*

(i) *For* $y \neq 0$,

$$\langle \delta_{c_y}, \varphi \rangle = \frac{1}{2}\int_0^{+\infty} 1_+(\rho_t^2 - y)\rho_t^{p-1}(\sqrt{\rho_t^2 - y})^{q-2}\psi^{S,S}(\rho_t, \sqrt{\rho_t^2 - y})d\rho_t \tag{20}$$

$$= \frac{1}{2}\int_0^{+\infty} 1_+(\rho_s^2 + y)(\sqrt{\rho_s^2 + y})^{p-2}\rho_s^{q-1}\psi^{S,S}(\sqrt{\rho_s^2 + y}, \rho_s)d\rho_s. \tag{21}$$

(ii) *For* $y=0$,

$$\langle \delta_{c_0}, \varphi \rangle = \frac{1}{2}\int_0^{+\infty} \rho^{n-3}\psi^{S,S}(\rho,\rho)d\rho. \tag{22}$$

Herein is $\psi^{S,S}$: $\mathbf{R}^2 \to \mathbf{R}$ *such that* $(t,s) \mapsto \psi^{S,S}(t,s)$, *with*

$$\psi^{S,S}(t,s) \triangleq \langle S_t^{p-1} \times S_s^{q-1}, \varphi(t\omega_t, s\omega_s)(\omega_{S_t^{p-1}} \wedge \omega_{S_s^{q-1}})\rangle_{dR}. \tag{23}$$

Proof: Introduce bispherical coordinates $(\rho_t \in \mathbf{R}_+, \omega_t \in S_t^{p-1}; \rho_s \in \mathbf{R}_+, \boldsymbol{\omega}_s \in S_s^{q-1})$ in X. Let $\omega_{S_t^{p-1}}$ and $\omega_{S_s^{q-1}}$ be the volume forms on the unit spheres S_t^{p-1} and S_s^{q-1} of the time and space sections, respectively. Now, $P=\rho_t^2-\rho_s^2$. The volume form ω_X becomes (suppressing the Levi–Civita pseudotensor field ϵ which numerically amounts to a factor 1)

$$\omega_X = (\rho_t^{p-1}d\rho_t \wedge \omega_{S_t^{p-1}}) \wedge (\rho_s^{q-1}d\rho_s \wedge \omega_{S_s^{q-1}}).$$

The mapping (23) from $\varphi \to \psi^{S,S}$ is sequentially continuous and $\psi^{S,S} \in \mathcal{D}(\mathbf{R}^2)$, $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$. Moreover, $\psi^{S,S}$ is even in t and in s (Ref. 4, p. 250).

Form (a). In terms of the coordinates $P=y \in \mathbf{R}$, $\rho_t \in \mathbf{R}_+$, and ω_t, ω_s, we get for the volume form

$$\omega_X = \rho_t^{p-1}d\rho_t \wedge \omega_{S_t^{p-1}} \wedge \tfrac{1}{2}(\sqrt{\rho_t^2 - y})^{q-2}dy \wedge \omega_{S_s^{q-1}}, \tag{24}$$

with the argument of the square root restricted to positive values. The Leray form is read off as

$$\omega_P = \tfrac{1}{2}\rho_t^{p-1}(\sqrt{\rho_t^2 - y})^{q-2}d\rho_t \wedge \omega_{S_t^{p-1}} \wedge \omega_{S_s^{q-1}}. \tag{25}$$

Notwithstanding ω_X and ω_P are odd forms, any permutation of the basis forms never generates a change in sign because this transformation involves the modulus of the Jacobian determinant of the permutation. For the same reason, any square root in (24) and (25) must be taken as positive. Substituting (25) in (4) and using (3) gives

$$(\Sigma_P\varphi)(y) = \frac{1}{2}\int_0^{+\infty} 1_+(\rho_t^2 - y)\rho_t^{p-1}(\sqrt{\rho_t^2 - y})^{q-2}\psi^{S,S}(\rho_t, \sqrt{\rho_t^2 - y})d\rho_t. \tag{26}$$

Form (b). In terms of the coordinates $P=y \in \mathbf{R}$, $\rho_s \in \mathbf{R}_+$, and ω_t, ω_s, we get for the volume form

$$\omega_X = \tfrac{1}{2}(\sqrt{\rho_s^2 + y})^{p-2}dy \wedge \rho_s^{q-1}d\rho_s \wedge \omega_{S_t^{p-1}} \wedge \omega_{S_s^{q-1}}, \tag{27}$$

with the argument of the square root again restricted to positive values, and for the Leray form

$$\omega_P = \tfrac{1}{2}(\sqrt{\rho_s^2 + y})^{p-2}\rho_s^{q-1}d\rho_s \wedge \omega_{S_t^{p-1}} \wedge \omega_{S_s^{q-1}}. \tag{28}$$

Substituting (28) in (4) gives

$$(\Sigma_P\varphi)(y)=\frac{1}{2}\int_0^{+\infty}1_+(\rho_s^2+y)(\sqrt{\rho_s^2+y})^{p-2}\rho_s^{q-1}\psi^{S,S}(\sqrt{\rho_s^2+y},\rho_s)d\rho_s. \tag{29}$$

(i) For $y\neq 0$, either (26) or (29) defines $\Sigma_P\varphi$, $\forall\varphi\in\mathcal{D}(\mathbf{R}^n)$. Both (26) and (29) as well as the mapping (23) are sequentially continuous linear operators, acting on φ. Then, δ_{c_y}, defined by (2) and either (26) or (29) together with (23), is a distribution based on $\mathbf{R}^n$.

(ii) If $y=0$ we get, from either (26) or (29),

$$(\Sigma_P\varphi)(0)=\frac{1}{2}\int_0^{+\infty}\rho^{n-3}\psi^{S,S}(\rho,\rho)d\rho. \tag{30}$$

Reordering the limit $y\to 0$ and the integration in (26) and (29) is permitted since the integrands are jointly continuous in y and the integration variable. Hence, $(\Sigma_P\varphi)(0)=\langle x_+^{n-3},\varphi_1\rangle$ with $\varphi_1(x)\triangleq\frac{1}{2}\psi^{S,S}(x,x)$ and $\varphi_1\in\mathcal{D}(\mathbf{R})$. Since $n\geqslant 4$, $\langle x_+^{n-3},\varphi_1\rangle$ exists. Then δ_{c_0}, defined by (2) together with (30) and (23), is a distribution based on $\mathbf{R}^n$ due to the sequential continuity of the mapping (23) and the sequential continuity of the one-dimensional distribution x_+^{n-3}. □

Expressions (26) and (29) are easily shown to be equivalent. However, we will see in the next section that continuing with both forms (a) and (b) is not superfluous, as they can give rise to different regularizations for certain multiplet delta distributions (a fact also observed in Ref. 4, p. 251). This is also made explicit in Sec. V.

From (22) readily follows (with $1_{(\omega)}$ the 1-distribution on S^{m-1})

$$\delta_{c_0}=\tfrac{1}{2}(\delta_{(t-s)}\otimes s_+^{n-3}\otimes(1_{(\omega_t)}\otimes 1_{(\omega_s)})), \tag{31}$$

$$=\tfrac{1}{2}(t_+^{n-3}\otimes\delta_{(s-t)}\otimes(1_{(\omega_t)}\otimes 1_{(\omega_s)})). \tag{32}$$

C. Case p=1 and q>1

Proposition 4: *The delta distribution δ_{c_y}, concentrated at $P(\mathbf{x})=y$, is for $p=1$, $q>1$ and $\forall\varphi\in\mathcal{D}(\mathbf{R}^n)$ given by the following expressions.*

(i) For $y\neq 0$,

$$\langle\delta_{c_y},\varphi\rangle=\frac{1}{2}\int_{-\infty}^{+\infty}1_+(t^2-y)(\sqrt{t^2-y})^{q-2}\psi^{\cdot,S}(t,\sqrt{t^2-y})dt, \tag{33}$$

$$=\frac{1}{2}\int_0^{+\infty}1_+(\rho_s^2+y)\frac{\rho_s^{q-1}}{\sqrt{\rho_s^2+y}}\Psi_s(\sqrt{\rho_s^2+y},\rho_s)d\rho_s, \tag{34}$$

with Ψ_s defined as

$$\Psi_s(t,\rho_s)\triangleq\psi^{\cdot,S}(+t,\rho_s)+\psi^{\cdot,S}(-t,\rho_s). \tag{35}$$

(ii) For $y=0$,

$$\langle\delta_{c_0},\varphi\rangle=\frac{1}{2}\int_0^{+\infty}\rho^{q-2}\psi^{\cdot,S}(+\rho,\rho)d\rho+\frac{1}{2}\int_0^{+\infty}\rho^{q-2}\psi^{\cdot,S}(-\rho,\rho)d\rho. \tag{36}$$

Herein is $\psi^{\cdot,S}$: $\mathbf{R}^2\to\mathbf{R}$ such that $(t,s)\mapsto\psi^{\cdot,S}(t,s)$, given by

$$\psi^{\cdot,S}(t,s)\triangleq\langle S_s^{q-1},\varphi(t,s\omega_s)\omega_{S_s^{q-1}}\rangle_{dR}. \tag{37}$$

Proof: Introduce spherical coordinates $(\rho_s \in \mathbf{R}_+, \omega_s \in S_s^{q-1})$ for the spatial section of X. Now, $P=(x^1)^2-\rho_s^2$ and the volume form ω_X is $\omega_X = dx^1 \wedge \rho_s^{q-1} d\rho_s \wedge \omega_{S_s^{q-1}}$.

Form (a). Introduce the change in variables $(x^1,\rho_s) \to (t=x^1, y=(x^1)^2-\rho_s^2)$ having Jacobian determinant $-2\sqrt{t^2-y}$. In terms of the coordinates $t \in \mathbf{R}$, $P=y \in \mathbf{R}$, and ω_s, the volume form becomes

$$\omega_X = \tfrac{1}{2}(\sqrt{t^2-y})^{q-2} dy \wedge dt \wedge \omega_{S_s^{q-1}}, \tag{38}$$

with the argument of the square root restricted to positive values, and the Leray form is

$$\omega_P = \tfrac{1}{2}(\sqrt{t^2-y})^{q-2} dt \wedge \omega_{S_s^{q-1}}. \tag{39}$$

Substituting (39) in (4) gives

$$(\Sigma_P \varphi)(y) = \frac{1}{2}\int_{-\infty}^{+\infty} 1_+(t^2-y)(\sqrt{t^2-y})^{q-2} \psi^{\bar{t},S}(t,\sqrt{t^2-y})dt. \tag{40}$$

The mapping (37) from $\varphi \to \psi^{\bar{t},S}$ is sequentially continuous and $\psi^{\bar{t},S} \in \mathcal{D}(\mathbf{R}^2)$, $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$. Moreover, $\psi^{\bar{t},S}$ is even in s (Ref. 4, p. 250).

Form (b). Introduce the change in variables $(x^1,\rho_s) \to (y=(x^1)^2-\rho_s^2, \rho_s)$ with Jacobian determinant $2(\rho_s^2+y)^{1/2}$. In terms of the coordinates $P=y \in \mathbf{R}$, $\rho_s \in \mathbf{R}_+$, and ω_s, the volume form is

$$\omega_X = \frac{1}{2}\frac{1}{\sqrt{\rho_s^2+y}} dy \wedge \rho_s^{q-1} d\rho_s \wedge \omega_{S_s^{q-1}}, \tag{41}$$

with the argument of the square root again restricted to positive values, and the Leray form is

$$\omega_P = \frac{1}{2}\frac{\rho_s^{q-1}}{\sqrt{\rho_s^2+y}} d\rho_s \wedge \omega_{S_s^{q-1}}. \tag{42}$$

Substituting (42) in (4) gives (since $x^1 \in \mathbf{R}$)

$$(\Sigma_P \varphi)(y) = \frac{1}{2}\int_0^{+\infty} 1_+(\rho_s^2+y)\frac{\rho_s^{q-1}}{\sqrt{\rho_s^2+y}} \Psi_s(\sqrt{\rho_s^2+y},\rho_s)d\rho_s. \tag{43}$$

(i) For $y \neq 0$, either (40) or (43) defines $\Sigma_P \varphi$, $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$. Then (2), together with either (40) or (43), defines δ_{c_y} as a distribution based on $\mathbf{R}^n$.

