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On a Number Pyramid Related to the Binomial, Deleham, Eulerian, MacMahon and Stirling number triangles

Ghislain R. Franssens
Belgian Institute for Space Aeronomy
Ringlaan 3
B-1180 Brussels
BELGIUM
ghislain.franssens@oma.be

Abstract

We study a particular number pyramid $b_{n,k,l}$ that relates the binomial, Deleham, Eulerian, MacMahon-type and Stirling number triangles. The numbers $b_{n,k,l}$ are generated by a function $B^c(x,y,t)$, $c \in \mathbb{C}$, that appears in the calculation of derivatives of a class of functions whose derivatives can be expressed as polynomials in the function itself or a related function. Based on the properties of the numbers $b_{n,k,l}$, we derive several new relations related to these triangles. In particular, we show that the number triangle $T_{n,k}$, recently constructed by Deleham (Sloane's A088874), is generated by the Maclaurin series of $\operatorname{sech}^c t$, $c \in \mathbb{C}$. We also give explicit expressions and various partial sums for the triangle $T_{n,k}$. Further, we find that e^m_{2p}, the numbers appearing in the Maclaurin series of $\cosh^m t$, for all $m \in \mathbb{N}$, equal the number of closed walks, based at a vertex, of length $2p$ along the edges of an m-dimensional cube.

1 Introduction

In this work we study a function $B^c(x,y,t)$, the c-th power of $B(x,y,t)$ defined in Eq. (3.1), that plays a central role in the calculation of derivatives, of a class of functions whose derivatives can be expressed as polynomials in the function itself or a related function. The construction of these polynomials, in terms of the function $B^c(x,y,t)$, is treated in a separate paper [3]. Here we focus on $B^c(x,y,t)$ as a generating function in its own right, and derive from it some interesting number-theoretic results.

We show that the function $B^c(x, y, t)$ generates a number pyramid $b_{n,k,l}$, of which various partial sums are closely related to some important number triangles, including the binomial coefficients $\binom{n}{k}$, a number triangle $T_{n,k}$ recently constructed by Deleham [11, A088874], the Eulerian numbers $A_{n,k}$ [2], a particular kind of MacMahon numbers $B_{n,k}$ [5, p. 331], and Stirling numbers of the first kind $s(n, k)$ [1, p. 824, 24.1.3].

We derive several new expressions related to these triangles. For the triangles $A_{n,k}$ and $B_{n,k}$, we obtain new generating functions. We show in particular that the so far unstudied triangle $T_{n,k}$ is generated by the Maclaurin series of $\operatorname{sech}^c t$, for all $c \in \mathbb{C}$. The numbers $T_{n,k}$ are thus as fundamental for $\operatorname{sech}^c t$ as the Euler numbers E_n are for $\operatorname{sech} t$ [1, p. 804, 23.1.2]. We give explicit expressions and various partial sums for the numbers $T_{n,k}$.

Moreover, the special cases $c = m \in \mathbb{Z}_+$ and $c = -m \in \mathbb{Z}_-$ give rise to a particular generalization of the Euler numbers E_n, here denoted E_n^m and called "multinomial Euler numbers", and a generalization of even parity numbers e_n (defined in Eq. (2.2)), here denoted e_n^m and called "even multinomial parity numbers", respectively. The E_n^m are generated by the Maclaurin series of $\operatorname{sech}^m t$ (so $E_n^1 = E_n$) and the e_n^m by the Maclaurin series of $\cosh^m t$ (so $e_n^1 = e_n$). Obviously, $E_{2p+1}^m = 0$ and $e_{2p+1}^m = 0$, for all $p \in \mathbb{N}$, because $\operatorname{sech}^m t$ and $\cosh^m t$ are even functions of t. We obtain explicit formulas for the numbers E_{2p}^m and e_{2p}^m, as well as relations between them. The numbers e_{2p}^m turn out to have as combinatorial interpretation, the number of closed walks, based at a vertex, of length $2p$ along the edges of an m-dimensional cube.

2 Notation and definitions

1. Define the sets of positive odd and even integers $\mathbb{Z}_{o,+}$ and $\mathbb{Z}_{e,+}$, the negative odd and even integers $\mathbb{Z}_{o,-}$ and $\mathbb{Z}_{e,-}$, the odd integers $\mathbb{Z}_o \triangleq \mathbb{Z}_{o,-} \cup \mathbb{Z}_{o,+}$ and the even integers $\mathbb{Z}_e \triangleq \mathbb{Z}_{e,-} \cup \{0\} \cup \mathbb{Z}_{e,+}$, the positive integers $\mathbb{Z}_+ \triangleq \mathbb{Z}_{o,+} \cup \mathbb{Z}_{e,+}$ and negative integers $\mathbb{Z}_- \triangleq \mathbb{Z}_{o,-} \cup \mathbb{Z}_{e,-}$, the natural numbers $\mathbb{N} \triangleq \{0\} \cup \mathbb{Z}_+$ and the integers $\mathbb{Z} \triangleq \mathbb{Z}_- \cup \{0\} \cup \mathbb{Z}_+$. Let $\mathbb{Z}_{+,n} \triangleq \{1, 2, ..., n\}$, $\mathbb{Z}_{-,n} \triangleq \{-n, -(n-1), ..., -1\}$, $\mathbb{N}_n \triangleq \{0\} \cup \mathbb{Z}_{+,n}$ and denote by $\mathbb{C}$ the complex numbers.

2. Define
$$\delta_{condition} \triangleq \begin{cases} 1, & \text{if } condition \text{ is true;} \\ 0, & \text{if } condition \text{ is false,} \end{cases} \tag{2.1}$$
and for all $n \in \mathbb{Z}$ the even and odd parity numbers
$$e_n \triangleq \delta_{n \in \mathbb{Z}_e}, \tag{2.2}$$
$$o_n \triangleq \delta_{n \in \mathbb{Z}_o}. \tag{2.3}$$

3. Denote the n-th derivative with respect to x by D_x^n.

4. We define $0^n \triangleq \delta_{n=0}$, for all $n \in \mathbb{N}$, and $z^0 \triangleq 1$, for all $z \in \mathbb{C}$.

5. Let $n \in \mathbb{N}$ and $z \in \mathbb{C}$. Denote by

$$\begin{aligned} z^{(n)} &\triangleq \delta_{n=0} + \delta_{n>0}\, z(z+1)(z+2)...(z+(n-1)), && (2.4)\\ &= \frac{\Gamma(z+n)}{\Gamma(z)}, && (2.5)\\ &= \sum_{k=0}^{n} (-1)^{n-k} s\,(n,k)\, z^k, && (2.6) \end{aligned}$$

the rising factorial polynomial (Pochhammer's symbol). In particular, $0^{(n)} = \delta_{n=0}$ and $m^{(n)} = (m-1+n)!/(m-1)!$ for $m \in \mathbb{Z}_+$.

Also, denote by

$$\begin{aligned} z_{(n)} &\triangleq \delta_{n=0} + \delta_{n>0}\, z(z-1)(z-2)...(z-(n-1)), && (2.7)\\ &= \frac{\Gamma(z+1)}{\Gamma(z+1-n)}, && (2.8)\\ &= \sum_{k=0}^{n} s\,(n,k)\, z^k, && (2.9) \end{aligned}$$

the falling factorial polynomial. In particular, $0_{(n)} = \delta_{n=0}$ and $m_{(n)} = (m!/(m-n)!)\,\delta_{n\le m}$ for $m \in \mathbb{Z}_+$. In Eqs. (2.6) and (2.9), $s\,(n,k)$ are Stirling numbers of the first kind. We have $z_{(n)} = (-1)^n\,(-z)^{(n)}$.

6. We will need

$$\begin{aligned} \frac{1}{(1-z)^c} &= \sum_{n=0}^{+\infty} c^{(n)} \frac{z^n}{n!}, && (2.10)\\ (1+z)^c &= \sum_{n=0}^{+\infty} c_{(n)} \frac{z^n}{n!}, && (2.11) \end{aligned}$$

being absolutely and uniformly convergent series for all $z \in \{z \in \mathbb{C} : |z| < 1\}$ and for all $c \in \mathbb{C}$. We have for all $n \in \mathbb{N}$ and for all $a, b \in \mathbb{C}$,

$$\begin{aligned} (a+b)^{(n)} &= \sum_{k=0}^{n} \tbinom{n}{k} a^{(n-k)} b^{(k)}, && (2.12)\\ (a+b)_{(n)} &= \sum_{k=0}^{n} \tbinom{n}{k} a_{(n-k)} b_{(k)}. && (2.13) \end{aligned}$$

In particular, for $a = c = -b$, we get the orthogonality relations, for all $n \in \mathbb{N}$ and for all $c \in \mathbb{C}$,

$$\begin{aligned} \sum_{k=0}^{n} \tbinom{n}{k} c^{(n-k)}\,(-c)^{(k)} &= \delta_{n=0}, && (2.14)\\ \sum_{k=0}^{n} \tbinom{n}{k} c_{(n-k)}\,(-c)_{(k)} &= \delta_{n=0}. && (2.15) \end{aligned}$$

7. With $m, n \in \mathbb{N}$ and $K \triangleq \{k_1, k_2, ..., k_m \in \mathbb{N}\}$, define $|K| \triangleq k_1+k_2+...+k_m$, $\#(K) \triangleq m$ and $\binom{n}{K} \triangleq n!/(k_1!k_2!...k_m!)$, expressions that are used in the last section.

3 The generating function $B^c(x, y, t)$

For all $x, y, t \in \mathbb{C}$ define

$$B(x,y,t) \triangleq \begin{cases} \frac{x-y}{xe^{-\frac{x-y}{2}t} - ye^{+\frac{x-y}{2}t}}, & \text{if } x \neq y; \\ \frac{1}{1-xt}, & \text{if } x = y. \end{cases} \tag{3.1}$$

Proposition 3.1. *The Maclaurin series of the c-th power of $B(x, y, t)$, for all $c \in \mathbb{C}$, is given by*

$$B^c(x,y,t) = \sum_{n=0}^{+\infty} 2^{-n} B_n(x,y;c) \frac{t^n}{n!}, \tag{3.2}$$

and converges absolutely and uniformly for $|t| < \left|\frac{\ln x - \ln y}{x-y}\right|$. For all $n \in \mathbb{N}$,

$$B_n(x,y;c) = \sum_{k=0}^{n} B_{n,k}(c) x^{n-k} y^k, \tag{3.3}$$

with the coefficients $B_{n,k}(c)$ satisfying, for all $k \in \mathbb{N}_n$,

$$B_{n+1,k+1}(c) = (2(k+1)+c)\, B_{n,k+1}(c) + (2(n-k)+c)\, B_{n,k}(c), \tag{3.4}$$

with $B_{0,0}(c) = 1$ and we define $B_{n,k}(c) \triangleq 0$, for all $k \notin \mathbb{N}_n$.

Proof. The point $t = 0$ is an ordinary point of $B^c(x, y, t)$, so $B^c(x, y, t)$ has a Maclaurin power series, converging absolutely and uniformly for $|t| < \left|\frac{\ln x - \ln y}{x-y}\right|$.

Define the partial differential operator

$$D(x,y,t;c) \triangleq \left(1 - \frac{x+y}{2}t\right)\frac{\partial}{\partial t} + \frac{x-y}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) - c\frac{x+y}{2}. \tag{3.5}$$

A direct calculation shows that

$$D(x,y,t;c) B^c(x,y,t) = 0. \tag{3.6}$$

Substituting in Eq. (3.6) for $B^c(x, y, t)$ the uniformly convergent series (3.2) gives

$$\sum_{n=0}^{+\infty} 2^{-n} \left(D_n(x,y;c) B_n(x,y;c)\right) \frac{t^n}{n!} = 0,$$

wherein

$$D_n(x,y;c) \triangleq \frac{1}{2}T_1 + \frac{x-y}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) - (n+c)\frac{x+y}{2} \tag{3.7}$$

and T_p is the difference shift operator such that $T_p B_n(x,y;c) = B_{n+p}(x,y;c)$. This holds for any t, so we have

$$D_n(x,y;c)B_n(x,y;c) = 0. \tag{3.8}$$

Substituting in Eq. (3.8) for $B_n(x,y;c)$ the bivariate homogeneous polynomial (3.3) gives

$$\sum_{k=0}^{n} \left(D_{n,k}(x,y;c)B_{n,k}(c)\right) x^{n-k}y^k = 0,$$

wherein

$$D_{n,k}(x,y;c) \triangleq \frac{1}{2}T_{1,0} - \left((n-k+1)+c/2\right)T_{0,-1} - (k+c/2) \tag{3.9}$$

and $T_{p,q}$ is the bivariate difference shift operator such that $T_{p,q}B_{n,k}(c) = B_{n+p,k+q}(c)$. This holds for any x and y, so we have

$$D_{n,k}(x,y;c)B_{n,k}(c) = 0, \tag{3.10}$$

which is just Eq. (3.4).

¿From the fact that $B^c(x,y,0) = 1$, we obtain $B_0(x,y;c) = B_{0,0}(c) = 1$. □

We have that $B(x,y,t) = B(y,x,t)$ for all $x,y,t \in \mathbb{C}$, hence $B_n(x,y;c) = B_n(y,x;c)$ for all $n \in \mathbb{N}$, and $B_{n,k}(c) = B_{n,n-k}(c)$ for all $c \in \mathbb{C}$.

3.1 Special cases

(i) For $x = 0$ or $y = 0$, we get

$$B^c(0,z,t) = B^c(z,0,t) = e^{\frac{c}{2}zt}. \tag{3.11}$$

This implies that

$$B_n(0,z;c) = B_n(z,0;c) = (cz)^n, \tag{3.12}$$

and this yields in turn that

$$B_{n,k}(c) = c^n \delta_{n=k}. \tag{3.13}$$

(ii) For $y = \pm x$, we get

$$B^c(x,x,t) = \frac{1}{(1-xt)^c}, \tag{3.14}$$

$$B^c(x,-x,t) = \operatorname{sech}^c(xt). \tag{3.15}$$

This gives

$$B_n(x,x;c) = 2^n c^{(n)} x^n, \tag{3.16}$$

$$B_n(x,-x;c) = 2^n \left(\lim_{t\to 0} D_t^n \operatorname{sech}^c t\right) x^n, \tag{3.17}$$

and this yields in turn

$$\sum_{k=0}^{n} B_{n,k}(c) = 2^n c^{(n)}, \tag{3.18}$$

$$\sum_{k=0}^{n} (-1)^k B_{n,k}(c) = 2^n \left(\lim_{t\to 0} D_t^n \operatorname{sech}^c t\right). \tag{3.19}$$

n\ k	1	2	3	4	5	6
1	1					
2	1	1				
3	1	6	1			
4	1	23	23	1		
5	1	76	230	76	1	
6	1	237	1682	1682	237	1

Table 1: The number triangle $B_{n,k}$

3.2 The numbers $B_{n,k}(1)$ and $B_{n,k}(2)$

Putting $c = 0$ in Eq. (3.2) shows that $B_n(x, y; 0) = \delta_{n=0}$, so $B_{n,k}(0) = \delta_{n=0}$.

(i) For $c = 1$, Eq. (3.4) becomes

$$B_{n+1,k+1}(1) = (2(k+1)+1)\, B_{n,k+1}(1) + (2(n-k)+1)\, B_{n,k}(1),$$

so

$$B_{n,k}(1) = B_{n+1,k+1}, \tag{3.20}$$

with $B_{n,k}$ the numbers derived by MacMahon [5, p. 331], (Sloane's A060187), cf. Table 1.

In this case, Eqs. (3.18) and (3.19) become

$$\sum_{k=0}^{n} B_{n+1,k+1} = 2^n n!, \tag{3.21}$$

$$\sum_{k=0}^{n} (-1)^k B_{n+1,k+1} = 2^n E_n, \tag{3.22}$$

with E_n the Euler (or secant) numbers [1, p. 804, 23.1.2], ($|E_{2n}|$ are Sloane's A000364). The numbers $B_{n,k}$ are thus (also) generated by (for $|t| < \frac{1}{2}\left|\frac{\ln y}{1-y}\right|$ and $y \neq 1$)

$$\frac{1-y}{e^{-(1-y)t} - y e^{+(1-y)t}} = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} B_{n+1,k+1} y^k \frac{t^n}{n!}. \tag{3.23}$$

We can also obtain from Eq. (3.23) the following more standard generating function for the $B_{n,k}$, (i.e., on the same footing as Eq. (3.29) below), (for $|t| < \left|\frac{\ln y}{1-y}\right|$ and $y \neq 1$)

$$\frac{1}{2} \ln \frac{\frac{e^{-\frac{1-y^2}{2}t} + y e^{+\frac{1-y^2}{2}t}}{1+y}}{\frac{e^{-\frac{1-y^2}{2}t} - y e^{+\frac{1-y^2}{2}t}}{1-y}} = \sum_{n=1}^{+\infty} \sum_{k=1}^{n} B_{n,k} y^{2k-1} \frac{t^n}{n!}. \tag{3.24}$$

Eqs. (3.23) and (3.24) appear to be new.

(ii) For $c = 2$, Eq. (3.4) becomes

$$B_{n+1,k+1}(2) = (k+2)\, 2B_{n,k+1}(2) + (n-k+1)\, 2B_{n,k}(2),$$

n\ k	1	2	3	4	5	6
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

Table 2: The number triangle $A_{n,k}$

so

$$B_{n,k}(2) = 2^n A_{n+1,k+1}, \tag{3.25}$$

with $A_{n,k}$ the Eulerian numbers [2], (Sloane's A008292), cf. Table 2. Another notation for the Eulerian numbers is $\left\langle {n \atop k} \right\rangle = A_{n,k+1}$.

In this case, Eqs. (3.18) and (3.19) become

$$\sum_{k=0}^{n} A_{n+1,k+1} = (n+1)!, \tag{3.26}$$

$$\sum_{k=0}^{n} (-1)^k A_{n+1,k+1} = 2^{n+2}\left(2^{n+2}-1\right)\frac{B_{n+2}}{n+2}, \tag{3.27}$$

with B_n the Bernoulli numbers [1, p. 804, 23.1.2], ($|B_n|$ are Sloane's A027641 and A027642). In Eq. (3.27) we used $D_t^n \operatorname{sech}^2 t = D_t^{n+1} \tanh t$. The Eulerian numbers $A_{n,k}$ are thus (also) generated by

$$\frac{1}{\left(\frac{e^{-\frac{1-y}{2}t} - ye^{+\frac{1-y}{2}t}}{1-y}\right)^2} = \sum_{n=0}^{+\infty}\sum_{k=0}^{n} A_{n+1,k+1} y^k \frac{t^n}{n!}. \tag{3.28}$$

The well-known standard generating function for the Eulerian numbers is

$$\frac{1-y}{1-ye^{(1-y)t}} = 1 + \sum_{n=1}^{+\infty}\sum_{k=1}^{n} A_{n,k} y^k \frac{t^n}{n!}. \tag{3.29}$$

For further convenience we define $A_{n,k} \triangleq 0$ and $B_{n,k} \triangleq 0$, for all $k \notin \mathbb{Z}_{+,n}$.

3.3 Examples of some $B_n(x, y; c)$

The first six $B_n(x, y; c)$ are:

$$B_0(x, y; c) = 1,$$

$$B_1(x, y; c) = cx + cy,$$

$$B_2(x, y; c) = c^2x^2 + 2c\,(2+c)\,xy + c^2y^2,$$

$$B_3(x, y; c) = c^3x^3 + c\left(3c^2 + 12c + 8\right)x^2y + c\left(3c^2 + 12c + 8\right)xy^2 + c^3y^3,$$

$$B_4(x,y;c) = \begin{array}{c} c^4x^4 \\ +4c\,(c^3+6c^2+8c+4)\,x^3y \\ +2c\,(3c^3+24c^2+56c+32)\,x^2y^2 \\ +4c\,(c^3+6c^2+8c+4)\,xy^3 \\ +c^4y^4 \end{array},$$

$$B_5(x,y;c) = \begin{array}{c} c^5x^5 \\ +c\,(5c^4+40c^3+80c^2+80c+32)\,x^4y \\ +c\,(10c^4+120c^3+480c^2+720c+352)\,x^3y^2 \\ +c\,(10c^4+120c^3+480c^2+720c+352)\,x^2y^3 \\ +c\,(5c^4+40c^3+80c^2+80c+32)\,xy^4 \\ +c^5y^5 \end{array}.$$

4 Properties of the $B_n(x,y;c)$

4.1 Additive property with respect to the parameter c

Obviously, for all $a, b \in \mathbb{C}$,

$$B^{a+b}(x,y,t) = B^a(x,y,t)B^b(x,y,t), \tag{4.1}$$

and from this follows, for all $n \in \mathbb{N}$,

$$B_n(x,y;a+b) = \sum_{k=0}^{n} \tbinom{n}{k} B_{n-k}(x,y;a)B_k(x,y;b). \tag{4.2}$$

Substituting Eq. (3.3) in Eq. (4.2) gives

$$B_{n,k}(a+b) = \sum_{p=0}^{n} \tbinom{n}{p} \sum_{q=0}^{k} B_{n-p,k-q}(a)B_{p,q}(b). \tag{4.3}$$

For instance, by letting $a = b = 1$, Eq. (4.3) yields the following quadratic expansion of Eulerian numbers $A_{n,k}$ in the MacMahon numbers $B_{n,k}$,

$$A_{n+1,k+1} = \frac{1}{2^n} \sum_{p=0}^{n} \tbinom{n}{p} \sum_{q=0}^{k} B_{n-p+1,k-q+1}B_{p+1,q+1}. \tag{4.4}$$

4.2 Infinite series

Proposition 4.1. *For all $c \in \mathbb{C}$ and for all $n \in \mathbb{N}$,*

$$B_n(x,y;c) = \begin{cases} \frac{(x-y)^{n+c}}{x^c} \sum_{k=0}^{+\infty} c^{(k)}\,(2k+c)^n\,\frac{(y/x)^k}{k!} & \text{if} \quad |y| < |x|\,; \\ \frac{(y-x)^{n+c}}{y^c} \sum_{k=0}^{+\infty} c^{(k)}\,(2k+c)^n\,\frac{(x/y)^k}{k!} & \text{if} \quad |x| < |y|\,, \end{cases} \tag{4.5}$$

where $x, y \in \mathbb{C}$ and the series converges absolutely.