(ii) If $y=0$ we get, from either (40) or (43),

$$(\Sigma_P \varphi)(0) = \frac{1}{2}\int_0^{+\infty} \rho^{q-2}\psi^{\bar{t},S}(+\rho,\rho)d\rho + \frac{1}{2}\int_0^{+\infty} \rho^{q-2}\psi^{\bar{t},S}(-\rho,\rho)d\rho. \tag{44}$$

Hence, $(\Sigma_P\varphi)(0) = \langle x_+^{q-2}, \varphi_1\rangle$ with $\varphi_1(x) \triangleq \frac{1}{2}\Psi_s(x,x)$ and $\varphi_1 \in \mathcal{D}(\mathbf{R})$, $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$. Since $q \geqslant 2$, $\langle x_+^{q-2}, \varphi_1\rangle$ exists. Then δ_{c_0}, defined by (2) together with (44), is a distribution based on $\mathbf{R}^n$. □

Equation (36) corrects Ref. 4, p. 252, Eq. (14′) with $k=0$, where the second integral has the wrong sign due to not applying the modulus to the Jacobian determinant when transforming from $(x^1,\rho_s) \to (y,\rho_s)$.

The temporal future and past hyperboloid sheets (or if $y=0$, half null cones) with respect to the origin are the respective sets, $\forall y \in \mathbf{R}$,

$$c_y^{t+} \triangleq \{\mathbf{x} \in \mathbf{R}^n : (x^1)^2 - \rho_s^2 = y, 0 \leqslant x^1\}, \tag{45}$$

$$c_y^{t-} \triangleq \{\mathbf{x} \in \mathbf{R}^n : (x^1)^2 - \rho_s^2 = y, x^1 \leqslant 0\}. \tag{46}$$

If $y<0$, (45) represents the $0 \leqslant x^1$ part of a hyperboloid of one sheet, while (46) represents its $x^1 \leqslant 0$ part. If $y>0$, (45) represents the $0 \leqslant x^1$ sheet of a hyperboloid of two sheets, while (46) represents its $x^1 \leqslant 0$ sheet.

Corollary 5:

(i) *Let $y \neq 0$. The temporal causal delta distribution $\delta_{c_y^{t+}}$, concentrated at a future timelike mass hyperboloid sheet c_y^{t+}, and temporal anticausal delta distribution $\delta_{c_y^{t-}}$, concentrated at a past timelike mass hyperboloid sheet c_y^{t-}, are given by*

$$\langle \delta_{c_y^{t+}}, \varphi \rangle = \frac{1}{2} \int_0^{+\infty} 1_+(t^2 - y)(\sqrt{t^2 - y})^{q-2} \bar{\psi}^{,S}(t, \sqrt{t^2 - y}) dt, \tag{47}$$

$$= \frac{1}{2} \int_0^{+\infty} 1_+(\rho_s^2 + y) \frac{\rho_s^{q-1}}{\sqrt{\rho_s^2 + y}} \bar{\psi}^{,S}(+\sqrt{\rho_s^2 + y}, \rho_s) d\rho_s, \tag{48}$$

and

$$\langle \delta_{c_y^{t-}}, \varphi \rangle = \frac{1}{2} \int_{-\infty}^{0} 1_+(t^2 - y)(\sqrt{t^2 - y})^{q-2} \bar{\psi}^{,S}(t, \sqrt{t^2 - y}) dt, \tag{49}$$

$$= \frac{1}{2} \int_0^{+\infty} 1_+(\rho_s^2 + y) \frac{\rho_s^{q-1}}{\sqrt{\rho_s^2 + y}} \bar{\psi}^{,S}(-\sqrt{\rho_s^2 + y}, \rho_s) d\rho_s. \tag{50}$$

(ii) *Let $y=0$. The temporal causal delta distribution $\delta_{c_0^{t+}}$, concentrated at the future half null cone c_0^{t+}, and temporal anticausal delta distribution $\delta_{c_0^{t-}}$, concentrated at the past half null cone c_0^{t-}, are given by*

$$\langle \delta_{c_0^{t+}}, \varphi \rangle = \frac{1}{2} \int_0^{+\infty} \rho^{q-2} \bar{\psi}^{,S}(+\rho, \rho) d\rho, \tag{51}$$

$$\langle \delta_{c_0^{t-}}, \varphi \rangle = \frac{1}{2} \int_0^{+\infty} \rho^{q-2} \bar{\psi}^{,S}(-\rho, \rho) d\rho. \tag{52}$$

Proof: The delta distribution $\delta_{c_y^{t+}}$ ($\delta_{c_y^{t-}}$) is obtained by restricting to $0 \leqslant x^1$ ($x^1 \leqslant 0$) and by integrating in (33) from $0 \to +\infty$ ($-\infty \to 0$). Equivalently, we only use the first (second) integral in (34). □

From (51) and (52) readily follows that

$$\delta_{c_0^{t\pm}} = \tfrac{1}{2}(\delta_{(t-(\pm s))} \otimes s_+^{q-2} \otimes 1_{(\omega_s)}). \tag{53}$$

D. Case p>1 and q=1

Proposition 6: *The delta distribution δ_{c_y}, concentrated at $P(\mathbf{x})=y$, is for $p>1$, $q=1$ and $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$ given by the following expressions.*

(i) *For* $y\neq 0$,

$$\langle \delta_{c_y},\varphi\rangle = \frac{1}{2}\int_0^{+\infty} 1_+(\rho_t^2 - y)\frac{\rho_t^{p-1}}{\sqrt{\rho_t^2 - y}}\Psi_t(\rho_t,\sqrt{\rho_t^2 - y})d\rho_t, \tag{54}$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} 1_+(s^2 + y)(\sqrt{s^2+y})^{p-2}\psi^{S,-}(\sqrt{s^2+y},s)ds, \tag{55}$$

with Ψ_t *defined as*

$$\Psi_t(\rho_t,s) \triangleq \psi^{S,-}(\rho_t,+s) + \psi^{S,-}(\rho_t,-s). \tag{56}$$

(ii) *For* $y=0$,

$$\langle \delta_{c_0},\varphi\rangle = \frac{1}{2}\int_0^{+\infty} \rho^{p-2}\psi^{S,-}(\rho,+\rho)d\rho + \frac{1}{2}\int_0^{+\infty} \rho^{p-2}\psi^{S,-}(\rho,-\rho)d\rho. \tag{57}$$

Herein is $\psi^{S,-}$: $\mathbf{R}^2\to\mathbf{R}$ *such that* $(t,s)\mapsto\psi^{S,-}(t,s)$, *given by*

$$\psi^{S,-}(t,s) \triangleq \langle S_t^{p-1},\varphi(t\omega_t,s)\omega_{S_t^{p-1}}\rangle_{dR}. \tag{58}$$

Proof: The proof is *mutatis mutandis* similar to the proof of Proposition 4. Form (a) now becomes

$$(\Sigma_P\varphi)(y) = \frac{1}{2}\int_0^{+\infty} 1_+(\rho_t^2 - y)\frac{\rho_t^{p-1}}{\sqrt{\rho_t^2 - y}}\Psi_t(\rho_t,\sqrt{\rho_t^2 - y})d\rho_t, \tag{59}$$

while form (b) is

$$(\Sigma_P\varphi)(y) = \frac{1}{2}\int_{-\infty}^{+\infty} 1_+(s^2 + y)(\sqrt{s^2+y})^{p-2}\psi^{S,-}(\sqrt{s^2+y},s)ds. \tag{60}$$

(i) For $y\neq 0$, δ_{c_y}, defined by (2) together with either (59) and (60), is a distribution based on $\mathbf{R}^n$.

(ii) If $y=0$ we get, from either (59) and (60),

$$(\Sigma_P\varphi)(0) = \frac{1}{2}\int_0^{+\infty} \rho^{p-2}\psi^{S,-}(\rho,+\rho)d\rho + \frac{1}{2}\int_0^{+\infty} \rho^{p-2}\psi^{S,-}(\rho,-\rho)d\rho. \tag{61}$$

Again δ_{c_0}, defined by (2) together with (61), is a distribution based on $\mathbf{R}^n$. □

Equation (61) corrects Ref. 4, p. 252, Eq. (15′) with $k=0$, where the second integral has the wrong sign due to not applying the modulus to the Jacobian determinant when transforming from $(\rho_t,x^n)\to(\rho_t,y)$.

The spatial forward and backward hyperboloid sheets (or if $y=0$, half null cones) with respect to the origin are the respective sets, $\forall y\in\mathbf{R}$,

$$c_y^{s+} \triangleq \{\mathbf{x}\in\mathbf{R}^n:\rho_t^2 - (x^n)^2 = y, 0\leqslant x^n\}, \tag{62}$$

$$c_y^{s-} \triangleq \{\mathbf{x}\in\mathbf{R}^n:\rho_t^2 - (x^n)^2 = y, x^n\leqslant 0\}. \tag{63}$$

If $y>0$, (62) represents the $0\leqslant x^n$ part of a hyperboloid of one sheet, while (63) represents its $x^n\leqslant 0$ part. If $y<0$, (62) represents the $0\leqslant x^n$ sheet of a hyperboloid of two sheets, while (63) represents its $x^n\leqslant 0$ sheet.

Corollary 7:

(i) *Let $y\neq 0$. The spatial causal delta distribution $\delta_{c_y^{s+}}$, concentrated at a forward spacelike mass hyperboloid sheet c_y^{s+}, and spatial anticausal delta distribution $\delta_{c_y^{s-}}$, concentrated at a backward spacelike mass hyperboloid sheet c_y^{s+}, are given by*

$$\langle \delta_{c_y^{s+}},\varphi\rangle = \frac{1}{2}\int_0^{+\infty} 1_+(\rho_t^2-y)\frac{\rho_t^{p-1}}{\sqrt{\rho_t^2-y}}\psi^{S,-}(\rho_t,+\sqrt{\rho_t^2-y})d\rho_t, \tag{64}$$

$$=\frac{1}{2}\int_0^{+\infty} 1_+(s^2+y)(\sqrt{s^2+y})^{p-2}\psi^{S,-}(\sqrt{s^2+y},s)ds, \tag{65}$$

and

$$\langle \delta_{c_y^{s-}},\varphi\rangle = \frac{1}{2}\int_0^{+\infty} 1_+(\rho_t^2-y)\frac{\rho_t^{p-1}}{\sqrt{\rho_t^2-y}}\psi^{S,-}(\rho_t,-\sqrt{\rho_t^2-y})d\rho_t, \tag{66}$$

$$=\frac{1}{2}\int_{-\infty}^{0} 1_+(s^2+y)(\sqrt{s^2+y})^{p-2}\psi^{S,-}(\sqrt{s^2+y},s)ds. \tag{67}$$

(ii) *Let $y=0$. The spatial causal delta distribution $\delta_{c_0^{s+}}$, concentrated at the forward half null cone c_0^{s+}, and spatial anticausal delta distribution $\delta_{c_0^{s-}}$, concentrated at the backward half null cone c_0^{s-}, are given by*

$$\langle \delta_{c_0^{s+}},\varphi\rangle = \frac{1}{2}\int_0^{+\infty} \rho^{p-2}\psi^{S,-}(\rho,+\rho)d\rho, \tag{68}$$

$$\langle \delta_{c_0^{s-}},\varphi\rangle = \frac{1}{2}\int_0^{+\infty} \rho^{p-2}\psi^{S,-}(\rho,-\rho)d\rho. \tag{69}$$

Proof: The delta distribution $\delta_{c_y^{s+}}$ ($\delta_{c_y^{s-}}$) is obtained by restricting to $0\leq x^n$ ($x^n\leq 0$) and integrating in (55) from $0\to+\infty$ ($-\infty\to 0$). Equivalently, we only use the first (second) integral in (54). □

From (68) and (69) readily follows that

$$\delta_{c_0^{s\pm}} = \tfrac{1}{2}(t_+^{p-2}\otimes\delta_{(s-(\pm t))}\otimes 1_{(\omega_t)}). \tag{70}$$

E. Case p=1 and q=1

Proposition 8: *The delta distribution δ_{c_y}, concentrated at $P(\mathbf{x})=y$, is for $p=1=q$ and $\forall\varphi\in\mathcal{D}(\mathbf{R}^n)$ given by the following expressions.*

(i) *For $y\neq 0$,*

$$\langle \delta_{c_y},\varphi\rangle = \frac{1}{2}\int_{-\infty}^{+\infty} 1_+(t^2-y)\frac{\varphi(t,\sqrt{t^2-y})}{\sqrt{t^2-y}}dt + \frac{1}{2}\int_{-\infty}^{+\infty} 1_+(t^2-y)\frac{\varphi(t,-\sqrt{t^2-y})}{\sqrt{t^2-y}}dt, \tag{71}$$

$$=\frac{1}{2}\int_{-\infty}^{+\infty} 1_+(s^2+y)\frac{\varphi(\sqrt{s^2+y},s)}{\sqrt{s^2+y}}ds + \frac{1}{2}\int_{-\infty}^{+\infty} 1_+(s^2+y)\frac{\varphi(-\sqrt{s^2+y},s)}{\sqrt{s^2+y}}ds. \tag{72}$$