Proof. (i) Applying Eq. (2.10) to $B^c(x,y,t)$ gives, for all $(x,y) \in D_{|y|<|x|} \triangleq \{(x,y) \in \mathbb{C}^2 : |y| < |x|\}$ and for all $t \in \Lambda_t(x,y) \triangleq \{t \in \mathbb{C} : \mathrm{Re}((1-y/x)t) < (\ln|x| - \ln|y|)\}$, the absolutely convergent series

$$B^c(x,y,t) = \frac{(x-y)^c}{x^c} \sum_{k=0}^{+\infty} c^{(k)} \frac{(y/x)^k}{k!} e^{+(k+c/2)(x-y)t}.$$

Expanding herein $e^{+(k+c/2)(x-y)t}$ in Maclaurin series gives

$$B^c(x,y,t) = \frac{(x-y)^c}{x^c} \sum_{k=0}^{+\infty} c^{(k)} \frac{(y/x)^k}{k!} \sum_{n=0}^{+\infty} (k+c/2)^n (x-y)^n \frac{t^n}{n!}.$$

Both series are absolutely convergent, so we may interchange the order of summation [10, p. 175, Theorem 8.3], yielding

$$B^c(x,y,t) = \sum_{n=0}^{+\infty} \left(\frac{(x-y)^{n+c}}{x^c} \sum_{k=0}^{+\infty} c^{(k)} (k+c/2)^n \frac{(y/x)^k}{k!} \right) \frac{t^n}{n!}.$$

On the other hand holds by Proposition 3.1, for all $t \in \Omega_t(x,y) \triangleq \left\{ t \in \mathbb{C} : |t| < \left| \frac{\ln x - \ln y}{x-y} \right| \right\}$, that

$$B^c(x,y,t) = \sum_{n=0}^{+\infty} 2^{-n} B_n(x,y;c) \frac{t^n}{n!}.$$

For $(x,y) \in D_{|y|<|x|}$, $\Lambda_t(x,y) \cap \Omega_t(x,y) \neq \varnothing$. Then for all $t \in \Lambda_t(x,y) \cap \Omega_t(x,y)$ holds

$$\sum_{n=0}^{+\infty} 2^{-n} B_n(x,y;c) \frac{t^n}{n!} = \sum_{n=0}^{+\infty} \left(\frac{(x-y)^{n+c}}{x^c} \sum_{k=0}^{+\infty} c^{(k)} (k+c/2)^n \frac{(y/x)^k}{k!} \right) \frac{t^n}{n!},$$

and the first part of Eq. (4.5) follows.

(ii) Similar. □

In particular, Eq. (4.5) becomes, for $c = m \in \mathbb{Z}_+$,

$$B_n(x,y;m) = (x-y)^{n+m} \sum_{k=0}^{+\infty} \binom{m-1+k}{k} (2k+m)^n x^{-(k+m)} y^k, \tag{4.6}$$

and for $c = -m \in \mathbb{Z}_-$,

$$B_n(x,y;-m) = (x-y)^{n-m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (2k-m)^n x^{m-k} y^k. \tag{4.7}$$

Moreover, Eq. (4.5) reduces to the following special form, for all $c \in \mathbb{C}$,

$$\sum_{k=0}^{+\infty} c^{(k)} (2k+c)^n \frac{z^k}{k!} = \frac{B_n(1,z;c)}{(1-z)^{n+c}}, |z| < 1, \tag{4.8}$$

$$\sum_{k=0}^{+\infty} c^{(k)} (2k+c)^n \frac{z^{-k}}{k!} = z^c \frac{B_n(z,1;c)}{(z-1)^{n+c}}, |z| > 1. \tag{4.9}$$

In particular, Eqs. (4.8) and (4.9) become,
(i) for $c = 1$, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_{k=0}^{+\infty} (2k+1)^n z^k &= \frac{B_n(1,z)}{(1-z)^{n+1}}, |z| < 1, && (4.10)\\
\sum_{k=0}^{+\infty} (2k+1)^n z^{-k} &= z\frac{B_n(z,1)}{(z-1)^{n+1}}, |z| > 1. && (4.11)
\end{aligned}$$

Herein is $B_n(x, y)$ the MacMahon homogeneous bivariate polynomial,

$$B_n(x, y; 1) \triangleq B_n(x, y) = \sum_{k=0}^{n} B_{n+1,k+1} x^{n-k} y^k. \tag{4.12}$$

(ii) for $c = 2$, for all $n \in \mathbb{Z}_+$,

$$\begin{aligned}
\sum_{l=1}^{+\infty} l^n z^l &= \frac{z}{(1-z)^{n+1}} A_{n-1}(1,z), |z| < 1, && (4.13)\\
\sum_{l=1}^{+\infty} l^n z^{-l} &= \frac{z}{(z-1)^{n+1}} A_{n-1}(z,1), |z| > 1. && (4.14)
\end{aligned}$$

Herein is $A_n(x, y)$ the Eulerian homogeneous bivariate polynomial,

$$2^{-n} B_n(x, y; 2) \triangleq A_n(x, y) = \sum_{k=0}^{n} A_{n+1,k+1} x^{n-k} y^k. \tag{4.15}$$

Notice that the left hand side of Eq. (4.13) is by definition the polylogarithm of negative integer order, $\mathrm{Li}_{-n}(z)$ [6]. Further, combining Eq. (4.13) with [13, Eq. (14)], we get the interesting identity, for all $n \in \mathbb{N}$,

$$\sum_{p=1}^{n} (-1)^{n-p} p! S(n,p) z^{p-1} = A_{n-1}(z, z-1). \tag{4.16}$$

Taking in Eq. (4.7) the $\lim_{y \to -x}$ and using Eq. (3.15) we obtain, for all $m, n \in \mathbb{N}$,

$$\lim_{t \to 0} D_t^n \cosh^m t = \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} (2k-m)^n. \tag{4.17}$$

Taking in Eq. (4.7) the $\lim_{y \to x}$ and using Eq. (3.16) yields, for all $m \in \mathbb{N}$,

$$\frac{1}{2^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (2k-m)^m = (-1)^m m!. \tag{4.18}$$

4.3 Generating expression

Proposition 4.2. *For all $n \in \mathbb{N}$ and for all $c, z \in \mathbb{C}$,*

$$B_n(1, z; c) = (1-z)^{n+c} (c + 2zD_z)^n (1-z)^{-c}. \tag{4.19}$$

Proof. It is easy to show from Eq. (2.11) that, for all $n \in \mathbb{N}$, for all $b, x \in \mathbb{C}$ and for all $z \in \{z \in \mathbb{C} : |z| < 1\}$, the following identity holds

$$(x + 2zD_z)^n (1+z)^b = \sum_{k=0}^{+\infty} (2k + x)^n b_{(k)} \frac{z^k}{k!}.$$

Then

$$(1+z)^a (x + 2zD_z)^n (1+z)^b = \sum_{k=0}^{+\infty} \left(\sum_{l=0}^{k} \binom{k}{l} a_{(k-l)} b_{(l)} (2l + x)^n \right) \frac{z^k}{k!}.$$

Putting $x = c$, $a = n + c$, $b = -c$ and substituting $z \to -z$, we get

$$\begin{aligned} & (1-z)^{n+c} (c + 2zD_z)^n (1-z)^{-c} \\ = & \sum_{k=0}^{+\infty} \left((-1)^k \sum_{l=0}^{k} \binom{k}{l} (n+c)_{(k-l)} (-c)_{(l)} (2l + c)^n \right) \frac{z^k}{k!}. \end{aligned}$$

Due to the fact that $(1-z)^{n+c} (c + 2zD_z)^n (1-z)^{-c}$ is a polynomial of degree n in z, we must have that

$$(-1)^k \sum_{l=0}^{k} \binom{k}{l} (n+c)_{(k-l)} (-c)_{(l)} (2l + c)^n = 0,$$

for all $k \notin \mathbb{Z}_{+,n}$. Hence using Eq. (5.5) below, Eq. (3.3) and the fact that $B_{n,k}(c) \triangleq 0$, for all $k \notin \mathbb{Z}_{+,n}$, Eq. (4.19) follows. □

In particular, for $c = 1$, we obtain

$$(1-z)^{n+1} (1 + 2zD_z)^n (1-z)^{-1} = \sum_{k=0}^{n} B_{n+1,k+1} z^k, \tag{4.20}$$

and for $c = 2$, we obtain

$$(1-z)^{n+2} (1 + zD_z)^n (1-z)^{-2} = \sum_{k=0}^{n} A_{n+1,k+1} z^k. \tag{4.21}$$

These appear to be new generating expressions for the MacMahon and Eulerian numbers.

5 Properties of the $B_{n,k}(c)$

Proposition 5.1. *For all $m, n \in \mathbb{N}$ and for all $c \in \mathbb{C}$,*

$$\sum_{k=0}^{m} \binom{m}{k} (n+c)^{(m-k)} (k! B_{n,k}(c)) = c^{(m)} (2m+c)^n . \tag{5.1}$$

Proof. For all $c \in \mathbb{C}$ and for all $z \in \mathbb{C}$ such that $|z| < 1$ we have the absolutely convergent series (4.8). As $|z| < 1$, we can apply Eq. (2.10) and get

$$\sum_{m=0}^{+\infty} c^{(m)} (2m+c)^n \frac{z^m}{m!} = \sum_{k=0}^{n} k! B_{n,k}(c) \frac{z^k}{k!} \sum_{l=0}^{+\infty} (n+c)^{(l)} \frac{z^l}{l!}.$$

Interchanging the summation order gives

$$\sum_{m=0}^{+\infty} c^{(m)} (2m+c)^n \frac{z^m}{m!} = \sum_{l=0}^{+\infty} \sum_{k=0}^{n} \binom{k+l}{k} (n+c)^{(l)} k! B_{n,k}(c) \frac{z^{k+l}}{(k+l)!}.$$

With the definition $B_{n,k}(c) \triangleq 0$, for all $k \notin \mathbb{Z}_{+,n}$, we can write this as

$$\sum_{m=0}^{+\infty} c^{(m)} (2m+c)^n \frac{z^m}{m!} = \sum_{l=0}^{+\infty} \sum_{k=0}^{+\infty} \binom{k+l}{k} (n+c)^{(l)} k! B_{n,k}(c) \frac{z^{k+l}}{(k+l)!}.$$

This is equivalent to

$$\sum_{m=0}^{+\infty} c^{(m)} (2m+c)^n \frac{z^m}{m!} = \sum_{m=0}^{+\infty} \sum_{k=0}^{m} \binom{m}{k} (n+c)^{(m-k)} k! B_{n,k}(c) \frac{z^m}{m!},$$

and since z is arbitrary, Eq. (5.1) follows. □

In particular, for $c = 1$, we obtain

$$\sum_{k=0}^{m} \binom{n+m-k}{n} B_{n+1,k+1} = (2m+1)^n , \tag{5.2}$$

and for $c = 2$, we obtain

$$\sum_{k=0}^{m} \binom{n+1+m-k}{n+1} A_{n+1,k+1} = (m+1)^{n+1} . \tag{5.3}$$

These are well-known partial sums of the MacMahon and Eulerian number triangles [5, p. 328 and p. 331].

5.1 Expressions

Proposition 5.2. *For all $n \in \mathbb{N}$, for all $k \in \mathbb{N}_n$ and for all $c \in \mathbb{C}$,*

$$k!B_{n,k}(c) = \sum_{l=0}^{k} \binom{k}{l} \left(-\left(n+c\right)\right)^{(k-l)} c^{(l)} \left(2l+c\right)^n , \tag{5.4}$$

$$= (-1)^k \sum_{l=0}^{k} \binom{k}{l} \left(n+c\right)_{(k-l)} \left(-c\right)_{(l)} \left(2l+c\right)^n . \tag{5.5}$$

Proof. (i) We will show that Eq. (5.4) is a solution of Eq. (5.1). Substitute Eq. (5.4) in Eq. (5.1) and get

$$\sum_{k=0}^{m} \binom{m}{k} \left(n+c\right)^{(m-k)} \sum_{l=0}^{k} \binom{k}{l} \left(-\left(n+c\right)\right)^{(k-l)} c^{(l)} \left(2l+c\right)^n = c^{(m)} \left(2m+c\right)^n .$$

Interchanging the summation order gives

$$\sum_{l=0}^{m} \binom{m}{l} \left(\sum_{q=0}^{m-l} \binom{m-l}{q} \left(n+c\right)^{(m-l-q)} \left(-\left(n+c\right)\right)^{(q)} \right) c^{(l)} \left(2l+c\right)^n = c^{(m)} \left(2m+c\right)^n .$$

Due to the orthogonality relation (2.14) this simplifies to

$$\sum_{l=0}^{m} \binom{m}{l} \delta_{l=m} c^{(l)} \left(2l+c\right)^n = c^{(m)} \left(2m+c\right)^n ,$$

and this is an identity.