(ii) *For $y=0$,*

$$\langle(\delta_{c_0})_e,\varphi\rangle=\frac{1}{2}\int_0^{+\infty}\frac{\varphi(x,x)+\varphi(-x,x)+\varphi(x,-x)+\varphi(-x,-x)-4\varphi(0,0)1_+(1-x)}{x}dx + c\varphi(0,0). \tag{73}$$

Herein $(\delta_{c_0})_e$ *is an extension of the partial distribution* δ_{c_0} *and* $c\in\mathbf{C}$ *arbitrary.*

Proof: Put $x^1=t$ and $x^2=s$. Now, $P=t^2-s^2$ and the volume form is $\omega_X=dt\wedge ds$. *Form (a)*. In terms of the coordinates $P=y\in\mathbf{R}$ and $t\in\mathbf{R}$ is

$$(\Sigma_P\varphi)(y)=\frac{1}{2}\int_{-\infty}^{+\infty}1_+(t^2-y)\frac{\varphi(t,\sqrt{t^2-y})}{\sqrt{t^2-y}}dt+\frac{1}{2}\int_{-\infty}^{+\infty}1_+(t^2-y)\frac{\varphi(t,-\sqrt{t^2-y})}{\sqrt{t^2-y}}dt. \tag{74}$$

Form (b). In terms of the coordinates $P=y\in\mathbf{R}$ and $s\in\mathbf{R}$ is

$$(\Sigma_P\varphi)(y)=\frac{1}{2}\int_{-\infty}^{+\infty}1_+(s^2+y)\frac{\varphi(\sqrt{s^2+y},s)}{\sqrt{s^2+y}}ds+\frac{1}{2}\int_{-\infty}^{+\infty}1_+(s^2+y)\frac{\varphi(-\sqrt{s^2+y},s)}{\sqrt{s^2+y}}ds. \tag{75}$$

(i) For $y\neq 0$, δ_{c_y} exists as a distribution based on $\mathbf{R}^2$, given by (2) together with either (74) or (75).

(ii) If $y=0$, expressions (74) and (75) reduce to, respectively,

$$(\Sigma_P\varphi)(0)=\int_{-\infty}^{+\infty}|t|^{-1}\Psi_t(t)dt, \tag{76}$$

$$(\Sigma_P\varphi)(0)=\int_{-\infty}^{+\infty}|s|^{-1}\Psi_s(s)ds, \tag{77}$$

with $\Psi_t(x)\triangleq\varphi(x,+x)+\varphi(x,-x)$ and $\Psi_s(x)\triangleq\varphi(+x,x)+\varphi(-x,x)$. Reordering the limit $y\to 0$ and the integration in (74) and (75) is permitted, provided we restrict φ to having a zero at the origin $\mathbf{0}\in\mathbf{R}^2$. Thus $(\Sigma_P\varphi)(0)$ is the functional value $\langle|x|^{-1},\Psi\rangle$, with $\Psi\in\{\Psi_t,\Psi_s\}$, for the partial distribution $|x|^{-1}$, only defined for test functions Ψ having a zero at the origin $0\in\mathbf{R}$. By extending the partial distribution $|x|^{-1}$ to the distribution $|x|_e^{-1}=|x|_0^{-1}+c\delta$, $c\in\mathbf{C}$ arbitrary, we also extend the existence of $(\Sigma_P\varphi)(0)$, so that it now becomes defined $\forall\varphi\in\mathcal{D}(\mathbf{R}^2)$. This produces the following two extensions of $(\Sigma_P\varphi)(0)$:

$$(\Sigma_P\varphi)_t(0)\triangleq\langle|t|_0^{-1},\Psi_t\rangle+c_t\Psi_t(0), \tag{78}$$

$$(\Sigma_P\varphi)_s(0)\triangleq\langle|s|_0^{-1},\Psi_s\rangle+c_s\Psi_s(0), \tag{79}$$

with $c_t,c_s\in\mathbf{C}$ arbitrary. Using Ref. 2 Eq. (173), Eqs. (78) and (79) are more explicitly

$$(\Sigma_P\varphi)_t(0)=\int_0^{+\infty}\frac{\Psi_t(t)+\Psi_t(-t)-2\Psi_t(0)1_+(1-t)}{t}dt+c_t\Psi_t(0), \tag{80}$$

$$(\Sigma_P\varphi)_s(0)=\int_0^{+\infty}\frac{\Psi_s(s)+\Psi_s(-s)-2\Psi_s(0)1_+(1-s)}{s}ds+c_s\Psi_s(0). \tag{81}$$

In terms of φ, Eqs. (80) and (81) reduce to the common equation,

$$(\Sigma_P\varphi)_e(0)=(\Sigma_P\varphi)_0(0)+c\varphi(0,0), \tag{82}$$

with $(\Sigma_P\varphi)_e(0)\triangleq(\Sigma_P\varphi)_t(0)=(\Sigma_P\varphi)_s(0)$, $c\in\mathbf{C}$ arbitrary, and wherein

$$(\Sigma_P\varphi)_0(0) \triangleq \frac{1}{2}\int_0^{+\infty} \frac{\varphi(x,x)+\varphi(-x,x)+\varphi(x,-x)+\varphi(-x,-x)-4\varphi(0,0)1_+(1-x)}{x}dx. \quad (83)$$

This shows that the distribution $(\delta_{c_0})_e$, given by (73), is an extension of the partial distribution δ_{c_0}, which is only defined for test functions having a zero at the origin $\mathbf{0}\in\mathbf{R}^2$. □

From (78) and (79), and together with (35) and (56) readily follows

$$(\delta_{c_0})_e = \tfrac{1}{2}(|t|_e^{-1}\otimes(\delta_{(s-|t|)}+\delta_{(s+|t|)})), \quad (84)$$

$$=\tfrac{1}{2}((\delta_{(t-|s|)}+\delta_{(t+|s|)})\otimes|s|_e^{-1}). \quad (85)$$

The temporal future and past (spatial forward and backward) hyperboloid sheets (or if $y=0$, half null cones) with respect to the origin are the sets, $\forall y\in\mathbf{R}$,

$$c_y^{t+}\triangleq\{\mathbf{x}\in\mathbf{R}^2 : t^2-s^2=y, 0\leqslant t\}, \quad (86)$$

$$c_y^{t-}\triangleq\{\mathbf{x}\in\mathbf{R}^2 : t^2-s^2=y, t\leqslant 0\}, \quad (87)$$

respectively,

$$c_y^{s+}\triangleq\{\mathbf{x}\in\mathbf{R}^2 : t^2-s^2=y, 0\leqslant s\}, \quad (88)$$

$$c_y^{s-}\triangleq\{\mathbf{x}\in\mathbf{R}^2 : t^2-s^2=y, s\leqslant 0\}. \quad (89)$$

Corollary 9:

(i) *Let $y\neq 0$. The temporal causal delta distribution $\delta_{c_y^{t+}}$, concentrated at a future timelike mass hyperboloid sheet c_y^{t+}, and temporal anticausal delta distribution $\delta_{c_y^{t-}}$, concentrated at a past timelike mass hyperboloid sheet c_y^{t-}, are given by*

$$\langle\delta_{c_y^{t+}},\varphi\rangle = \frac{1}{2}\int_{-\infty}^{+\infty} 1_+(s^2+y)\frac{\varphi(\sqrt{s^2+y},s)}{\sqrt{s^2+y}}ds, \quad (90)$$

$$=\int_0^{+\infty} 1_+(t^2-y)\frac{\frac{1}{2}(\varphi(t,\sqrt{t^2-y})+\varphi(t,-\sqrt{t^2-y}))}{\sqrt{t^2-y}}dt, \quad (91)$$

and

$$\langle\delta_{c_y^{t-}},\varphi\rangle = \frac{1}{2}\int_{-\infty}^{+\infty} 1_+(s^2+y)\frac{\varphi(-\sqrt{s^2+y},s)}{\sqrt{s^2+y}}ds, \quad (92)$$

$$=\int_{-\infty}^{0} 1_+(t^2-y)\frac{\frac{1}{2}(\varphi(t,\sqrt{t^2-y})+\varphi(t,-\sqrt{t^2-y}))}{\sqrt{t^2-y}}dt. \quad (93)$$

The spatial causal delta distribution $\delta_{c_y^{s+}}$, concentrated at a forward spacelike mass hyperboloid sheet c_y^{s+}, and spatial anticausal delta distribution $\delta_{c_y^{s-}}$, concentrated at a backward spacelike mass hyperboloid sheet c_y^{s-}, are given by

$$\langle \delta_{c_y^{s+}}, \varphi \rangle = \frac{1}{2}\int_{-\infty}^{+\infty} 1_+(t^2-y)\frac{\varphi(t,\sqrt{t^2-y})}{\sqrt{t^2-y}}dt, \tag{94}$$

$$= \int_0^{+\infty} 1_+(s^2+y)\frac{\frac{1}{2}(\varphi(\sqrt{s^2+y},s)+\varphi(-\sqrt{s^2+y},s))}{\sqrt{s^2+y}}ds, \tag{95}$$

and

$$\langle \delta_{c_y^{s-}}, \varphi \rangle = \frac{1}{2}\int_{-\infty}^{+\infty} 1_+(t^2-y)\frac{\varphi(t,-\sqrt{t^2-y})}{\sqrt{t^2-y}}dt, \tag{96}$$

$$= \int_{-\infty}^{0} 1_+(s^2+y)\frac{\frac{1}{2}(\varphi(\sqrt{s^2+y},s)+\varphi(-\sqrt{s^2+y},s))}{\sqrt{s^2+y}}ds. \tag{97}$$

(ii) *Let $y=0$. The temporal causal delta distribution $(\delta_{c_0^{t+}})_e$, concentrated at the future half null cone c_0^{t+}, and temporal anticausal delta distribution $(\delta_{c_0^{t-}})_e$, concentrated at the past half null cone c_0^{t-}, are given by*

$$\langle (\delta_{c_0^{t+}})_e, \varphi \rangle = \int_0^{+\infty} \frac{\frac{1}{2}(\varphi(+s,+s)+\varphi(+s,-s)) - \varphi(0,0)1_+(1-s)}{s}ds + c^{t+}\varphi(0,0), \tag{98}$$

$$\langle (\delta_{c_0^{t-}})_e, \varphi \rangle = \int_0^{+\infty} \frac{\frac{1}{2}(\varphi(-s,+s)+\varphi(-s,-s)) - \varphi(0,0)1_+(1-s)}{s}ds + c^{t-}\varphi(0,0), \tag{99}$$

with $c^{t\pm} \in \mathbf{C}$ arbitrary.

The spatial causal delta distribution $(\delta_{c_0^{s+}})_e$, concentrated at the forward half null cone c_0^{s+}, and spatial anticausal delta distribution $(\delta_{c_0^{s-}})_e$, concentrated at the backward half null cone c_0^{s-}, are given by

$$\langle (\delta_{c_0^{s+}})_e, \varphi \rangle = \int_0^{+\infty} \frac{\frac{1}{2}(\varphi(+t,+t)+\varphi(-t,+t)) - \varphi(0,0)1_+(1-t)}{t}dt + c^{s+}\varphi(0,0), \tag{100}$$

$$\langle (\delta_{c_0^{s-}})_e, \varphi \rangle = \int_0^{+\infty} \frac{\frac{1}{2}(\varphi(+t,-t)+\varphi(-t,-t)) - \varphi(0,0)1_+(1-t)}{t}dt + c^{s-}\varphi(0,0), \tag{101}$$

with $c^{s\pm} \in \mathbf{C}$ *arbitrary.*

Proof: The delta distribution with support the temporal future (temporal past) hyperboloid sheet is obtained by restricting to $0 \leq t$ ($t \leq 0$) in (71) or only use the first (second) integral in (72). The delta distribution with support the spatial forward (spatial backward) hyperboloid sheet is obtained by restricting to $0 \leqslant s$ ($s \leqslant 0$) in (72) or only use the first (second) integral in (71). □

From (72) readily follows (with positive square root branch)

$$\delta_{c_y} = \frac{1}{2}((\delta_{(t-\sqrt{s^2+y})} + \delta_{(t+\sqrt{s^2+y})}) \otimes (s^2+y)_+^{-1/2}). \tag{102}$$

Equation (102) is immediately recognized as a familiar expression, if we denote δ_{c_y} by $\delta_{(t^2-(s^2+y))}$. The delta distribution with support the temporal future (temporal past) hyperboloid sheet is obtained by restricting to $0 \leqslant t$ ($t \leqslant 0$) and only use the first (second) delta distribution in the second parentheses in (102).

From (71) readily follows (with positive square root branch)

$$\delta_{c_y} = \frac{1}{2}((t^2-y)_+^{-1/2} \otimes (\delta_{(s-\sqrt{(t^2-y)_+})} + \delta_{(s+\sqrt{(t^2-y)_+})})). \tag{103}$$

Also (103) is immediately understood, if we use the even parity of the delta distribution and denote δ_{c_y} by $\delta_{(s^2-(t^2-y))}$. The delta distribution with support the spatial forward (spatial backward) hyperboloid sheet is obtained by restricting to $0 \leqslant s$ ($s \leqslant 0$) and only use the first (second) delta distribution in the second parentheses in (103).

IV. THE MULTIPLET DELTA DISTRIBUTIONS $\delta_{c_y}^{(k)}$

A. Summary of results

The distributions, $\forall k \in \mathbf{Z}_+$,

$$\delta_{c_y}^{(k)} \triangleq P^* \delta_y^{(k)}, \tag{104}$$

with $\langle \delta_y^{(k)}, \varphi \rangle \triangleq (-1)^k \varphi^{(k)}(y)$, $\forall \varphi \in \mathcal{D}(\mathbf{R})$, represent multiplet delta distributions having $\operatorname{supp}(\delta_{c_y}^{(k)}) = c_y$. They are given according to (2) and (3) by, $\forall k \in \mathbf{Z}_+$, $\forall y \in \mathbf{R}$, and $\forall \varphi \in \mathcal{D}(\mathbf{R}^n \backslash \{\mathbf{0}\})$,

$$\langle \delta_{c_y}^{(k)}, \varphi \rangle = (-1)^k (d^k \Sigma_P \varphi)(y). \tag{105}$$

In this section, we will obtain the following results.

(i) For $y \neq 0$, $\delta_{c_y}^{(k)}$, $\forall k \in \mathbf{Z}_+$, always exists as a distribution in $\mathcal{D}'(\mathbf{R}^n)$.

(ii) For $y=0$, we have the following subcases.

(a) If $k < (n-2)/2$, then $\delta_{c_0}^{(k)} \in \mathcal{D}'(\mathbf{R}^n)$.

(b) If $(n-2)/2 \leqslant k$, then we have the following subsubcases.

(1) If n is odd, then $\delta_{c_0}^{(k)} \in \mathcal{D}'(\mathbf{R}^n)$ is an *analytic continuation*.

(2) If n is even, then $\delta_{c_0}^{(k)}$ is a *partial distribution* defined on $\mathcal{D}'(\mathbf{R}^n \backslash \{\mathbf{0}\})$.

(iii) For $y=0$ and $p=1$, we have the following subsubcases.

(a) If $k < (q-1)/2$, then $\delta_{c_0^{t\pm}}^{(k)} \in \mathcal{D}'(\mathbf{R}^n)$.

(b) If $(q-1)/2 \leqslant k$, then $\delta_{c_0^{t\pm}}^{(k)}$ is a *partial distribution* defined on $\mathcal{D}'(\mathbf{R}^n \backslash \{\mathbf{0}\})$, irrespective of the parity of q.

(iv) For $y=0$ and $q=1$, we have the following subsubcases.

(a) If $k < (p-1)/2$, then $\delta_{c_0^{s\pm}}^{(k)} \in \mathcal{D}'(\mathbf{R}^n)$.

(b) If $(p-1)/2 \leqslant k$, then $\delta_{c_0^{s\pm}}^{(k)}$ is a *partial distribution* defined on $\mathcal{D}'(\mathbf{R}^n \backslash \{\mathbf{0}\})$, irrespective of the parity of p.

In (ii) (b) (2) our method converts the partial distribution $\delta_{c_0}^{(k)}$ into an (in general, nonunique) *extension* $(\delta_{c_0}^{(k)})_e \in \mathcal{D}'(\mathbf{R}^n)$, which coincides with $\delta_{c_0}^{(k)}$ in the sense that $\langle \delta_{c_0}^{(k)}, \psi \rangle = \langle (\delta_{c_0}^{(k)})_e, \psi \rangle$, $\forall \psi \in \mathcal{D}(\mathbf{R}^n \backslash \{\mathbf{0}\})$. Similarly for (iii) (b) and (iv) (b).

B. Functions $\Psi^k_{t,\pm}$ and $\Psi^k_{s,\pm}$

Let $y \in \mathbf{R}$ and $m \in \mathbf{Z}$. To structure the calculations, it is convenient to define the following functions (with parameters y and m), $\forall t \in \mathbf{R}: t^2 - y > 0$:

$$\Psi^k_{t,\pm}(t,y;m) \triangleq t^{2k} d^k_y \left(\frac{1}{2} \left(\frac{\sqrt{t^2 - y}}{|t|} \right)^m \psi(t, \pm \sqrt{t^2 - y}) \right) \tag{106}$$

and, $\forall s \in \mathbf{R}: s^2 + y > 0$,

$$\Psi^k_{s,\pm}(s,y;m) \triangleq s^{2k} d^k_y \left(\frac{1}{2} \left(\frac{\sqrt{s^2 + y}}{|s|} \right)^m \psi(\pm \sqrt{s^2 + y}, s) \right), \tag{107}$$

with $\psi \in \{\psi^{S,S}, \psi^{-,S}, \psi^{S,-}, \varphi\}$, cf. (23), (37), and (58). Using Lemmas 18 and 19, these functions are more explicitly

$$\Psi^k_{t,\pm}(t,y;m) = \frac{(-1)^k}{2} \left(\frac{\sqrt{t^2 - y}}{|t|} \right)^{m-2k} \frac{1}{2^k} \sum_{r=0}^{k} b^k_r(m) (\pm \sqrt{t^2 - y})^r \psi_{0,r}(t, \pm \sqrt{t^2 - y}), \tag{108}$$

$$\Psi^k_{s,\pm}(s,y;m) = \frac{1}{2} \left(\frac{\sqrt{s^2 + y}}{|s|} \right)^{m-2k} \frac{1}{2^k} \sum_{r=0}^{k} b^k_r(m) (\pm \sqrt{s^2 + y})^r \psi_{r,0}(\pm \sqrt{s^2 + y}, s), \tag{109}$$

with

$$\psi_{i,j}(t,s) \triangleq (\partial^i_t \partial^j_s \psi)(t,s). \tag{110}$$

We will also need the functions $\Psi^k_{t,\pm}$ and $\Psi^k_{s,\pm}$ for $y=0$, which we now extend to $\mathbf{R}$ as follows:

$$\Psi^k_{t,\pm}(t,0;m) = \frac{(-1)^k}{2} \frac{1}{2^k} \sum_{r=0}^{k} b^k_r(m) (\pm t)^r \psi_{0,r}(t, \pm t), \tag{111}$$

$$\Psi^k_{s,\pm}(s,0;m) = \frac{1}{2} \frac{1}{2^k} \sum_{r=0}^{k} b^k_r(m) (\pm s)^r \psi_{r,0}(\pm s, s). \tag{112}$$

With this extension, $\Psi^k_{t,\pm}, \Psi^k_{s,\pm} \in \mathcal{D}(\mathbf{R})$.

Lemma 10: *The derivatives at the origin $0 \in \mathbf{R}$ of the functions $\Psi^k_{t,\pm}$ and $\Psi^k_{s,\pm}$ are given in terms of partial derivatives of φ at the origin $\mathbf{0} \in \mathbf{R}^n$ by, $\forall k \in \mathbf{Z}_+$, $\forall l \in \mathbf{Z}_{e,[+}$, and $\forall m \in \mathbf{Z}$,*

$$\lim_{x \to 0} \frac{d^l_x}{l!} \Psi^k_{t,\pm}(x,0;m) = \frac{(-1)^k}{2} \frac{1}{2^k} \sum_{r=0}^{k} b^k_r(m) ((\beta^l_r)_{p,q}(\Delta_t, \Delta_s)\varphi)(\mathbf{0}), \tag{113}$$

$$\lim_{x \to 0} \frac{d^l_x}{l!} \Psi^k_{s,\pm}(x,0;m) = \frac{1}{2} \frac{1}{2^k} \sum_{r=0}^{k} b^k_r(m) ((\beta^l_r)_{p,q}(\Delta_s, \Delta_t)\varphi)(\mathbf{0}), \tag{114}$$

wherein

$$(\beta^l_r)_{p,q}(u,v) \triangleq 1_{0 \le r \le l} e_l a_{p,q} \frac{\pi^{n/2}}{2^l} \sum_{i=0}^{l-r} e_i \frac{\dfrac{(l-i)!}{(l-r-i)!}}{\Gamma\left(\dfrac{p}{2} + \dfrac{i}{2}\right)\Gamma\left(\dfrac{q}{2} + \dfrac{l-i}{2}\right)} \frac{u^{i/2}}{(i/2)!} \frac{v^{(l-i)/2}}{((l-i)/2)!}, \tag{115}$$

with $a_{p,q} \triangleq (1_{p=1} + 1_{p>1}2)(1_{q=1} + 1_{q>1}2)$. *If* $p=1$, *the time Laplacian* Δ_t *is to be replaced by* ∂_t^2. *If* $q=1$, *the space Laplacian* Δ_s *is to be replaced by* ∂_s^2.

Proof: By direct calculation, together with (A11). □

C. Case p>1 and q>1

Proposition 11: *The delta distribution* $\delta_{c_y}^{(k)}$, $\forall k \in \mathbf{Z}_+$, *concentrated at* $P(\mathbf{x})=y$, *is for* $p>1$, $q>1$ *and* $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$ *given by the following expressions.*

(i) *For* $y \neq 0$ *or for* $y=0$ *and* $k<(n-2)/2$,

$$\langle \delta_{c_y}^{(k)}, \varphi \rangle = (-1)^k \int_0^{+\infty} 1_+(\rho_t^2 - y)\rho_t^{(n-2-2k)-1} \Psi_{t,+}^k(\rho_t, y; q-2) d\rho_t, \tag{116}$$

$$= (-1)^k \int_0^{+\infty} 1_+(\rho_s^2 + y)\rho_s^{(n-2-2k)-1} \Psi_{s,+}^k(\rho_s, y; p-2) d\rho_s. \tag{117}$$

(ii) *For* $y=0$ *and* $(n-2)/2 \leq k$.

(a) *For n odd, as the analytic continuation,*

$$\langle \delta_{c_0}^{(k)}, \varphi \rangle = (-1)^k \int_0^{+\infty} \rho_t^{-(2k-(n-2)+1)} (T_{2k-(n-2),0} \Psi_{t,+}^k)(\rho_t, 0; q-2) d\rho_t, \tag{118}$$

$$= (-1)^k \int_0^{+\infty} \rho_s^{-(2k-(n-2)+1)} (T_{2k-(n-2),0} \Psi_{s,+}^k)(\rho_s, 0; p-2) d\rho_s. \tag{119}$$

(b) *For n even, as the, in general, nonequivalent extensions,*

$$\langle (\delta_{c_0}^{(k)})_{t,e}, \varphi \rangle = (-1)^k \int_0^{+\infty} \rho_t^{-(2k-(n-2)+1)} (T_{2k-(n-2),0} \Psi_{t,+}^k)(\rho_t, 0; q-2) d\rho_t + c_t (-1)^k \lim_{\rho_t \to 0} d_{\rho_t}^{2k-(n-2)} \Psi_{t,+}^k(\rho_t, 0; q-2), \tag{120}$$

$$\langle (\delta_{c_0}^{(k)})_{s,e}, \varphi \rangle = (-1)^k \int_0^{+\infty} \rho_s^{-(2k-(n-2)+1)} (T_{2k-(n-2),0} \Psi_{s,+}^k)(\rho_s, 0; p-2) d\rho_s + c_s (-1)^k \lim_{\rho_s \to 0} d_{\rho_s}^{2k-(n-2)} \Psi_{s,+}^k(\rho_s, 0; p-2), \tag{121}$$

with $c_t, c_s \in \mathbf{C}$ *arbitrary.*

Herein are $\Psi_{t,+}^k, \Psi_{s,+}^k$ *given by (108) and (109) for* $y \neq 0$ *and (111) and (112) for* $y=0$, *together with (110), and with* ψ *standing for* $\psi^{S,S}$, *given by (23). The projection operator* $T_{2k-(n-2),0}$ *is defined by (18).*

Proof: The functions $d^k\Sigma_P\varphi$, $\forall k\in\mathbf{Z}_+$, in (105) are obtained from the equivalent expressions (26) and (29) as, respectively,

$$(d^k\Sigma_P\varphi)(y)=\int_0^{+\infty}1_+(\rho_t^2-y)\rho_t^{(n-2-2k)-1}\Psi_{t,+}^k(\rho_t,y;q-2)d\rho_t, \tag{122}$$

$$=\int_0^{+\infty}1_+(\rho_s^2+y)\rho_s^{(n-2-2k)-1}\Psi_{s,+}^k(\rho_s,y;p-2)d\rho_s, \tag{123}$$

with ψ in (106) and (107) here standing for $\psi^{S,S}$, given by (23). Differentiation under the integral sign of (26) and (29) is permitted since the integrands are jointly smooth in y and the integration variable.