(ii) Use $z_{(n)} = (-1)^n \left(-z\right)^{(n)}$. □

In particular, for $c = -m \in \mathbb{Z}_-$,
(i) for $n \geq m$

$$B_{n,k}(-m) = (-1)^k \sum_{l=\max(0,k+m-n)}^{\min(k,m)} \binom{n-m}{k-l} \binom{m}{l} \left(2l-m\right)^n , \tag{5.6}$$

(ii) for $n < m$

$$B_{n,k}(-m) = \sum_{l=0}^{\min(k,m)} (-1)^l \binom{m-n-1+(k-l)}{k-l} \binom{m}{l} \left(2l-m\right)^n . \tag{5.7}$$

An equivalent form of Eqs. (5.4) and (5.5) is

$$k!B_{n,k}(c) = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \frac{\Gamma\left(n+c+1\right) \Gamma\left(c+l\right)}{\Gamma\left(n+c+1-(k-l)\right) \Gamma\left(c\right)} \left(2l+c\right)^n . \tag{5.8}$$

In particular, for $c=1$, we obtain

$$B_{n+1,k+1}=\sum_{l=0}^{k}(-1)^{k-l}\binom{n+1}{k-l}(2l+1)^{n}\,. \tag{5.9}$$

Expression (5.9) coincides with that given by MacMahon [5, p. 331]. For $c=2$ and using Eq. (3.25), we get the familiar result

$$A_{n+1,k+1}=\sum_{l=0}^{k}(-1)^{k-l}\binom{n+2}{k-l}(l+1)^{n+1}\,, \tag{5.10}$$

or equivalently, for all $n-1\in\mathbb{Z}_{+}$ and for all $k\in\mathbb{Z}_{+,n-1}$,

$$\left\langle {n-1 \atop k-1} \right\rangle = A_{n-1,k}=\sum_{l=0}^{k}(-1)^{l}\binom{n}{l}(k-l)^{n-1}\,. \tag{5.11}$$

Let $S(j,i)$ denote the Stirling numbers of the second kind (Sloane's A008277).

Proposition 5.3. *For all $n\in\mathbb{N}$, for all $k\in\mathbb{N}_{n}$ and for all $c\in\mathbb{C}$,*

$$B_{n,k}(c)=(-1)^{k}\sum_{j=0}^{n}\binom{n}{j}2^{j}c^{n-j}\sum_{i=0}^{\min(k,j)}(-1)^{i}\binom{n-i}{k-i}S(j,i)\,c^{(i)}. \tag{5.12}$$

Proof. Using Eqs. (5.15) and (5.16) from Proposition 5.4 below, we have

$$\begin{aligned}
B_{n,k}(c) &= \sum_{l=0}^{n}b_{n,k,l}c^{l},\\
&= (-1)^{k}\sum_{l=0}^{n}(-1)^{l}\sum_{p=0}^{l}(-1)^{p}\binom{n}{p}2^{n-p}\sum_{i=l-p}^{\min(k,n-p)}\binom{n-i}{k-i}S(n-p,i)\,s(i,l-p)\,c^{l}.
\end{aligned}$$

Interchanging the order of the first two summation gives

$$\begin{aligned}
B_{n,k}(c) &= (-1)^{k}\sum_{p=0}^{n}\sum_{l=p}^{n}(-1)^{l-p}\binom{n}{p}2^{n-p}\sum_{i=l-p}^{\min(k,n-p)}\binom{n-i}{k-i}S(n-p,i)\,s(i,l-p)\,c^{l},\\
&= (-1)^{k}\sum_{p=0}^{n}\sum_{l-p=0}^{n-p}(-1)^{l-p}\binom{n}{n-p}2^{n-p}\sum_{i=l-p}^{\min(k,n-p)}\binom{n-i}{k-i}S(n-p,i)\,s(i,l-p)\,c^{l-p+p},\\
&= (-1)^{k}\sum_{p=0}^{n}\sum_{j=0}^{n-p}(-1)^{j}\binom{n}{n-p}2^{n-p}\sum_{i=j}^{\min(k,n-p)}\binom{n-i}{k-i}S(n-p,i)\,s(i,j)\,c^{j+p},\\
&= (-1)^{k}\sum_{q=0}^{n}\binom{n}{q}2^{q}c^{n-q}\sum_{j=0}^{q}(-c)^{j}\sum_{i=j}^{\min(k,q)}\binom{n-i}{k-i}S(q,i)\,s(i,j)\,.
\end{aligned}$$

Taking into account that $S(n,k) = s(n,k) \triangleq 0$, for all $k \notin \mathbb{N}_n$, we can write this as

$$B_{n,k}(c) = (-1)^k \sum_{q=0}^{n} \binom{n}{q} 2^q c^{n-q} \sum_{j=0}^{q} (-c)^j \sum_{i=0}^{k} \binom{n-i}{k-i} S(q,i)\, s(i,j)\,.$$

Interchanging the two last summations yields

$$B_{n,k}(c) = (-1)^k \sum_{q=0}^{n} \binom{n}{q} 2^q c^{n-q} \sum_{i=0}^{k} \binom{n-i}{k-i} S(q,i) \sum_{j=0}^{i} s(i,j)\,(-c)^j.$$

Using the fundamental property of the Stirling numbers of the first kind [1, p. 824, 24.1.3, I, B, 1],

$$(-c)_{(i)} = \sum_{j=0}^{i} s(i,j)\,(-c)^j,$$

we obtain

$$B_{n,k}(c) = (-1)^k \sum_{q=0}^{n} \binom{n}{q} 2^q c^{n-q} \sum_{i=0}^{k} \binom{n-i}{k-i} S(q,i)\,(-c)_{(i)}.$$

By using the identity $(-c)_{(i)} = (-1)^i c^{(i)}$, writing j for q and replacing the upper limit in the second sum with $\min(k,j)$, Eq. (5.12) follows. □

In particular, for $c = 1$, we get

$$B_{n+1,k+1} = (-1)^k \sum_{j=0}^{n} \binom{n}{j} 2^j \sum_{i=0}^{\min(k,j)} (-1)^i \binom{n-i}{k-i} i!\,S(j,i)\,, \tag{5.13}$$

and for $c = 2$, we get

$$A_{n+1,k+1} = (-1)^k \sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{\min(k,j)} (-1)^i \binom{n-i}{k-i} (i+1)!\,S(j,i)\,. \tag{5.14}$$

These appear to be new expressions for the MacMahon and Eulerian numbers, in terms of Stirling numbers of the second kind.

5.2 Polynomial expression

Proposition 5.4. *For all $n \in \mathbb{N}$ and for all $z \in \mathbb{C}$,*

$$B_{n,k}(c) = \sum_{l=0}^{n} b_{n,k,l}\, c^l, \tag{5.15}$$

where, for all $k, l \in \mathbb{N}_n$,

$$b_{n,k,l} = (-1)^{k+l} \sum_{p=\max(0,l-k)}^{l} (-1)^p \binom{n}{p} 2^{n-p} \sum_{i=l-p}^{\min(k,n-p)} \binom{n-i}{k-i} s(i,l-p)\, S(n-p,i)\,. \tag{5.16}$$

Proof. Using

$$(c+2zD_z)^n = \sum_{l=0}^{n} \binom{n}{l} c^{n-l} 2^l (zD_z)^l ,$$

we get

$$(1-z)^{n+c}(c+2zD_z)^n (1-z)^{-c} = (1-z)^{n+c} \sum_{l=0}^{n} \binom{n}{l} c^{n-l} 2^l (zD_z)^l (1-z)^{-c} .$$

Using herein the formula [9, p. 144],

$$(zD_z)^l = \sum_{p=0}^{l} S(l,p) z^p D_z^p,$$

we obtain

$$\begin{aligned}
& (1-z)^{n+c}(c+2zD_z)^n (1-z)^{-c} \\
= \;& (1-z)^{n+c} \sum_{l=0}^{n} \binom{n}{l} c^{n-l} 2^l \sum_{p=0}^{l} S(l,p) z^p D_z^p (1-z)^{-c} , \\
= \;& \sum_{l=0}^{n} \binom{n}{l} c^{n-l} 2^l \sum_{p=0}^{l} S(l,p) z^p c^{(p)} (1-z)^{n-p} , \\
= \;& \sum_{l=0}^{n} \binom{n}{l} c^{n-l} 2^l \sum_{p=0}^{l} S(l,p) (-c)_{(p)} (1-z)^{n-p} (-z)^p , \\
= \;& \sum_{p=0}^{n} \sum_{l=p}^{n} \binom{n}{l} c^{n-l} 2^l S(l,p) (-c)_{(p)} (1-z)^{n-p} (-z)^p .
\end{aligned}$$

Substituting herein the binomial expansion for $(1-z)^{n-p}$ gives

$$\begin{aligned}
& (1-z)^{n+c}(c+2zD_z)^n (1-z)^{-c} \\
= \;& \sum_{p=0}^{n} \sum_{l=p}^{n} \binom{n}{l} c^{n-l} 2^l S(l,p) (-c)_{(p)} \sum_{q=0}^{n-p} \binom{n-p}{q} (-z)^q (-z)^p , \\
= \;& \sum_{p=0}^{n} \sum_{l=p}^{n} \sum_{m=p}^{n} \binom{n}{l} \binom{n-p}{m-p} c^{n-l} 2^l (-c)_{(p)} S(l,p) (-z)^m , \\
= \;& \sum_{p=0}^{n} \sum_{m=p}^{n} \sum_{l=p}^{n} \binom{n}{l} \binom{n-p}{m-p} c^{n-l} 2^l (-c)_{(p)} S(l,p) (-z)^m , \\
= \;& \sum_{m=0}^{n} \sum_{p=0}^{m} \binom{n-p}{m-p} (-c)_{(p)} \sum_{l=p}^{n} \binom{n}{l} c^{n-l} 2^l S(l,p) (-z)^m .
\end{aligned}$$

Making use of the fundamental relation of the Stirling numbers of the first kind,

$$(-c)_{(p)} = \sum_{k=0}^{p} s(p,k) (-c)^k ,$$

we get

$$
\begin{aligned}
& (1-z)^{n+c}(c+2zD_z)^n(1-z)^{-c} \\
= & \sum_{m=0}^{n}\sum_{p=0}^{m}\tbinom{n-p}{m-p}\sum_{k=0}^{p}(-1)^k s(p,k)\sum_{q=p}^{n}\tbinom{n}{q}2^q S(q,p)\,c^{n-q+k}(-z)^m, \\
= & \sum_{m=0}^{n}\sum_{p=0}^{m}\tbinom{n-p}{m-p}\sum_{k=0}^{p}(-1)^k s(p,k)\sum_{l=0}^{n-p}\tbinom{n}{n-l}2^{n-l} S(n-l,p)\,c^{l+k}(-z)^m.
\end{aligned}
$$

Summing over diagonals in the two last summations and putting $S(n,k)=s(n,k)\triangleq 0$, for all $k\notin\mathbb{N}_n$, this becomes

$$
\begin{aligned}
& (1-z)^{n+c}(c+2zD_z)^n(1-z)^{-c} \\
= & \sum_{m=0}^{n}\sum_{p=0}^{m}\tbinom{n-p}{m-p}\sum_{q=0}^{n}\sum_{l=0}^{n-p}(-1)^{q-l}\tbinom{n}{l}2^{n-l}S(n-l,p)\,s(p,q-l)\,c^q(-z)^m, \\
= & \sum_{m=0}^{n}\sum_{q=0}^{n}\sum_{p=0}^{m}\sum_{l=0}^{n-p}\tbinom{n-p}{m-p}(-1)^{q-l}\tbinom{n}{l}2^{n-l}S(n-l,p)\,s(p,q-l)\,c^q(-z)^m, \\
= & \sum_{m=0}^{n}\sum_{q=0}^{n}\sum_{p=0}^{m}\sum_{l=0}^{n}\tbinom{n-p}{m-p}(-1)^{q-l}\tbinom{n}{l}2^{n-l}S(n-l,p)\,s(p,q-l)\,c^q(-z)^m, \\
= & \sum_{m=0}^{n}\sum_{q=0}^{n}\sum_{l=0}^{n}\sum_{p=0}^{m}\tbinom{n-p}{m-p}(-1)^{q-l}\tbinom{n}{l}2^{n-l}S(n-l,p)\,s(p,q-l)\,c^q(-z)^m, \\
= & \sum_{m=0}^{n}\sum_{q=0}^{n}\sum_{l=0}^{n}(-1)^{q-l}\tbinom{n}{l}2^{n-l}\sum_{p=\max(0,q-l)}^{\min(m,n-l)}\tbinom{n-p}{m-p}S(n-l,p)\,s(p,q-l)\,c^q(-z)^m.
\end{aligned}
$$

Renaming indexes gives

$$
\begin{aligned}
& (1-z)^{n+c}(c+2zD_z)^n(1-z)^{-c} \\
= & \sum_{k=0}^{n}(-1)^k\sum_{l=0}^{n}(-1)^l\sum_{j=0}^{n}\tbinom{n}{j}(-1)^j \\
& \left(2^{n-j}\sum_{i=\max(0,l-n+n-j)}^{\min(k,n-j)}\tbinom{n-i}{k-i}S(n-j,i)\,s(i,l-n+n-j)\right)c^l z^k, \\
= & \sum_{k=0}^{n}(-1)^k\sum_{l=0}^{n}(-1)^l\sum_{j=0}^{n}\tbinom{n}{j}(-1)^{n-j} \\
& \left(2^{j}\sum_{i=\max(0,j+l-n)}^{\min(k,j)}\tbinom{n-i}{k-i}S(j,i)\,s(i,j+l-n)\right)c^l z^k.
\end{aligned}
$$

Applying Proposition 4.2 and identification with

$$B_n(1,z;c) = \sum_{k=0}^{n} \left(\sum_{l=0}^{n} b_{n,k,l} c^l \right) z^k,$$

yields, for all $k, l \in \mathbb{N}_n$,

$$b_{n,k,l} = (-1)^{n+k+l} \sum_{j=n-l}^{n} (-1)^j \binom{n}{j} \left(2^j \sum_{i=j-(n-l)}^{\min(k,j)} \binom{n-i}{k-i} S(j,i)\, s(i, j-(n-l)) \right).$$

This can be rewritten as

$$\begin{aligned}
b_{n,k,l} &= (-1)^{n+l+k} \sum_{j=n-l}^{n} (-1)^j \binom{n}{j} 2^j \sum_{i=j-(n-l)}^{\min(k,j)} \binom{n-i}{k-i} S(j,i)\, s(i, j-(n-l)), \\
&= (-1)^k \sum_{j-(n-l)=0}^{l} (-1)^{j-(n-l)} \binom{n}{j-(n-l)+(n-l)} \\
&\quad 2^{j-(n-l)+(n-l)} \sum_{i=j-(n-l)}^{\min(k,j-(n-l)+(n-l))} \binom{n-i}{k-i} S(j-(n-l)+(n-l), i)\, s(i, j-(n-l)), \\
&= (-1)^k \sum_{q=0}^{l} (-1)^q \binom{n}{q+(n-l)} \\
&\quad 2^{q+(n-l)} \sum_{i=q}^{\min(k,q+(n-l))} \binom{n-i}{k-i} S(q+(n-l), i)\, s(i,q), \\
&= (-1)^k \sum_{q=0}^{l} (-1)^q \binom{n}{l-q} 2^{n-(l-q)} \sum_{i=q}^{\min(k,n-(l-q))} \binom{n-i}{k-i} S(n-(l-q), i)\, s(i,q), \\
&= (-1)^k \sum_{p=\max(0,l-k)}^{l} (-1)^{l-p} \binom{n}{p} 2^{n-p} \sum_{i=l-p}^{\min(k,n-p)} \binom{n-i}{k-i} S(n-p, i)\, s(i, l-p).
\end{aligned}$$

□

Using basic properties of the Stirling numbers, it is easy to derive the following special values for the $b_{n,k,l}$,

$$b_{n,n,l} = b_{n,0,l} = \delta_{n=l}, \tag{5.17}$$

$$b_{n,k,n} = \binom{n}{k}, \tag{5.18}$$

$$b_{n,k,0} = \delta_{k=0}\delta_{n=0}. \tag{5.19}$$

5.3 Symmetry

Recall Eq. (5.5) and a variant of it obtained by replacing k with $n-k$,

$$\begin{aligned} k!B_{n,k}(c) &= (-1)^k \sum_{l=0}^{k} \tbinom{k}{l} \, (n+c)_{(k-l)} \, (-c)_{(l)} \, (2l+c)^n \,, \\ (n-k)!B_{n,n-k}(c) &= (-1)^{n-k} \sum_{l=0}^{n-k} \tbinom{n-k}{l} \, (n+c)_{(n-k-l)} \, (-c)_{(l)} \, (2l+c)^n \,. \end{aligned}$$

Due to the symmetry $B_{n,k}(c) = B_{n,n-k}(c)$, there must hold, for all $n \in \mathbb{N}$, for all $k \in \mathbb{N}_n$ and for all $c \in \mathbb{C}$, that

$$\begin{aligned} & \frac{(-1)^k}{k!} \sum_{l=0}^{k} \tbinom{k}{l} \, (n+c)_{(k-l)} \, (-c)_{(l)} \, (2l+c)^n \\ = \; & \frac{(-1)^{n-k}}{(n-k)!} \sum_{l=0}^{n-k} \tbinom{n-k}{l} \, (n+c)_{(n-k-l)} \, (-c)_{(l)} \, (2l+c)^n \,. \end{aligned} \tag{5.20}$$

In particular, for $k=0$, Eq. (5.20) yields

$$\frac{(-1)^n}{n!} \sum_{l=0}^{n} (-1)^l \tbinom{n}{l} \frac{c+n}{c+l} \, (c+2l)^n = \frac{c^n}{c^{(n)}}, \tag{5.21}$$

and for $k=1$,

$$\begin{aligned} & \frac{(-1)^n}{n!} \sum_{l=0}^{n} (-1)^l \tbinom{n}{l} \frac{(c+n+1)\,(c+n)}{(c+l+1)(c+l)} \, (c+2l)^{n+1} \\ = \; & \frac{c\,(c+2)^{n+1} - (c+n+1)\,c^{n+1}}{c^{(n)}}. \end{aligned} \tag{5.22}$$

5.4 A result of Ruiz

Summing Eq. (5.8) over k from 0 to n and using result (3.18), we get, for all $n \in \mathbb{N}$ and for all $c \in \mathbb{C}$,

$$\begin{aligned} c^{(n)} &= 2^{-n} \sum_{k=0}^{n} B_{n,k}(c), \\ &= \sum_{k=0}^{n} \sum_{l=0}^{k} (-1)^{k-l} \tbinom{n+c}{k-l} \tbinom{c-1+l}{l} \, (l+c/2)^n \,, \end{aligned}$$

or

$$\sum_{l=0}^{n} \left(\sum_{q=0}^{n-l} (-1)^q \tbinom{n+c}{q} \right) \tbinom{c-1+l}{l} \, (l+c/2)^n = c^{(n)}.$$

Using the binomial identity

$$\sum_{q=0}^{n-l}(-1)^q\binom{n+c}{q} = (-1)^{n-l}\binom{n+c-1}{n-l},$$

we get

$$\sum_{l=0}^{n}(-1)^{n-l}\binom{n+c-1}{n-l}\binom{c-1+l}{l}(l+c/2)^n = c^{(n)},$$

or

$$\binom{n+c-1}{n}\sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}(l+c/2)^n = c^{(n)},$$

or

$$\sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}(l+c/2)^n = n!,$$

or finally

$$\frac{1}{n!}\sum_{l=0}^{n}(-1)^{l}\binom{n}{l}((-c/2)-l)^n = 1. \tag{5.23}$$

Identity (5.23) is a result of Ruiz [8]. Written in the form

$$\frac{1}{n!}\sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}(c+2l)^n = 2^n, \tag{5.24}$$

it can be regarded as the first identity in a series of identities of which Eqs. (5.21) and (5.22) are the next two successors. Ruiz's result however is special because the sum in Eq. (5.24) is independent of c.

In addition, applying D_c^m to Eq. (5.24) we obtain the following derived identities, for all $n \in \mathbb{N}$, for all $m \in \mathbb{N}_n$ and for all $c \in \mathbb{C}$,

$$\sum_{l=0}^{n}(-1)^{l}\binom{n}{l}(c+2l)^{n-m} = (-1)^n\, 2^n n!\delta_{m=0}. \tag{5.25}$$

6 Properties of the $b_{n,k,l}$

Due to the symmetry relation $B_{n,k}(c) = B_{n,n-k}(c)$, we have that $b_{n,n-k,l} = b_{n,k,l}$, for all $n \in \mathbb{N}$ and for all $k, l \in \mathbb{N}_n$.

6.1 Recursion relation for the $b_{n,k,l}$

Proposition 6.1. *For all $n \in \mathbb{N}$ and for all $k, l \in \mathbb{N}_n$,*

$$b_{n+1,k+1,l+1} = 2(k+1)b_{n,k+1,l+1} + b_{n,k+1,l} + 2(n-k)b_{n,k,l+1} + b_{n,k,l}, \tag{6.1}$$

with $b_{0,0,0} = 1$ and $b_{n,k,l} \triangleq 0$ if $k, l \notin \mathbb{Z}_{+,n}$.

Proof. ¿From the fact that $B_{n,k}(0) = \delta_{n=0}$ we find that $b_{n,k,0} = \delta_{k=0}\delta_{n=0}$ and hence $b_{0,0,0} = 1$.

Substituting Eq. (5.15) in Eq. (3.4), we get

$$\sum_{l=0}^{n+1} b_{n+1,k+1,l}c^l = (2(k+1)+c)\sum_{l=0}^{n} b_{n,k+1,l}c^l + (2(n-k)+c)\sum_{l=0}^{n} b_{n,k,l}c^l.$$

This is equivalent to

$$\begin{aligned}\sum_{l=0}^{n+1} b_{n+1,k+1,l}c^l &= \sum_{l=0}^{n} b_{n,k+1,l}c^{l+1} + \sum_{l=0}^{n} b_{n,k,l}c^{l+1} \\ &\quad +2(k+1)\sum_{l=0}^{n} b_{n,k+1,l}c^l + 2(n-k)\sum_{l=0}^{n} b_{n,k,l}c^l,\end{aligned}$$

or

$$\begin{aligned}\sum_{l=0}^{n+1} b_{n+1,k+1,l}c^l &= \sum_{q=1}^{n+1} b_{n,k+1,q-1}c^q + \sum_{q=1}^{n+1} b_{n,k,q-1}c^q \\ &\quad +2(k+1)\sum_{l=0}^{n} b_{n,k+1,l}c^l + 2(n-k)\sum_{l=0}^{n} b_{n,k,l}c^l,\end{aligned}$$

or, because $b_{n,k,l} \triangleq 0$ if $k, l \notin \mathbb{Z}_{+,n}$, we get

$$\begin{aligned}\sum_{l=0}^{n+1} b_{n+1,k+1,l}c^l &= \sum_{q=0}^{n+1} b_{n,k+1,q-1}c^q + \sum_{q=0}^{n+1} b_{n,k,q-1}c^q \\ &\quad +2(k+1)\sum_{l=0}^{n+1} b_{n,k+1,l}c^l + 2(n-k)\sum_{l=0}^{n+1} b_{n,k,l}c^l.\end{aligned}$$

This holds for all c, so we obtain

$$b_{n+1,k+1,l} = 2(k+1)b_{n,k+1,l} + b_{n,k+1,l-1} + 2(n-k)b_{n,k,l} + b_{n,k,l-1}.$$

□

6.2 Partial sums of the $b_{n,k,l}$

The following result shows that various partial sums over the number pyramid $b_{n,k,l}$ are related to several important number triangles.