(i) If $y<0$ the integral in (122) clearly exists and if $0<y$ the integral in (123) clearly exists, $\forall\varphi\in\mathcal{D}(\mathbf{R}^n)$ and $\forall k\in\mathbf{Z}_+$. Both (122) and (123), as well as the mapping (23), are sequentially continuous linear operators. The distributions $\delta_{c_y}^{(k)}$ are thus in $\mathcal{D}'(\mathbf{R}^n)$ and given by (105) and either (122), (108) or (123), (109), together with (23) and (110) for $\psi=\psi^{S,S}$.

(ii) Let $y=0$. Equations (122) and (123) show that $(d^k\Sigma_P\varphi)(0)$ is given by the functional value $\langle x_+^z,\Psi\rangle$ for $\Psi(x)\triangleq\Psi_{t,+}^k(x,0;q-2)$ or equivalently for $\Psi(x)\triangleq\Psi_{s,+}^k(x,0;p-2)$ at $z=(n-2-2k)-1$. Reordering the limit $y\to 0$ and the integration in (122) and (123) is permitted, provided we restrict φ to an element of $\mathcal{D}(\mathbf{R}^n\setminus\{\mathbf{0}\})$.

(a) *Subcase* $k<(n-2)/2$. The integrals in (122) and (123) still exist and again $\delta_{c_0}^{(k)}\in\mathcal{D}'(\mathbf{R}^n)$.

(b) *Subcase* $(n-2)/2\leq k$. Now the integrals in (122) and (123) do not exist, in general, which is also clear from the fact that $\langle x_+^z,\Psi\rangle$ has simple poles at $z=-(2k-(n-2)+1)\in\mathbf{Z}_-$ with residue $x_{+,-1}^{-(2k-(n-2)+1)}=[(-1)^{2k-(n-2)}/(2k-(n-2))!]\delta^{(2k-(n-2))}$. At its simple poles, x_+^z is a partial distribution, which can be extended to a nonunique distribution $x_{+,e}^z$ given by

$$x_{+,e}^{-(2k-(n-2)+1)}=x_{+,0}^{-(2k-(n-2)+1)}+c\,\delta^{(2k-(n-2))}, \tag{124}$$

with analytic finite part $x_{+,0}^{-(2k-(n-2)+1)}$ and $c\in\mathbf{C}$ arbitrary. We now extend $(d^k\Sigma_P\varphi)(0)$ to all test functions in $\mathcal{D}(\mathbf{R}^n)$ by defining $(d^k\Sigma_P\varphi)(0)\triangleq\langle x_{+,e}^{-(2k-(n-2)+1)},\Psi\rangle$. Then, (105) and (124) define distributions $(\delta_{c_0}^{(k)})_e$ given by

$$\langle(\delta_{c_0}^{(k)})_e,\varphi\rangle\triangleq(-1)^k\langle x_{+,0}^{-(2k-(n-2)+1)},\Psi\rangle+c(-1)^k\langle\delta^{(2k-(n-2))},\Psi\rangle. \tag{125}$$

Substituting herein (17) gives

$$\langle(\delta_{c_0}^{(k)})_e,\varphi\rangle=(-1)^k\int_0^{+\infty}x^{-(2k-(n-2)+1)}(T_{2k-(n-2),0}\Psi)(x)dx+c(-1)^{k+n}\Psi^{(2k-(n-2))}(0), \tag{126}$$

with the projection operator $T_{2k-(n-2),0}$ defined in (18).

(1) *Subsubcase n odd*. Then $l=2k-(n-2)$ is odd so that $\Psi^{(2k-(n-2))}(0)=0$ and the nonuniqueness term in (126) will not be present. This agrees with the finding in Ref. 4 p. 251. In this case, $\delta_{c_0}^{(k)}$ is uniquely given by the analytic continuation

$$\langle\delta_{c_0}^{(k)},\varphi\rangle=(-1)^k\int_0^{+\infty}x^{-(2k-(n-2)+1)}(T_{2k-(n-2),0}\Psi)(x)dx, \tag{127}$$

with $\Psi(x)=\Psi_{t,+}^k(x,0;q-2)$ or equivalently $\Psi(x)=\Psi_{s,+}^k(x,0;p-2)$.

(2) *Subsubcase n even.* Equation (126) defines, by choosing $\Psi=\Psi^k_{t,+}$, a distribution $(\delta^{(k)}_{c_0})_{t,e}$ given by

$$\langle(\delta^{(k)}_{c_0})_{t,e},\varphi\rangle \triangleq (-1)^k\int_0^{+\infty}\rho_t^{-(2k-(n-2)+1)}(T_{2k-(n-2),0}\Psi^k_{t,+})(\rho_t)d\rho_t$$

$$+c_t(-1)^k\lim_{\rho_t\to 0} d^{2k-(n-2)}_{\rho_t}\Psi^k_{t,+}(\rho_t,0;q-2), \tag{128}$$

and, by choosing $\Psi=\Psi^k_{s,+}$, a distribution $(\delta^{(k)}_{c_0})_{s,e}$ given by

$$\langle(\delta^{(k)}_{c_0})_{s,e},\varphi\rangle \triangleq (-1)^k\int_0^{+\infty}\rho_s^{-(2k-(n-2)+1)}(T_{2k-(n-2),0}\Psi^k_{s,+})(\rho_s)d\rho_s$$

$$+c_s(-1)^k\lim_{\rho_s\to 0} d^{2k-(n-2)}_{\rho_s}\Psi^k_{s,+}(\rho_s,0;p-2), \tag{129}$$

with $c_t,c_s\in\mathbf{C}$ arbitrary.□

The integrals in (128) and (129) generally define different distributions $(\delta^{(k)}_{c_0})_{t,0}$ and $(\delta^{(k)}_{c_0})_{s,0}$, which, however, only can differ by a distribution having as support $\{\mathbf{0}\in\mathbf{R}^n\}$. The indeterminate terms in (120) and (121), expressed in terms of the original test function φ, are given by (113)–(115). This precise nature of the so arising extra distribution with support $\{\mathbf{0}\in\mathbf{R}^n\}$ was not made explicit in Ref. 4.

The Gel'fand and Shilov distributions $\delta^{(k)}_1(P)$ and $\delta^{(k)}_2(P)$ correspond with $c_t=0=c_s$ in (128) and (129), due to our definition (18) of the projection operator $T_{2k-(n-2),0}$ and the choice made in Ref. 4 pp. 250–251. In Ref. 4 p. 278, two more distributions, $\delta^{(k)}(P_+)$ and $\delta^{(k)}(P_-)$, were introduced as a "more natural" definition for $\delta^{(k)}_{c_0}$, apparently because they contain an additional term proportional to $\Box^{k-(n-2)/2}\delta$ (Ref. 4 pp. 268–269). The above approach shows that any of the extensions $(\delta^{(k)}_{c_0})_{t,e}$ or $(\delta^{(k)}_{c_0})_{s,e}$, of the common partial distribution $\delta^{(k)}_{c_0}$, defines a multiplet delta distribution on c_0 and no particular extension can be singled out as being the most natural one.

D. Case p=1 and q>1

Proposition 12: *The delta distribution* $\delta^{(k)}_{c_y}$, $\forall k\in\mathbf{Z}_+$, *concentrated at* $P(\mathbf{x})=y$, *is for* $p=1$, $q>1$ *and* $\forall\varphi\in\mathcal{D}(\mathbf{R}^n)$ *given by the following expressions.*

(i) For $y\neq 0$ *or for* $y=0$ *and* $k<(q-1)/2$,

$$\langle\delta^{(k)}_{c_y},\varphi\rangle = (-1)^k\int_{-\infty}^{+\infty} 1_+(t^2-y)|t|^{(q-1-2k)-1}\Psi^k_{t,+}(t,y;q-2)dt, \tag{130}$$

$$=(-1)^k\int_0^{+\infty}1_+(\rho_s^2+y)\rho_s^{(q-1-2k)-1}\Psi^k_s(\rho_s,y;-1)d\rho_s, \tag{131}$$

with Ψ^k_s *defined as*

$$\Psi^k_s(\rho_s,y;-1)\triangleq\Psi^k_{s,+}(\rho_s,y;-1)+\Psi^k_{s,-}(\rho_s,y;-1). \tag{132}$$

(ii) For $y=0$ and $(q-1)/2\leqslant k$.

(a) For q even, as the analytic continuation,

$$\langle\delta_{c_0}^{(k)},\varphi\rangle=(-1)^k\int_{-\infty}^{+\infty}|t|^{-(2k-(q-1)+1)}(T_{2k-(q-1),0}\Psi_{t,+}^k)(t,0;q-2)dt, \tag{133}$$

$$=(-1)^k\int_0^{+\infty}\rho_s^{-(2k-(q-1)+1)}(T_{2k-(q-1),0}\Psi_s^k)(\rho_s,0;-1)d\rho_s. \tag{134}$$

(b) For q odd, as the, in general, nonequivalent extensions

$$\langle(\delta_{c_0}^{(k)})_{t,e},\varphi\rangle=(-1)^k\int_{-\infty}^{+\infty}|t|^{-(2k-(q-1)+1)}(T_{2k-(q-1),0}\Psi_{t,+}^k)(t,0;q-2)dt$$
$$+c_t(-1)^k\lim_{t\to 0}d_t^{2k-(q-1)}\Psi_{t,+}^k(t,0;q-2), \tag{135}$$

$$\langle(\delta_{c_0}^{(k)})_{s,e},\varphi\rangle=(-1)^k\int_0^{+\infty}\rho_s^{-(2k-(q-1)+1)}(T_{2k-(q-1),0}\Psi_s^k)(\rho_s,0;-1)d\rho_s$$
$$+c_s(-1)^k\lim_{\rho_s\to 0}d_{\rho_s}^{2k-(q-1)}\Psi_s^k(\rho_s,0;-1), \tag{136}$$

with $c_t,c_s\in\mathbf{C}$ arbitrary.

Herein are $\Psi_{t,+}^k,\Psi_{s,\pm}^k$ *given by* (108) and (109) *for $y\neq 0$ and* (111) and (112) *for $y=0$, together with (110), and with ψ standing for $\psi^{-,S}$, given by* (37). *The projection operator $T_{2k-(q-1),0}$ is defined by (18).*

Proof: The functions $d^k\Sigma_P\varphi$, $\forall k\in\mathbf{Z}_+$, in (105) are obtained from the equivalent expressions (40) and (43) as, respectively,

$$(d^k\Sigma_P\varphi)(y)=\int_{-\infty}^{+\infty}1_+(t^2-y)|t|^{(q-1-2k)-1}\Psi_{t,+}^k(t,y;q-2)dt, \tag{137}$$

$$=\int_0^{+\infty}1_+(\rho_s^2+y)\rho_s^{(q-1-2k)-1}\Psi_s^k(\rho_s,y;-1)d\rho_s, \tag{138}$$

with ψ in (106) and (107) here standing for $\psi^{-,S}$, given by (37), and Ψ_s^k defined by (132).

(i) Let $y\neq 0$. Similarly as in the proof of Proposition 11, the distributions $\delta_{c_y}^{(k)}$ are given by (105) and either (137), (108), or (138), (109), (132), together with (37) and (110) for $\psi=\psi^{-,S}$, $\forall k\in\mathbf{Z}_+$.

(ii) Let $y=0$. Equations (137) and (138) show that $(d^k\Sigma_P\varphi)(0)$ equals the functional value $\langle|t|^z,\Psi_{t,+}^k(t,0;q-2)\rangle$ or equivalently $\langle x_+^z,\Psi_s^k(x,0;-1)\rangle$, both at $z=(q-1-2k)-1$.

(a) *Subcase* $k<(q-1)/2$. The integrals in (137) and (138) still exist, so $\delta_{c_0}^{(k)}\in\mathcal{D}'(\mathbf{R}^n)$.