Proposition 6.2. *For all $n \in \mathbb{N}$ and for all $k, l \in \mathbb{N}_n$,*

$$\sum_{l=0}^{n} b_{n,k,l} = B_{n+1,k+1}, \tag{6.2}$$

$$\sum_{l=0}^{n} (-1)^l b_{n,k,l} = (-1)^{n-k} \left(\binom{n}{k} - 2 o_n \delta_{k>0} \binom{n-1}{k-1} \right), \tag{6.3}$$

$$\sum_{k=0}^{n} b_{n,k,l} = (-1)^{n-l} 2^n s(n,l), \tag{6.4}$$

$$\sum_{k=0}^{n} (-1)^k b_{n,k,l} = e_n (-1)^{n/2} 2^n \delta_{l \le n/2} T_{n/2,l}. \tag{6.5}$$

Proof. (i) This immediately follows from Eq. (3.20).

(ii) We have, for all $z, t \in \mathbb{C}$,

$$\begin{aligned}
& B^{-1}(1,z,t) \\
= \; & \frac{1}{1-z} e^{-\frac{1-z}{2}t} - \frac{z}{1-z} e^{+\frac{1-z}{2}t}, \\
= \; & \sum_{n=0}^{+\infty} 2^{-n} (-1)^n (1-z)^{n-1} \frac{t^n}{n!} - z \sum_{n=0}^{+\infty} 2^{-n} (1-z)^{n-1} \frac{t^n}{n!}, \\
= \; & \sum_{n=0}^{+\infty} 2^{-n} \left((-1)^n - z \right) (1-z)^{n-1} \frac{t^n}{n!}, \\
= \; & \sum_{n=0}^{+\infty} 2^{-n} \left(\delta_{n=0} + \delta_{n>0} \left((-1)^n - z \right) \sum_{k=0}^{n-1} \binom{n-1}{k} (-z)^k \right) \frac{t^n}{n!}, \\
= \; & \sum_{n=0}^{+\infty} 2^{-n} \left(\delta_{n=0} + \delta_{n>0} \left((-1)^n \sum_{k=0}^{n-1} \binom{n-1}{k} (-z)^k + \sum_{k=0}^{n-1} \binom{n-1}{k} (-z)^{k+1} \right) \right) \frac{t^n}{n!}, \\
= \; & \sum_{n=0}^{+\infty} 2^{-n} \left(\delta_{n=0} + \delta_{n>0} \left((-1)^n \sum_{k=0}^{n-1} \binom{n-1}{k} (-z)^k + \sum_{l=1}^{n} \binom{n-1}{l-1} (-z)^l \right) \right) \frac{t^n}{n!},
\end{aligned}$$

or

$$\begin{aligned}
& B^{-1}(1,z,t) \\
= \; & \sum_{n=0}^{+\infty} 2^{-n} \left(\delta_{n=0} + \delta_{n>0} \left(\begin{array}{c} (-1)^n \\ + \sum_{k=1}^{n-1} \left((-1)^n \binom{n-1}{k} + \binom{n-1}{k-1} \right) (-z)^k \\ +(-z)^n \end{array} \right) \right) \frac{t^n}{n!}.
\end{aligned}$$

With the identity

$$(-1)^n \binom{n-1}{k} + \binom{n-1}{k-1} = \left((-1)^n + (1 - (-1)^n) \frac{k}{n} \right) \binom{n}{k},$$

the expression inside the first pair of parentheses becomes

$$
\begin{aligned}
&\delta_{n=0} + \delta_{n>0} \left((-1)^n + \sum_{k=1}^{n-1} \left((-1)^n + (1-(-1)^n)\, \frac{k}{n} \right) \tbinom{n}{k} (-z)^k + (-z)^n \right) \\
= \;& \sum_{k=0}^{n} \left(\delta_{n=0} + \delta_{n>0} \left((-1)^n + (1-(-1)^n)\, \frac{k}{n} \right) \right) \tbinom{n}{k} (-z)^k, \\
= \;& \sum_{k=0}^{n} (-1)^{n-k} \tbinom{n}{k} \left(1 - \delta_{n>0} \, (1-(-1)^n)\, \frac{k}{n} \right) z^k,
\end{aligned}
$$

and we get for the full expression

$$
B^{-1}(1,z,t) = \sum_{n=0}^{+\infty} 2^{-n} \sum_{k=0}^{n} (-1)^{n-k} \tbinom{n}{k} \left(1 - \delta_{n>0} \, (1-(-1)^n)\, \frac{k}{n} \right) z^k \frac{t^n}{n!},
$$

or

$$
B^{-1}(1,z,t) = \sum_{n=0}^{+\infty} 2^{-n} \sum_{k=0}^{n} (-1)^{n-k} \left(\tbinom{n}{k} - 2 o_n \delta_{k>0} \tbinom{n-1}{k-1} \right) z^k \frac{t^n}{n!}.
$$

Identifying this expression with

$$
B^{-1}(1,z,t) = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left(2^{-n} \sum_{l=0}^{n} b_{n,k,l} (-1)^l \right) z^k \frac{t^n}{n!}
$$

yields

$$
\sum_{l=0}^{n} (-1)^l b_{n,k,l} = (-1)^{n-k} \left(\tbinom{n}{k} - 2 o_n \delta_{k>0} \tbinom{n-1}{k-1} \right).
$$

(iii) Applying Eq. (3.18) to the following double sum we obtain, for all $c \in \mathbb{C}$,

$$
\sum_{k=0}^{n} \sum_{l=0}^{n} b_{n,k,l} c^l = \sum_{k=0}^{n} B_{n,k}(c) = 2^n c^{(n)},
$$

or

$$
\sum_{l=0}^{n} \left((-1)^{n-l} 2^{-n} \sum_{k=0}^{n} b_{n,k,l} \right) c^l = c_{(n)}.
$$

Identifying this with the definition of the Stirling numbers of the first kind gives

$$
(-1)^{n-l} 2^{-n} \sum_{k=0}^{n} b_{n,k,l} = s(n,l).
$$

(iv) This result is related to the Maclaurin series of $\operatorname{sech}^c t$, considered in the next section. There it is shown that (i)

$$
\sum_{k=0}^{2m+1} (-1)^k b_{2m+1,k,l} = 0,
$$

and (ii) (see Proposition 7.1 below)

$$\sum_{k=0}^{n}(-1)^k b_{n,k,l} = (-1)^{n/2}2^n e_n \delta_{l\le n/2} T_{n/2,l},$$

where the number triangle $T_{m,l}$ is Sloane's sequence A088874. □

We can add to the sums given by Proposition 6.2, Eq. (3.25), which expressed in terms of the $b_{n,k,l}$ reads

$$\sum_{l=0}^{n} b_{n,k,l}2^l = 2^n A_{n+1,k+1}. \tag{6.6}$$

6.3 The numbers $T_{n,k}$

We now give an explicit expression for the numbers A088874.

Proposition 6.3. *For all $n \in \mathbb{N}$ and for all $l \in \mathbb{N}_n$,*

$$e_n \delta_{l\le n/2}(-1)^{n/2-l} T_{n/2,l} = \sum_{p=0}^{l}(-1)^p \tbinom{n}{p} w_{n-p,l-p}, \tag{6.7}$$

wherein

$$w_{n,m} \triangleq 2^n \sum_{k=m}^{n} S(n,k)\, s(k,m)\, (1/2)^k. \tag{6.8}$$

Proof. Summing over k from 0 to n in the expression for $b_{n,k,l}$ given by Proposition 5.4, we get

$$\sum_{k=0}^{n}(-1)^k b_{n,k,l} = \sum_{p=0}^{l}(-1)^{l-p}\tbinom{n}{p}2^{n-p}\sum_{k=0}^{n}\sum_{i=l-p}^{\min(k,n-p)}\tbinom{n-i}{k-i} S(n-p,i)\, s(i,l-p).$$

Using $S(n,k) = s(n,k) \triangleq 0$, $\forall k \notin \mathbb{N}_n$, we can write this as

$$\sum_{k=0}^{n}(-1)^k b_{n,k,l} = \sum_{p=0}^{l}(-1)^{l-p}\tbinom{n}{p}2^{n-p}\sum_{k=0}^{n}\sum_{i=0}^{k}\tbinom{n-i}{k-i} S(n-p,i)\, s(i,l-p).$$

Interchanging the last two summations gives

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k b_{n,k,l} &= \sum_{p=0}^{l}(-1)^{l-p}\tbinom{n}{p}2^{n-p}\sum_{i=0}^{n}\left(\sum_{k=i}^{n}\tbinom{n-i}{k-i}\right) S(n-p,i)\, s(i,l-p),\\ &= \sum_{p=0}^{l}(-1)^{l-p}\tbinom{n}{p}2^{n-p}\sum_{i=0}^{n}2^{n-i} S(n-p,i)\, s(i,l-p),\end{aligned}$$

or

$$2^{-n}\sum_{k=0}^{n}(-1)^k b_{n,k,l} = \sum_{p=0}^{l}(-1)^{l-p}\tbinom{n}{p}\sum_{i=l-p}^{n-p}2^{n-p-i} S(n-p,i)\, s(i,l-p).$$

Using herein definition (6.8) and the result (7.8) from Proposition 7.1 below, Eq. (6.7) follows. □

n\ k	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	2	3						
3	0	16	30	15					
4	0	272	588	420	105				
5	0	7 936	18 960	16 380	6 300	945			
6	0	353 792	911 328	893 640	429 660	103 950	10 395		
7	0	22 368 256	61 152 000	65 825 760	36 636 600	11 351 340	1 891 890	135 135	
8	0	1 903 757 312	5 464 904 448	6 327 135 360	3 918 554 640	1 427 025 600	310 269 960	37 837 800	2 027 025

Table 3: The number triangle $T_{n,k}$

We give the onset of the number triangle $T_{n,k}$ in Table 3. In particular, $T_{n,0} = \delta_{n=0}$, $T_{n,1}$ are the tangent numbers (Sloane's A000182), and $T_{n,n} = 1.3.5...(2n-1)$ are the double factorial numbers (Sloane's A001147).

7 The Maclaurin series of $\operatorname{sech}^c t$

Substituting $x = -y = 1$ in Eqs. (3.2) and (3.3), and using Eq. (3.15), shows that the Maclaurin series of the c-th power of $\operatorname{sech} t$ is given by, for all $c \in \mathbb{C}$,

$$\operatorname{sech}^c t = \sum_{n=0}^{+\infty} p_n(c) \frac{t^n}{n!}, \tag{7.1}$$

where

$$\begin{aligned} p_n(c) &= 2^{-n} \sum_{k=0}^{n} (-1)^k B_{n,k}(c), &(7.2)\\ &= \sum_{l=0}^{n} \left(2^{-n} \sum_{k=0}^{n} (-1)^k b_{n,k,l} \right) c^l. &(7.3) \end{aligned}$$

The polynomials $p_n(c)$ have the following properties.