(b) *Subcase* $(q-1)/2\leqslant k$. Now the integrals in (137) and (138) do not exist. Recall that the distribution $|t|^z$ has simple poles at $z=-(2l+1)\in\mathbf{Z}_{o,-}$, but can be analytically continued to any $z=-(2l+2)\in\mathbf{Z}_{e,-}$. On the other hand, the distribution x_+^z has simple poles at each $z=-l\in\mathbf{Z}_-$. Remark, however, that $\Psi_s^k(x,0;-1)$ is even in x, since $(+x)^r\psi_{r,0}^{-,S}(+x,x)+(-x)^r\psi_{r,0}^{-,S}(-x,x)$ is even in x because $\psi_{r,0}^{-,S}(t,x)$ is even in x, $\forall r\in\mathbf{N}$. Therefore, also $\langle x_+^z,\Psi_s^k\rangle$ only has genuine poles when $z=-(2l+1)\in\mathbf{Z}_{o,-}$ and can be analytically continued to any $z=-(2l+2)\in\mathbf{Z}_{e,-}$.

(1) *Subsubcase q even*. Then $2k-(q-1)$ is odd so that, similarly as in the proof of Proposition 11, $\Psi_s^{(2k-(q-1))}(0,0;-1)=0$ and the nonuniqueness term will not be present. In this case, $\delta_{c_0}^{(k)}$ is uniquely given by the equivalent analytic continuations,

$$\langle \delta_{c_0}^{(k)}, \varphi \rangle = (-1)^k \int_{-\infty}^{+\infty} |t|^{-(2k-(q-1)+1)} (T_{2k-(q-1),0} \Psi_{t,+}^k)(t) dt, \tag{139}$$

$$= (-1)^k \int_0^{+\infty} \rho_s^{-(2k-(q-1)+1)} (T_{2k-(q-1),0} \Psi_s^k)(\rho_s) d\rho_s. \tag{140}$$

(2) *Subsubcase q odd*. The partial distributions $|t|^{-(2k-(q-1)+1)}$ and $x_+^{-(2k-(q-1)+1)}$ are extended to the respective distributions $|t|_e^{-(2k-(q-1)+1)}$ and $x_{+,e}^{-(2k-(q-1)+1)}$, given by

$$|t|_e^{-(2k-(q-1)+1)} = |t|_0^{-(2k-(q-1)+1)} + c_t \delta^{(2k-(q-1))}, \tag{141}$$

$$x_{+,e}^{-(2k-(q-1)+1)} = x_{+,0}^{-(2k-(q-1)+1)} + c_s \delta^{(2k-(q-1))}, \tag{142}$$

with analytic finite parts $|t|_0^{-(2k-(q-1)+1)}$ and $x_{+,0}^{-(2k-(q-1)+1)}$ and $c_t, c_s \in \mathbf{C}$ arbitrary. Extending $(d^k \Sigma_P \varphi)(0)$ to all $\varphi \in \mathcal{D}(\mathbf{R}^n)$ as in the proof of Proposition 11, (105) now defines distributions

$$\langle (\delta_{c_0}^{(k)})_{t,e}, \varphi \rangle \triangleq (-1)^k \int_{-\infty}^{+\infty} |t|^{-(2k-(q-1)+1)} (T_{2k-(q-1),0} \Psi_{t,+}^k)(t) dt + c_t (-1)^k \lim_{t \to 0} d_t^{2k-(q-1)} \Psi_{t,+}^k(t,0;q-2) \tag{143}$$

and

$$\langle (\delta_{c_0}^{(k)})_{s,e}, \varphi \rangle \triangleq (-1)^k \int_0^{+\infty} \rho_s^{-(2k-(q-1)+1)} (T_{2k-(q-1),0} \Psi_s^k)(\rho_s) d\rho_s + c_s (-1)^k \lim_{\rho_s \to 0} d_{\rho_s}^{2k-(q-1)} \Psi_s^k(\rho_s,0;-1). \tag{144}$$

□

Again, the distributions $(\delta_{c_0}^{(k)})_{t,e}$ and $(\delta_{c_0}^{(k)})_{s,e}$ will, in general, differ by a distribution with support the origin (e.g., see Sec. V). The indeterminate terms in (135) and (136), expressed in terms of the original test function φ, are given by (113)–(115), together with (132).

Corollary 13:

(i) *Let $y \neq 0$ or $y=0$ and $k<(q-1)/2$. The temporal causal multiplet delta distributions concentrated at the future hyperboloid sheet, $\delta_{c_y^{t+}}^{(k)}$, and temporal anticausal multiplet delta distributions concentrated at the past hyperboloid sheet, $\delta_{c_y^{t-}}^{(k)}$, are given by*

$$\langle \delta_{c_y^{t+}}^{(k)}, \varphi \rangle = (-1)^k \int_0^{+\infty} t^{(q-1-2k)-1} \Psi_{t,+}^k(t,y;q-2) dt, \tag{145}$$

$$= (-1)^k \int_0^{+\infty} \rho_s^{(q-1-2k)-1} \Psi_{s,+}^k(\rho_s,y;-1) d\rho_s, \tag{146}$$

and

$$\langle \delta^{(k)}_{c^{t-}_y}, \varphi \rangle = (-1)^k \int_{-\infty}^{0} |t|^{(q-1-2k)-1} \Psi^k_{t,+}(t,y;q-2)dt, \tag{147}$$

$$= (-1)^k \int_{0}^{+\infty} \rho_s^{(q-1-2k)-1} \Psi^k_{s,-}(\rho_s,y;-1)d\rho_s. \tag{148}$$

(ii) *Let $y=0$ and $(q-1)/2 \leqslant k$. The temporal causal multiplet delta distributions concentrated at the future half null cone, $(\delta^{(k)}_{c_0^{t+}})_{t,e}$ and $(\delta^{(k)}_{c_0^{t+}})_{s,e}$, and the temporal anticausal multiplet delta distributions concentrated at the past half null cone, $(\delta^{(k)}_{c_0^{t-}})_{t,e}$ and $(\delta^{(k)}_{c_0^{t-}})_{s,e}$, are given by*

$$\langle (\delta^{(k)}_{c_0^{t+}})_{t,e}, \varphi \rangle = (-1)^k \int_{0}^{+\infty} t^{-(2k-(q-1)+1)} (T_{2k-(q-1),0}\Psi^k_{t,+})(t,0;q-2)dt$$
$$+ c_{t+,t}(-1)^k \lim_{t\to 0} d_t^{2k-(q-1)} \Psi^k_{t,+}(t,0;q-2), \tag{149}$$

$$\langle (\delta^{(k)}_{c_0^{t+}})_{s,e}, \varphi \rangle = (-1)^k \int_{0}^{+\infty} \rho_s^{-(2k-(q-1)+1)} (T_{2k-(q-1),0}\Psi^k_{s,+})(\rho_s,0;-1)d\rho_s$$
$$+ c_{t+,s}(-1)^k \lim_{\rho_s\to 0} d_{\rho_s}^{2k-(q-1)} \Psi^k_{s,+}(\rho_s,0;-1), \tag{150}$$

and

$$\langle (\delta^{(k)}_{c_0^{t-}})_{t,e}, \varphi \rangle = (-1)^k \int_{-\infty}^{0} |t|^{-(2k-(q-1)+1)} (T_{2k-(q-1),0}\Psi^k_{t,+})(t,0;q-2)dt$$
$$+ c_{t-,t}(-1)^k \lim_{t\to 0} d_t^{2k-(q-1)} \Psi^k_{t,+}(t,0;q-2), \tag{151}$$

$$\langle (\delta^{(k)}_{c_0^{t-}})_{s,e}, \varphi \rangle = (-1)^k \int_{0}^{+\infty} \rho_s^{-(2k-(q-1)+1)} (T_{2k-(q-1),0}\Psi^k_{s,-})(\rho_s,0;-1)d\rho_s$$
$$+ c_{t-,s}(-1)^k \lim_{\rho_s\to 0} d_{\rho_s}^{2k-(q-1)} \Psi^k_{s,-}(\rho_s,0;-1), \tag{152}$$

with $c_{t\pm,t}, c_{t\pm,s} \in \mathbf{C}$ arbitrary

Proof:

(i) Let $y \neq 0$ or $y=0$ and $k<(q-1)/2$. The causal multiplet delta distributions are obtained by restricting to $0 \leqslant x^1$ and by integrating from 0 to $+\infty$ in (137) or only using the first integral in (138), involving $\Psi^k_{s,+}$. Similarly, the anticausal distributions are obtained by restricting to $x^1 \leqslant 0$ and by integrating from $-\infty$ to 0 or only using the second integral in (138), involving $\Psi^k_{s,-}$. Equations (145)–(148) still hold when $y\to 0$ if $k<(q-1)/2$.

(ii) Let $y=0$ and $(q-1)/2 \leqslant k$. Now, $|t|^z$ in (143) is to be replaced by t_+^z, which now has simple poles $\forall z=-l \in \mathbf{Z}_-$, so the $\delta^{(k)}_{c_0^{t+}}$ exist only as partial distributions, independent of the parity of q, and need an extension. The same holds for the anticausal partial distributions $\delta^{(k)}_{c_0^{t-}}$, which we obtain by replacing $|t|^z$ in (143) by t_-^z. □

Temporal causal and anticausal multiplet delta distributions concentrated at c_0 were not considered in Ref. 4.

E. Case p>1 and q=1

Proposition 14: *The delta distribution* $\delta_{c_y}^{(k)}$, $\forall k \in \mathbf{Z}_+$, *concentrated at* $P(\mathbf{x})=y$, *is for* $p>1$, $q=1$ *and* $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$ *given by the following expressions.*

(i) *For* $y \neq 0$ *or for* $y=0$ *and* $k<(p-1)/2$,

$$\langle \delta_{c_y}^{(k)}, \varphi \rangle = (-1)^k \int_0^{+\infty} 1_+(\rho_t^2 - y)\rho_t^{(p-1-2k)-1}\Psi_t^k(\rho_t, y; -1)d\rho_t, \tag{153}$$

$$= (-1)^k \int_{-\infty}^{+\infty} 1_+(s^2 + y)|s|^{(p-1-2k)-1}\Psi_{s,+}^k(s, y; p-2)ds, \tag{154}$$

with Ψ_t^k *defined as*

$$\Psi_t^k(\rho_t, y; -1) \triangleq \Psi_{t,+}^k(\rho_t, y; -1) + \Psi_{t,-}^k(\rho_t, y; -1). \tag{155}$$

(ii) *For* $y=0$ *and* $(p-1)/2 \leq k$.

(a) *For p even, as the analytic continuation,*

$$\langle \delta_{c_0}^{(k)}, \varphi \rangle = (-1)^k \int_0^{+\infty} \rho_t^{-(2k-(p-1)+1)}(T_{2k-(p-1),0}\Psi_t^k)(\rho_t, 0; -1)d\rho_t, \tag{156}$$

$$= (-1)^k \int_{-\infty}^{+\infty} |s|^{-(2k-(p-1)+1)}(T_{2k-(p-1),0}\Psi_{s,+}^k)(s, 0; p-2)ds. \tag{157}$$

(b) *For p odd, as the, in general, nonequivalent extensions,*

$$\langle (\delta_{c_0}^{(k)})_{t,e}, \varphi \rangle = (-1)^k \int_0^{+\infty} \rho_t^{-(2k-(p-1)+1)}(T_{2k-(p-1),0}\Psi_t^k)(\rho_t, 0; -1)d\rho_t + c_t(-1)^k \lim_{\rho_t \to 0} d_{\rho_t}^{2k-(p-1)}\Psi_t^k(\rho_t, 0; -1), \tag{158}$$

$$\langle (\delta_{c_0}^{(k)})_{s,e}, \varphi \rangle = (-1)^k \int_{-\infty}^{+\infty} |s|^{-(2k-(p-1)+1)}(T_{2k-(p-1),0}\Psi_{s,+}^k)(s, 0; p-2)ds + c_s(-1)^k \lim_{s \to 0} d_s^{2k-(p-1)}\Psi_{s,+}^k(s, 0; p-2), \tag{159}$$

with $c_t, c_s \in \mathbf{C}$ *arbitrary.*

Herein are $\Psi_{t,\pm}^k, \Psi_{s,+}^k$ *given by* (108) and (109) *for* $y \neq 0$ *and* (111) and (112) *for* $y=0$, *together with* (110), and with ψ standing for $\psi^{S,-}$, given by (58). The projection operator $T_{2k-(p-1),0}$ is defined by (18).

Proof: The proof is *mutatis mutandis* similar to the proof of Proposition 12. The functions $d^k\Sigma_P\varphi$, $\forall k \in \mathbf{Z}_+$, in (105) are obtained from the equivalent expressions (59) and (60) as, respectively,

$$(d^k\Sigma_P\varphi)(y) = \int_0^{+\infty} 1_+(\rho_t^2 - y)\rho_t^{(p-1-2k)-1}\Psi_t^k(\rho_t,y;-1)d\rho_t, \tag{160}$$

$$= \int_{-\infty}^{+\infty} 1_+(s^2 + y)|s|^{(p-1-2k)-1}\Psi_{s,+}^k(s,y;p-2)ds, \tag{161}$$

with ψ in (106) and (107) here standing for $\psi^{S,-}$, given by (58), and Ψ_t^k in (160) defined as (155).□

The indeterminate terms in (158) and (159), expressed in terms of the original test function φ, are given by (113)–(115), together with (155).