(i) $p_{2k+1}(c) = 0$, for all $k \in \mathbb{N}$ and for all $c \in \mathbb{C}$, because $\operatorname{sech}^c t$ is even in t.

(ii) $p_n(c)$ has degree $n/2$. This can be seen as follows. We have, for all $m \in \mathbb{N}$, $\lim_{c \to 0} D_c^m \operatorname{sech}^c t = (\ln \operatorname{sech} t)^m$ and $(\ln \operatorname{sech} t)^m = O(t^{2m})$. Hence $p_n(c)$ must have degree $n/2$.

(iii) $p_n(0) = \delta_{n=0}$.

Then due to (i) we have, for all $m \in \mathbb{N}$,

$$\sum_{k=0}^{2m+1} (-1)^k b_{2m+1,k,l} = 0, \tag{7.4}$$

and we define, in accordance with (ii),

$$\sum_{k=0}^{2m}(-1)^k b_{2m,k,l} \triangleq (-1)^m 2^{2m}\delta_{l\leq m}T_{m,l}. \tag{7.5}$$

Hence

$$p_{2m}(c) = \lim_{t\to 0} D_t^{2m}\operatorname{sech}^c t = (-1)^m \sum_{l=0}^{m} T_{m,l}c^l, \tag{7.6}$$

so

$$\operatorname{sech}^c t = \sum_{m=0}^{+\infty}\left((-1)^m \sum_{l=0}^{m} T_{m,l}c^l\right)\frac{t^{2m}}{(2m)!}. \tag{7.7}$$

Due to (iii), $T_{n,0} = \delta_{n=0}$, for all $n \in \mathbb{N}$.

We now clarify the nature of the numbers $T_{m,l}$ defined in Eq. (7.5).

Proposition 7.1. *The number triangle $T_{n,l}$, for all $n \in \mathbb{N}$ and for all $l \in \mathbb{N}_n$, satisfies the recursion relation*

$$T_{n+1,l+1} = \sum_{p=0}^{n-l} 2^p\left(\binom{p+l+1}{p+1} + \delta_{l>0}\binom{p+l}{p+1}\right)T_{n,l+p}, \tag{7.8}$$

with $T_{n,0} = \delta_{n=0}$.

Proof. Starting from the identity

$$D_t^2 \operatorname{sech}^c t = c^2 \operatorname{sech}^c t - c(c+1)\operatorname{sech}^{c+2} t,$$

and substituting herein the series expansion for $\operatorname{sech}^c t$, Eq. (7.7), we get

$$\begin{aligned} &\sum_{n=1}^{+\infty}(-1)^n\sum_{l=0}^{n}T_{n,l}c^l\frac{t^{2(n-1)}}{(2(n-1))!} \\ = \;& c^2\sum_{n=0}^{+\infty}(-1)^n\sum_{l=0}^{n}T_{n,l}c^l\frac{t^{2n}}{(2n)!} - c(c+1)\sum_{n=0}^{+\infty}(-1)^n\sum_{l=0}^{n}T_{n,l}(c+2)^l\frac{t^{2n}}{(2n)!}. \end{aligned}$$

As this holds for all t, we must have that

$$-\sum_{l=0}^{n+1}T_{n+1,l}c^l = c^2\sum_{l=0}^{n}T_{n,l}c^l - c(c+1)\sum_{l=0}^{n}T_{n,l}(c+2)^l.$$

This can be rearranged in the form

$$\sum_{l=0}^{n}T_{n+1,l+1}c^l = (c+1)\sum_{l=0}^{n}T_{n,l}(c+2)^l - \sum_{l=0}^{n}T_{n,l}c^{l+1}.$$

By the binomial theorem,

$$\begin{aligned}\sum_{l=0}^{n} T_{n+1,l+1}c^l &= \sum_{l=0}^{n} T_{n,l} \sum_{k=0}^{l} \tbinom{l}{k} 2^{l-k} c^{k+1} \\ &\quad + \sum_{l=0}^{n} T_{n,l} \sum_{k=0}^{l} \tbinom{l}{k} 2^{l-k} c^{k} - \sum_{l=0}^{n} T_{n,l}c^{l+1}.\end{aligned}$$

Interchanging the order of summation in the double sum terms gives

$$\begin{aligned}\sum_{l=0}^{n} T_{n+1,l+1}c^l &= \sum_{k=0}^{n}\sum_{q=0}^{n-k} 2^q \tbinom{q+k}{k} T_{n,q+k}c^{k+1} \\ &\quad + \sum_{k=0}^{n}\sum_{q=0}^{n-k} 2^q \tbinom{q+k}{k} T_{n,q+k}c^{k} - \sum_{k=1}^{n+1} T_{n,k-1}c^k.\end{aligned}$$

We can rearrange this further into

$$\begin{aligned}\sum_{l=0}^{n} T_{n+1,l+1}c^l &= \sum_{l=1}^{n+1}\sum_{q=0}^{n-(l-1)} 2^q \tbinom{q+l-1}{l-1} T_{n,q+l-1}c^{l} \\ &\quad + \sum_{l=0}^{n}\sum_{q=0}^{n-l} 2^q \tbinom{q+l}{l} T_{n,q+l}c^{l} - \sum_{l=1}^{n+1} T_{n,l-1}c^l.\end{aligned}$$

As this holds for all c, we must have that

$$\begin{aligned}T_{n+1,1} &= \sum_{q=0}^{n} 2^q T_{n,q}; \\ \sum_{l=1}^{n} T_{n+1,l+1}c^l &= \sum_{l=1}^{n} \left(\begin{array}{c} \sum_{q=0}^{n-(l-1)} 2^q \tbinom{q+l-1}{l-1} T_{n,q+l-1} \\ + \sum_{q=0}^{n-l} 2^q \tbinom{q+l}{l} T_{n,q+l} - T_{n,l-1} \end{array} \right) c^l \\ &\quad + T_{n,n}c^{n+1} - T_{n,n}c^{n+1},\end{aligned}$$

or

$$\begin{aligned}T_{n+1,1} &= \sum_{q=0}^{n} 2^q T_{n,q}; \\ T_{n+1,l+1} &= \sum_{q=0}^{n-(l-1)} 2^q \tbinom{q+l-1}{l-1} T_{n,q+l-1} + \sum_{q=0}^{n-l} 2^q \tbinom{q+l}{l} T_{n,q+l} - T_{n,l-1}, l \in \mathbb{Z}_{+,n},\end{aligned}$$

or

$$\begin{aligned}T_{n+1,1} &= \sum_{q=0}^{n} 2^q T_{n,q}; \\ T_{n+1,l+1} &= \sum_{q=1}^{n-l+1} 2^q \tbinom{q+l-1}{l-1} T_{n,q+l-1} + \sum_{q=0}^{n-l} 2^q \tbinom{q+l}{l} T_{n,q+l}, l \in \mathbb{Z}_{+,n},\end{aligned}$$

or

$$\begin{aligned} T_{n+1,1} &= \sum_{q=0}^{n} 2^q T_{n,q}; \\ T_{n+1,l+1} &= \sum_{p=0}^{n-l} 2^{p+1}\tbinom{p+l}{l-1} T_{n,p+l} + \sum_{p=0}^{n-l} 2^p \tbinom{p+l}{l} T_{n,p+l}, l \in \mathbb{Z}_{+,n}, \end{aligned}$$

or

$$\begin{aligned} T_{n+1,1} &= \sum_{p=0}^{n} 2^p T_{n,p}, l = 0; \\ T_{n+1,l+1} &= \sum_{p=0}^{n-l} 2^p \left(2\tbinom{p+l}{l-1} + \tbinom{p+l}{l}\right) T_{n,p+l}, l \in \mathbb{Z}_{+,n}, \end{aligned}$$

or

$$\begin{aligned} T_{n+1,1} &= \sum_{p=0}^{n} 2^p T_{n,p}, l = 0; \\ T_{n+1,l+1} &= \sum_{p=0}^{n-l} 2^p \left(2\tbinom{p+l}{p+1} + \tbinom{p+l}{p}\right) T_{n,p+l}, l \in \mathbb{Z}_{+,n}. \end{aligned}$$

With the basic additive (Pascal's first) binomial identity

$$\tbinom{p+l}{p+1} + \tbinom{p+l}{p} = \tbinom{p+l+1}{p+1},$$

we can write this also as

$$\begin{aligned} T_{n+1,1} &= \sum_{p=0}^{n} 2^p T_{n,p}; \\ T_{n+1,l+1} &= \sum_{p=0}^{n-l} 2^p \left(\tbinom{p+l+1}{p+1} + \tbinom{p+l}{p+1}\right) T_{n,l+p}, l \in \mathbb{Z}_{+,n}. \end{aligned}$$

Both these equations can be combined into the following single equation, for all $l \in \mathbb{N}_n$,

$$T_{n+1,l+1} = \sum_{p=0}^{n-l} 2^p \left(\tbinom{p+l+1}{p+1} + \delta_{l>0}\tbinom{p+l}{p+1}\right) T_{n,l+p}.$$

Finally, from the fact that $\operatorname{sech}^0 t = 1$ we conclude that $T_{n,0} = \delta_{n=0}$. □

Identifying our results with those given in Sloane [11, A085734 and A088874] we see that the $T_{n,k}$, satisfying recursion relation (7.8) and boundary condition $T_{n,0} = \delta_{n=0}$, are indeed Sloane's sequence A088874 (there constructed by Deleham using a delta operator). Thus

Eq. (7.7) shows that the number triangle $T_{n,k}$ is as fundamental for the Maclaurin series of $\operatorname{sech}^c t$ as the Euler numbers E_n are for the Maclaurin series of $\operatorname{sech} t$.

Eq. (7.7) also immediately leads to the orthogonality relation, for all $n, k \in \mathbb{N}$,

$$\sum_{p=0}^{n} \binom{2n}{2p} \sum_{l=\max(0,k+p-n)}^{\min(k,p)} (-1)^l \, T_{n-p,k-l} T_{p,l} = \delta_{n=0}\delta_{k=0}. \tag{7.9}$$

The series (7.7) shows that the numbers $E_{2m}^{(c)} \triangleq (-1)^m \sum_{l=0}^{m} T_{m,l} c^l$, for all $m \in \mathbb{N}$ and for all $c \in \mathbb{C}$, are a set of generalized Euler numbers (although they are polynomials in c) in the sense of Luo, et al. [7]. It seems more natural however, to regard the integer numbers $T_{m,l}$ as a more fundamental set, because they are independent of c.

7.1 Multinomial Euler numbers E_n^m

For the important special case that $c = m \in \mathbb{N}$, it might be convenient to introduce a generalization of the Euler numbers that are also integers.