Corollary 15:

(i) *Let $y\neq 0$ or $y=0$ and $k<(p-1)/2$. The spatial causal multiplet delta distributions concentrated at the forward hyperboloid sheet, $\delta_{c_y^{s+}}^{(k)}$, and spatial anticausal multiplet delta distributions concentrated at the backward hyperboloid sheet, $\delta_{c_y^{s-}}^{(k)}$, are given by*

$$\langle \delta_{c_y^{s+}}^{(k)},\varphi\rangle = (-1)^k\int_0^{+\infty} \rho_t^{(p-1-2k)-1}\Psi_{t,+}^k(\rho_t,y;-1)d\rho_t, \tag{162}$$

$$=(-1)^k\int_0^{+\infty} s^{(p-1-2k)-1}\Psi_{s,+}^k(s,y;p-2)ds, \tag{163}$$

and

$$\langle \delta_{c_y^{s-}}^{(k)},\varphi\rangle = (-1)^k\int_0^{+\infty} \rho_t^{(p-1-2k)-1}\Psi_{t,-}^k(\rho_t,y;-1)d\rho_t, \tag{164}$$

$$=(-1)^k\int_{-\infty}^{0} |s|^{(p-1-2k)-1}\Psi_{s,+}^k(s,y;p-2)ds. \tag{165}$$

(ii) *Let $y=0$ and $(p-1)/2\leq k$. The spatial causal multiplet delta distributions concentrated at the forward half null cone, $(\delta_{c_0^{s+}}^{(k)})_{t,e}$ and $(\delta_{c_0^{s+}}^{(k)})_{s,e}$, and the spatial anticausal multiplet delta distributions concentrated at the backward half null cone, $(\delta_{c_0^{s-}}^{(k)})_{t,e}$ and $(\delta_{c_0^{s-}}^{(k)})_{s,e}$, are given by*

$$\langle(\delta_{c_0^{s+}}^{(k)})_{t,e},\varphi\rangle = (-1)^k\int_0^{+\infty} \rho_t^{-(2k-(p-1)+1)}(T_{2k-(p-1),0}\Psi_{t,+}^k)(\rho_t,0;-1)d\rho_t$$
$$+ c_{s+,t}(-1)^k \lim_{\rho_t\to 0} d_{\rho_t}^{2k-(p-1)}\Psi_{t,+}^k(\rho_t,0;-1), \tag{166}$$

$$\langle(\delta_{c_0^{s+}}^{(k)})_{s,e},\varphi\rangle = (-1)^k\int_0^{+\infty} s^{-(2k-(p-1)+1)}(T_{2k-(p-1),0}\Psi_{s,+}^k)(s,0;p-2)ds$$
$$+ c_{s+,s}(-1)^k \lim_{s\to 0} d_s^{2k-(p-1)}\Psi_{s,+}^k(s,0;p-2), \tag{167}$$

and

$$\langle(\delta_{c_0^{s-}}^{(k)})_{t,e},\varphi\rangle = (-1)^k \int_0^{+\infty} \rho_t^{-(2k-(p-1)+1)} (T_{2k-(p-1),0}\Psi_{t,-}^k)(\rho_t,0;-1)d\rho_t + c_{s-,t}(-1)^k \lim_{\rho_t\to 0} d_{\rho_t}^{2k-(p-1)} \Psi_{t,-}^k(\rho_t,0;-1), \tag{168}$$

$$\langle(\delta_{c_0^{s-}}^{(k)})_{s,e},\varphi\rangle = (-1)^k \int_{-\infty}^0 |s|^{-(2k-(p-1)+1)} (T_{2k-(p-1),0}\Psi_{s,+}^k)(s,0;p-2)ds + c_{s-,s}(-1)^k \lim_{s\to 0} d_s^{2k-(p-1)} \Psi_{s,+}^k(s,0;p-2), \tag{169}$$

with $c_{s\pm,t}, c_{s\pm,s} \in \mathbf{C}$ *arbitrary.*

Proof: Similarly as for Corollary 13. □

Spatial causal and anticausal multiplet delta distributions concentrated at c_0 were also not considered in Ref. 4.

F. Case p=1 and q=1

Proposition 16: *The delta distribution* $\delta_{c_y}^{(k)}$, $\forall k \in \mathbf{Z}_+$, *concentrated at* $P(\mathbf{x})=y$, *is for* $p=1$, $q=1$ *and* $\forall \varphi \in \mathcal{D}(\mathbf{R}^n)$ *given by the following expressions.*

(i) *For* $y \neq 0$,

$$\langle \delta_{c_y}^{(k)},\varphi\rangle = (-1)^k \int_{-\infty}^{+\infty} 1_+(t^2-y)|t|^{(0-2k)-1} \Psi_t^k(t,y;-1)dt \tag{170}$$

$$= (-1)^k \int_{-\infty}^{+\infty} 1_+(s^2+y)|s|^{(0-2k)-1} \Psi_s^k(s,y;-1)ds, \tag{171}$$

with Ψ_t^k, Ψ_s^k *defined by* (155) and (132), *respectively.*

(ii) *For* $y=0$, *as the, in general, nonequivalent extensions,*

$$\langle(\delta_{c_0}^{(k)})_{t,e},\varphi\rangle \triangleq (-1)^k \int_{-\infty}^{+\infty} |t|^{-(2k+1)} (T_{2k,0}\Psi_t^k)(t,0;-1)dt + c_t(-1)^k \lim_{t\to 0} d_t^{2k}\Psi_t^k(t,0;-1), \tag{172}$$

$$\langle(\delta_{c_0}^{(k)})_{s,e},\varphi\rangle \triangleq (-1)^k \int_{-\infty}^{+\infty} |s|^{-(2k+1)} (T_{2k,0}\Psi_s^k)(s,0;-1)ds + c_s(-1)^k \lim_{s\to 0} d_s^{2k}\Psi_s^k(s,0;-1), \tag{173}$$

with $c_t, c_s \in \mathbf{C}$ *arbitrary.*

Herein are $\Psi_{t,\pm}^k$, $\Psi_{s,\pm}^k$ *given by* (108) and (109) *for* $y \neq 0$ *and* (111) and (112) *for* $y=0$, *together with* (110), *and with* ψ *standing for* φ. *The projection operator* $T_{2k,0}$ *is defined by* (18).

Proof: The functions $d^k\Sigma_P\varphi$, $\forall k\in\mathbf{Z}_+$, in (105) are obtained from the equivalent expressions (74) and (75) as, respectively,

$$(d^k\Sigma_P\varphi)(y)=\int_{-\infty}^{+\infty}1_+(t^2-y)|t|^{(0-2k)-1}\Psi_t^k(t,y;-1)dt, \tag{174}$$

$$=\int_{-\infty}^{+\infty}1_+(s^2+y)|s|^{(0-2k)-1}\Psi_s^k(s,y;-1)ds, \tag{175}$$

with ψ in (106) and (107) here standing for φ and Ψ_t^k, Ψ_s^k defined by (155) and (132), respectively.

(i) Let $y\neq 0$. Similarly as in the proof of Proposition 11, the distributions $\delta_{c_y}^{(k)}$ are in $\mathcal{D}'(\mathbf{R}^n)$ and given by (105) and either (174) and (108) or (175) and (109), together with (110) for $\psi=\varphi$.

(ii) Let $y=0$. Equations (174) and (175) show that $(d^k\Sigma_P\varphi)$ (0) is given by the functional value $\langle|x|^z,\Psi\rangle$ for $\Psi(x)\triangleq\Psi_t^k(x,0;-1)$ or equivalently for $\Psi(x)\triangleq\Psi_s^k(x,0;-1)$ at $z=(0-2k)-1$. The integrals in (174) and (175) do not exist, in general, which is also clear from the fact that $\langle|x|^z,\Psi\rangle$ has simple poles at $z=-(2k+1)\in\mathbf{Z}_-$. Similarly as in the proof of Proposition 11, (105) defines distributions

$$\langle(\delta_{c_0}^{(k)})_{t,e},\varphi\rangle\triangleq(-1)^k\int_{-\infty}^{+\infty}|t|^{-(2k+1)}(T_{2k,0}\Psi_t^k)(t)dt+c_t(-1)^k\lim_{t\to 0}d_t^{2k}\Psi_t^k(t,0;-1), \tag{176}$$

$$\langle(\delta_{c_0}^{(k)})_{s,e},\varphi\rangle\triangleq(-1)^k\int_{-\infty}^{+\infty}|s|^{-(2k+1)}(T_{2k,0}\Psi_s^k)(s)ds+c_s(-1)^k\lim_{s\to 0}d_s^{2k}\Psi_s^k(s,0;-1), \tag{177}$$

with $c_t,c_s\in\mathbf{C}$ arbitrary. Since $l=2k$ is $e_l=1$ in (115), so $(\beta_r^l)_{1,1}\neq 0$ and the indeterminate term in (176) and (177) is present, $\forall k\in\mathbf{Z}_+$. Then, $(\delta_{c_0}^{(k)})_{t,e}$ and $(\delta_{c_0}^{(k)})_{s,e}$ are given in terms of φ by (111) and (155) or (112), (132), and (113)–(115), together with (110) for $\psi=\varphi$. □

The indeterminate terms in (172) and (173), expressed in terms of the original test function φ, are given by (113)–(115), together with (155) and (132).

Corollary 17:

(i) *Let $y\neq 0$. The temporal causal multiplet delta distribution $\delta_{c_y^{t+}}^{(k)}$, concentrated at a future timelike mass hyperboloid sheet c_y^{t+}, and temporal anticausal multiplet delta distribution $\delta_{c_y^{t-}}^{(k)}$, concentrated at a past timelike mass hyperboloid sheet c_y^{t-}, are given by*

$$\langle\delta_{c_y^{t+}}^{(k)},\varphi\rangle=(-1)^k\int_{-\infty}^{+\infty}|t|^{(0-2k)-1}\Psi_{t,+}^k(t,y;-1)dt, \tag{178}$$

$$=(-1)^k\int_{0}^{+\infty}|s|^{(0-2k)-1}\Psi_s^k(s,y;-1)ds, \tag{179}$$

and

$$\langle \delta^{(k)}_{c^{t-}_y}, \varphi \rangle = (-1)^k \int_{-\infty}^{+\infty} |t|^{(0-2k)-1} \Psi^k_{t,-}(t,y;-1)dt, \tag{180}$$

$$= (-1)^k \int_{-\infty}^{0} |s|^{(0-2k)-1} \Psi^k_{s}(s,y;-1)ds. \tag{181}$$

The spatial causal multiplet delta distribution $\delta^{(k)}_{c^{s+}_y}$, *concentrated at a forward spacelike mass hyperboloid sheet* c^{s+}_y, *and spatial anticausal multiplet delta distribution* $\delta^{(k)}_{c^{s-}_y}$, *concentrated at a backward spacelike mass hyperboloid sheet* c^{s-}_y, *are given by*

$$\langle \delta^{(k)}_{c^{s+}_y}, \varphi \rangle = (-1)^k \int_{-\infty}^{+\infty} |s|^{(0-2k)-1} \Psi^k_{s,+}(s,y;-1)ds, \tag{182}$$

$$= (-1)^k \int_{0}^{+\infty} |t|^{(0-2k)-1} \Psi^k_{t}(t,y;-1)dt, \tag{183}$$

and

$$\langle \delta^{(k)}_{c^{s-}_y}, \varphi \rangle = (-1)^k \int_{-\infty}^{+\infty} |s|^{(0-2k)-1} \Psi^k_{s,-}(s,y;-1)ds, \tag{184}$$

$$= (-1)^k \int_{-\infty}^{0} |t|^{(0-2k)-1} \Psi^k_{t}(t,y;-1)dt. \tag{185}$$

(ii) *Let* $y=0$. *The temporal causal multiplet delta distributions,* $(\delta^{(k)}_{c^{t+}_0})_{t,e}$ *and* $(\delta^{(k)}_{c^{t+}_0})_{s,e}$, *concentrated at the future half null cone* c^{t+}_0, *are given by*

$$\langle (\delta^{(k)}_{c^{t+}_0})_{t,e}, \varphi \rangle = (-1)^k \int_{-\infty}^{+\infty} |t|^{-(2k+1)} (T_{2k,0}\Psi^k_{t,+})(t,0;-1)dt + c_{t+,t} \lim_{t\to 0} d^{2k}_t \Psi^k_{t,+}(t,0;-1), \tag{186}$$

$$\langle (\delta^{(k)}_{c^{t+}_0})_{s,e}, \varphi \rangle = (-1)^k \int_{0}^{+\infty} |s|^{-(2k+1)} (T_{2k,0}\Psi^k_{s})(s,0;-1)ds + c_{t+,s} \lim_{s\to 0} d^{2k}_s \Psi^k_{s}(s,0;-1). \tag{187}$$

The temporal anticausal multiplet delta distributions, $(\delta^{(k)}_{c^{t-}_0})_{t,e}$ *and* $(\delta^{(k)}_{c^{t-}_0})_{s,e}$, *concentrated at the past half null cone* c^{t-}_0, *are given by*

$$\langle (\delta^{(k)}_{c^{t-}_0})_{t,e}, \varphi \rangle = (-1)^k \int_{-\infty}^{+\infty} |t|^{-(2k+1)} (T_{2k,0}\Psi^k_{t,-})(t,0;-1)dt + c_{t-,t} \lim_{t\to 0} d^{2k}_t \Psi^k_{t,-}(t,0;-1), \tag{188}$$

$$\langle (\delta^{(k)}_{c^{t-}_0})_{s,e}, \varphi \rangle = (-1)^k \int_{-\infty}^{0} |s|^{-(2k+1)} (T_{2k,0}\Psi^k_{s})(s,0;-1)ds + c_{t-,s} \lim_{s\to 0} d^{2k}_s \Psi^k_{s}(s,0;-1). \tag{189}$$

The spatial causal multiplet delta distributions, $(\delta^{(k)}_{c^{s+}_0})_{s,e}$ *and* $(\delta^{(k)}_{c^{s+}_0})_{t,e}$, *concentrated at the forward half null cone* c^{s+}_0, *are given by*

$$\langle(\delta^{(k)}_{c_0^{s+}})_{t,e},\varphi\rangle=(-1)^k\int_0^{+\infty}|t|^{-(2k+1)}(T_{2k,0}\Psi^k_t)(t,0;-1)dt+c_{s+,t}\lim_{t\to 0}d_t^{2k}\Psi^k_t(t,0;-1), \tag{190}$$

$$\langle(\delta^{(k)}_{c_0^{s+}})_{s,e},\varphi\rangle=(-1)^k\int_{-\infty}^{+\infty}|s|^{-(2k+1)}(T_{2k,0}\Psi^k_{s,+})(s,0;-1)ds+c_{s+,s}\lim_{s\to 0}d_s^{2k}\Psi^k_{s,+}(s,0;-1). \tag{191}$$

The spatial anticausal multiplet delta distributions, $(\delta^{(k)}_{c_0^{s-}})_{s,e}$ and $(\delta^{(k)}_{c_0^{s-}})_{t,e}$, concentrated at the backward half null cone c_0^{s-}, are given by

$$\langle(\delta^{(k)}_{c_0^{s-}})_{t,e},\varphi\rangle=(-1)^k\int_{-\infty}^{0}|t|^{-(2k+1)}(T_{2k,0}\Psi^k_t)(t,0;-1)dt+c_{s-,t}\lim_{t\to 0}d_t^{2k}\Psi^k_t(t,0;-1), \tag{192}$$

$$\langle(\delta^{(k)}_{c_0^{s-}})_{s,e},\varphi\rangle=(-1)^k\int_{-\infty}^{+\infty}|s|^{-(2k+1)}(T_{2k,0}\Psi^k_{s,-})(s,0;-1)ds+c_{s-,s}\lim_{s\to 0}d_s^{2k}\Psi^k_{s,-}(s,0;-1). \tag{193}$$

Proof: Similarly as for Corollaries 13 and 15. □

V. EXAMPLE: THE DISTRIBUTIONS $(\delta^{(1)}_{c_0^{t\pm}})_e$

As practical example, we consider the temporal causal $(+)$ and anticausal $(-)$ distributions for the cases $p=1$, $q=3$, and $k=1$.

We have from (149)–(152) and (18), together with (111) and (112), (A7), and using $\psi^{-,S}(0,0)=A_{q-1}\varphi(\mathbf{0})$,

$$\langle(\delta^{(1)}_{c_0^{t+}})_{t,e},\varphi\rangle=\frac{1}{4}\int_0^{+\infty}\frac{\psi^{-,S}(t,t)+t\psi^{-,S}_{0,1}(t,t)-\psi^{-,S}(0,0)1_+(1-t^2)}{t}dt+c_{t+,t}\frac{A_{q-1}}{4}\varphi(\mathbf{0}), \tag{194}$$

$$\langle(\delta^{(1)}_{c_0^{t+}})_{s,e},\varphi\rangle=\frac{1}{4}\int_0^{+\infty}\frac{\psi^{-,S}(s,s)-s\psi^{-,S}_{1,0}(s,s)-\psi^{-,S}(0,0)1_+(1-s^2)}{s}ds+c_{t+,s}\frac{A_{q-1}}{4}\varphi(\mathbf{0}), \tag{195}$$

and

$$\langle(\delta^{(1)}_{c_0^{t-}})_{t,e},\varphi\rangle=\frac{1}{4}\int_{-\infty}^{0}\frac{\psi^{-,S}(t,t)+t\psi^{-,S}_{0,1}(t,t)-\psi^{-,S}(0,0)1_+(1-t^2)}{|t|}dt+c_{t-,t}\frac{A_{q-1}}{4}\varphi(\mathbf{0}), \tag{196}$$

$$\langle(\delta^{(1)}_{c_0^{t-}})_{s,e},\varphi\rangle=\frac{1}{4}\int_0^{+\infty}\frac{\psi^{-,S}(-s,s)+s\psi^{-,S}_{1,0}(-s,s)-\psi^{-,S}(0,0)1_+(1-s^2)}{s}ds+c_{t-,s}\frac{A_{q-1}}{4}\varphi(\mathbf{0}), \tag{197}$$

with $c_{t\pm,t},c_{t\pm,s}\in\mathbf{C}$ arbitrary. The distributions $(\delta^{(1)}_{c_0^{t\pm}})_{t,0}$ and $(\delta^{(1)}_{c_0^{t\pm}})_{s,0}$ are obtained by putting herein $c_{t\pm,t}=0=c_{t\pm,s}$. Since $\psi^{-,S}$ is even in its second argument and $\psi^{-,S}_{0,1}$ is odd in its second argument, we can write the integral in (196) as

$$\int_0^{+\infty}\frac{\psi^{-,S}(-t,t)+t\psi^{-,S}_{0,1}(-t,t)-\psi^{-,S}(0,0)1_+(1-t^2)}{t}dt.$$

Then, taking the difference of (194) and (195) and of (196) and (197), we get

$$(\delta^{(1)}_{c_0^{t\pm}})_{t,e} - (\delta^{(1)}_{c_0^{t\pm}})_{s,e} = \frac{A_{q-1}}{4}(c_{t\pm,t} - c_{t\pm,s} - 1)\delta_{\mathbf{0}}. \tag{198}$$

Equations (198) show that the $k=1$ causal $(+)$ and anticausal $(-)$ distributions for $p=1$ and $q=3$ cannot be canonically defined as a single distribution by putting $c_{t\pm,t}=0=c_{t\pm,s}$, since its definition, as a regularization, still depends on which of the two equivalent integrals we started from [t: Eq. (137) or s: Eq. (138)]. Moreover, the projection operator (18) is not unique and using a different projection operator amounts to a change in the constant $c_{t\pm,t}$ or $c_{t\pm,s}$ (Ref. 2, Eq. (34)).

The indeterminacy of the distribution $(\delta^{(1)}_{c_0^{t\pm}})_e$ thus has two causes: (i) the nonuniqueness resulting from extending the partial distribution $\delta^{(1)}_{c_0^{t\pm}}$, and (ii) the fact that two equivalent integrals such as (137) and (138), defining the same partial distribution $\delta^{(1)}_{c_0^{t\pm}}$, lead to different extensions (even with the same projection operator T). The nature of the indeterminacy in this example is such that we can regard the distributions $(\delta^{(1)}_{c_0^{t+}})_{t,e}$ and $(\delta^{(1)}_{c_0^{t+}})_{s,e}$ as a single equivalence set $(\delta^{(1)}_{c_0^{t+}})_e$, with equivalence relation

$$\sim:(\delta^{(1)}_{c_0^{t+}})_1 \sim (\delta^{(1)}_{c_0^{t+}})_2 \Leftrightarrow (\delta^{(1)}_{c_0^{t+}})_1 - (\delta^{(1)}_{c_0^{t+}})_2 = c^+\delta_{\mathbf{0}} \tag{199}$$

for some $c^+ \in \mathbf{C}$. Similarly for the distributions $(\delta^{(1)}_{c_0^{t-}})_{t,e}$ and $(\delta^{(1)}_{c_0^{t-}})_{s,e}$. It is thus natural to write

$$(\delta^{(1)}_{c_0^{t\pm}})_e = (\delta^{(1)}_{c_0^{t\pm}})_0 + c^{\pm}\delta_{\mathbf{0}}, \tag{200}$$

with $c^{\pm} \in \mathbf{C}$ arbitrary and wherein $(\delta^{(1)}_{c_0^{t\pm}})_0$ can be chosen equivalently as $(\delta^{(1)}_{c_0^{t\pm}})_{t,0}$ or $(\delta^{(1)}_{c_0^{t\pm}})_{s,0}$, and to call $(\delta^{(1)}_{c_0^{t\pm}})_e$, given by (200), the temporal causal and anticausal $k=1$ delta distributions concentrated at $c_0^{t\pm}$, respectively.

APPENDIX: SUPPORTING MATHEMATICAL RESULTS

The following lemmas were used to obtain the expressions (108) and (109).

Lemma 18: *Let $u \in \mathbf{R}_+$. For any C^∞ function $f: U \subseteq \mathbf{R} \to \mathbf{R}$ and $\forall k \in \mathbf{N}$ holds*

$$d_u^k f(u^{1/2}) = \frac{1}{2^k}\sum_{i=0}^{k} a_i^k (u^{1/2})^{i-2k} f^{(i)}(u^{1/2}), \tag{A1}$$

with

$$a_i^{i+r} = 1_{i=0}1_{r=0} + 1_{0<i}\frac{(-1)^r}{2^r r!}\frac{(i-1+2r)!}{(i-1)!}, \quad \forall\, i,r \in \mathbf{N}, \tag{A2}$$

$$a_i^k \triangleq 0, \quad \forall\, i \notin \mathbf{Z}_{[0,k]}. \tag{A3}$$

Lemma 19: *Let $u \in \mathbf{R}_+$. For any C^∞ function $f: U \subseteq \mathbf{R} \to \mathbf{R}$, $\forall m \in \mathbf{Z}$ and $\forall k \in \mathbf{N}$ holds*

$$d_u^k((u^{1/2})^m f(u^{1/2})) = \frac{1}{2^k}\sum_{i=0}^{k} b_i^k(m)(u^{1/2})^{i-2k+m} f^{(i)}(u^{1/2}). \tag{A4}$$

The constants $b_r^k(m)$, defined in (A4), are related to the a_i^k, given by (A2) and (A3), as, $\forall k \in \mathbf{N}$ and $\forall r \in \mathbf{Z}_{[0,k]}$,

$$b_r^k(m) = \sum_{i=r}^{k}\binom{i}{r} m_{(i-r)} a_i^k, \tag{A7}$$

with $z_{(k)} \triangleq 1_{k=0} + 1_{k>0} z(z-1)(z-2)\cdots(z-(k-1))$, *the falling factorial polynomial. They satisfy the recursion relation*

$$b_r^{k+1}(m) = (r - 2k + m) b_r^k(m) + b_{r-1}^k(m). \tag{A8}$$

In particular, $\forall k \in \mathbf{N}$,

$$b_k^k(m) = 1, \tag{A9}$$

$$b_0^k(m) = 2^k (m/2)_{(k)}. \tag{A10}$$

The following generalized Pizetti's formula (Ref. 4 p. 74) was used to obtain (113)–(115):

$$\frac{(\psi_{i,j}^{S,S})(0,0)}{i!j!} = e_i e_j \frac{A_{p+i-1}}{(4\pi)^{i/2}} \frac{A_{q+j-1}}{(4\pi)^{j/2}} \left(\frac{\Delta_t^{i/2}}{(i/2)!} \frac{\Delta_s^{j/2}}{(j/2)!} \varphi \right)(\mathbf{0}), \tag{A11}$$

with $A_{m-1} \triangleq 2\pi^{m/2}/\Gamma(m/2)$, the surface area of the $(m-1)$-dimensional unit sphere S^{m-1}.

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