By identifying Eq. (7.7) with the Maclaurin series of $\operatorname{sech}^m t$, written in the form

$$\operatorname{sech}^m t = \sum_{n=0}^{+\infty} E_n^m \frac{t^n}{n!}, \tag{7.10}$$

we get, for all $p \in \mathbb{N}$, $E_{2p+1}^m = 0$ and

$$E_{2p}^m = (-1)^p \sum_{l=0}^{p} T_{p,l} m^l. \tag{7.11}$$

On the other hand, we also have that $\operatorname{sech}^m t = \left(\sum_{n=0}^{+\infty} E_n \frac{t^n}{n!}\right)^m$, so we obtain,

$$E_n^m = \delta_{m=0}\delta_{n=0} + \delta_{m>0} \sum_{K:|K|=n} \binom{n}{K} \prod_{i=1}^{\#(K)} E_{k_i}. \tag{7.12}$$

Expression (7.12) suggests that the E_n^m be called *multinomial Euler numbers*. Equating the right hand sides of Eq. (7.11) and Eq. (7.12) gives

$$(-1)^n \sum_{l=0}^{n} T_{n,l} m^l = \delta_{m=0}\delta_{n=0} + \delta_{m>0} \sum_{K:|K|=n} \binom{n}{K} \prod_{i=1}^{\#(K)} E_{k_i}. \tag{7.13}$$

This reduces, for $m = 1$, to

$$(-1)^n \sum_{l=0}^{n} T_{n,l} = E_{2n}. \tag{7.14}$$

Some particular multinomial Euler numbers are mentioned in Sloane [11]. Eq. (7.11) reduces to the following particular cases: $E_0^m = 1$, $E_2^m = -m$, $E_4^m = m\,(3m+2)$ (rhombic matchstick numbers, Sloane's A045944) and $E_6^m = -m\,(15m^2 + 30m + 16)$ (not in Sloane). Also, $E_n^0 = \delta_{n=0}$, $E_n^1 = E_n$ (Euler or sech numbers, $|E_n|$ is Sloane's A000364), E_{n-1}^2 are the tanh numbers (due to $D_t^n \operatorname{sech}^2 t = D_t^{n+1} \tanh t$, with $\left|E_{n-1}^2\right|$ being Sloane's A000182).

We give the onset of the number square E_n^m, for even n, in Table 4.

n\ m	1	2	3	4	5	6
0	1	1	1	1	1	1
2	-1	-2	-3	-4	-5	-6
4	5	16	33	56	85	120
6	-61	-272	-723	-1 504	-2 705	-4 416
8	1 385	7 936	25 953	64 256	134 185	249 600
10	-50 521	-353 792	-1 376 643	-3 963 904	-9 451 805	-19 781 376

Table 4: The number square E_n^m

7.2 Even multinomial parity numbers e_n^m

By identifying Eq. (7.7) with the Maclaurin series of $\cosh^m t$, written in the form

$$\cosh^m t = \sum_{n=0}^{+\infty} e_n^m \frac{t^n}{n!}, \tag{7.15}$$

we get, for all $p \in \mathbb{N}$, $e_{2p+1}^m = 0$ and

$$e_{2p}^m = (-1)^p \sum_{l=0}^{p} T_{p,l}(-m)^l. \tag{7.16}$$

On the other hand, we also have that $\cosh^m t = \left(\sum_{n=0}^{+\infty} e_n \frac{t^n}{n!}\right)^m$, so we obtain,

$$e_n^m = \delta_{m=0}\delta_{n=0} + \delta_{m>0} \sum_{K:|K|=n} \binom{n}{K} \prod_{i=1}^{\#(K)} e_{k_i}. \tag{7.17}$$

Expression (7.17) suggests that the e_n^m be called *even multinomial parity numbers*. Eq. (4.17) gives us the following explicit expression for the e_n^m numbers,

$$e_n^m = \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} (m-2j)^n. \tag{7.18}$$

Equating the right hand sides of Eq. (7.16) and Eq. (7.18) gives

$$(-1)^n \sum_{l=0}^{n} T_{n,l}(-m)^l = \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} (m-2j)^{2n}. \tag{7.19}$$

In particular, for $m = 1$, Eq. (7.19) reduces to

$$(-1)^n \sum_{l=0}^{n} (-1)^l T_{n,l} = 1, \tag{7.20}$$

and, for $m = 2$, to

$$(-1)^n \sum_{l=0}^{n} T_{n,l}\,(-2)^l = 2^{2n-1}. \tag{7.21}$$

n\ m	1	2	3	4	5	6
0	1	1	1	1	1	1
2	1	2	3	4	5	6
4	1	8	21	40	65	96
6	1	32	183	544	1 205	2 256
8	1	128	1 641	8 320	26 465	64 896
10	1	512	14 763	131 584	628 805	2 086 656

Table 5: The number square e_n^m

Various particular even multinomial parity numbers are mentioned in Sloane [11]. For instance, $e_0^m = 1$, $e_2^m = m$, $e_4^m = m\,(3m-2)$ (octagonal numbers, Sloane's A000567) and $e_6^m = m\,(15m^2 - 30m + 16)$ (not in Sloane). Also, $e_n^0 = \delta_{n=0}$, $e_n^1 = e_n$, $e_n^2 = \delta_{n=0} + \delta_{n>0} e_n 2^{n-1}$ (e_{2p}^2 is Sloane's A009117), $e_n^3 = e_n(3^n+3)/4$ (e_{2p}^3 is Sloane's A054879), $e_n^4 = e_n\,(4^n + 4.2^n)\,/8$ (e_{2p}^4 is Sloane's A092812) and $e_n^5 = e_n\,(10 + 5.3^n + 5^n)\,/16$ (not in Sloane). In general, expression (7.18) shows that e_{2p}^m equals the number of closed walks, based at a vertex, of length $2p$ along the edges of an m-dimensional cube [12].

We give the onset of the number square e_n^m, for even n, in Table 5.

7.3 Relations between the E_n^m and the e_n^m numbers

(i) Evidently, due to the fact that $\cosh^m t \,\mathrm{sech}^m t = 1$, for all $m \in \mathbb{N}$, holds the following orthogonality relation, for all $n, m \in \mathbb{N}$,

$$\sum_{i=0}^{n} \binom{n}{i} e_{n-i}^m E_i^m = \delta_{n=0}. \tag{7.22}$$

Combining Eqs. (7.22) and (7.18), and using the fact that $e_0^m = 1$, for all $m \in \mathbb{N}$, we obtain the following recursion relation for the E_n^m,

$$E_n^m = \delta_{n=0} - \delta_{n>0} \sum_{i=0}^{n-1} \binom{n}{i} \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} (m-2j)^{n-i} E_i^m. \tag{7.23}$$

In particular, for $m = 1$, Eq. (7.23) yields

$$E_{2p} = \delta_{p=0} - \delta_{p>0} \sum_{j=0}^{p-1} \binom{2p}{2j} E_{2j}. \tag{7.24}$$

Due to the symmetry of the binomial expression in Eq. (7.22) and because $E_0^m = 1$, for all $m \in \mathbb{N}$, we get equivalently a recursion relation for the e_n^m in terms of the E_n^m numbers,

$$e_n^m = \delta_{n=0} - \delta_{n>0} \sum_{i=0}^{n-1} \binom{n}{i} E_{n-i}^m e_i^m. \tag{7.25}$$

In particular, for $m=1$, by using $e_n^1 = e_n$ and $e_0 = 1 = E_0$, we get

$$E_0 = \delta_{p=0} - \delta_{p>0} \sum_{j=0}^{p-1} \binom{2p}{2j} E_{2(p-j)}. \tag{7.26}$$

(ii) Comparing Eq. (7.11) with Eq. (7.16) we see that the numbers E_{2n}^m are the extension to negative integers m of the numbers e_{2n}^m, as expected from their generating functions given in Eqs. (7.10) and (7.15).

(iii) Recall Faa di Bruno's formula for the n-th derivative of a composition of two functions [1, p. 823, 24.1.2, II, C.], for all $n \in \mathbb{Z}_+$,

$$D_t^n f(g(t)) = \sum_{k=1}^{n} \left(D_g^k f(g)\right)(t) \sum_{P(n):|K|=k} n! \prod_{i=1}^{n} \frac{\left(D_t^i g(t)\right)^{k_i}}{(i!)^{k_i} k_i!},$$

where $P(n) \triangleq \Big\{K \triangleq \{k_1, k_2, ..., k_n \in \mathbb{N}\} : 1k_1 + 2k_2 + ... + nk_n = n\Big\}$. An element $K \in P(n)$ represents a partition of a set of cardinality n into k_1 classes of cardinality 1, k_2 classes of cardinality 2, up to k_n classes of cardinality n.

Applied to $f \circ g$, with $g(t) = \cosh^m t$ and $f(g) = 1/g$, we get

$$\begin{aligned} E_n^m &= \lim_{t\to 0} D_t^n f(g(t)), \\ &= \sum_{k=1}^{n} (-1)^k k! \sum_{P(n):|K|=k} n! \prod_{i=1}^{n} \frac{\left(e_i^m\right)^{k_i}}{(i!)^{k_i} k_i!}. \end{aligned}$$

Define, for all $n \in \mathbb{N}$, $S_e^m(n, 0) \triangleq \delta_{n=0}$ and if $n > 0$, for all $k \in \mathbb{Z}_{+,n}$,

$$S_e^m(n,k) \triangleq \sum_{P(n):|K|=k} n! \prod_{i=1}^{n} \frac{\left(e_i^m\right)^{k_i}}{(i!)^{k_i} k_i!}. \tag{7.27}$$

Then

$$E_n^m = \sum_{k=1}^{n} (-1)^k k! S_e^m(n,k). \tag{7.28}$$

Eq. (7.28) expresses the multinomial Euler numbers in terms of the even multinomial parity numbers through the intermediate numbers $S_e^m(n, k)$.

The even parity symbol e_i in the product in Eq. (7.27) makes that all k_i with odd index i must be taken zero, so $S_e^m(n, k)$ and hence E_n^m are both zero for odd n. With $n = 2p$, we get, for all $p \in \mathbb{Z}_+$,

$$S_e^m(2p,k) = \sum_{P_e(2p):|K_e|=k} (2p)! \prod_{j=1}^{p} \frac{\left(e_{2j}^m\right)^{k_{2j}}}{((2j)!)^{k_{2j}} k_{2j}!},$$

where $P_e(2p) \triangleq \Big\{K_e \triangleq \{k_2, k_4, ..., k_n \in \mathbb{N}\} : k_2 + 2k_4 + ... + pk_{2p} = p\Big\}$. An element $K_e \in P_e(2p)$ represents a partition of a set of $2p$ elements into 0 classes of cardinality 1, k_2 classes of cardinality 2, 0 classes of cardinality 3, k_4 classes of cardinality 4, up to k_{2p} classes of cardinality $2p$.

In particular, for $m = 1$, we get from Eq. (7.28), for all $p \in \mathbb{Z}_+$,

$$S_e(2p, k) \triangleq S_e^1(2p, k) = \sum_{P_e(2p):|K_e|=k} (2p)! \prod_{j=1}^{p} \frac{1}{((2j)!)^{k_{2j}}\, k_{2j}!},$$

i.e., the number of ways of partitioning a set of $2p$ elements into k non-empty subsets, each of even cardinality, and

$$E_{2p} = \sum_{k=1}^{2p} (-1)^k k! S_e(2p, k). \tag{7.29}$$

This seems to be a new expression for the (even) Euler numbers. Here the sum involves partitions into subsets of even cardinality. A similar sum, involving partitions into subsets of any cardinality, is the well-known result for the Stirling numbers of the second kind,

$$1 = \sum_{k=0}^{2p} (-1)^k k! S(2p, k).$$

It thus turns out that the numbers $S_e^m(n, k)$, (which by comparing Eq. (7.27) with Eq. (7.30) might be called "even multinomial Stirling numbers of the second kind"), are more natural to the E_n^m than the $S(n, k)$. This can be seen by applying Faa di Bruno's formula to $f \circ g$, with $g(t) = e^t$ and $f(g) = \left(\frac{1}{2}(g + 1/g)\right)^{-m}$, and using [1, p. 823, 24.1.2, II B],

$$S(n, k) = \sum_{P(n):|K|=k} \frac{n!}{\prod_{i=1}^{n} (i!)^{k_i}\, k_i!}. \tag{7.30}$$

We get

$$E_n^m = \sum_{k=1}^{n} \left(\lim_{t \to 0} D_t^k \operatorname{sech}^m (\ln(1 + t)) \right) S(n, k), \tag{7.31}$$

an expression more complicated than $E_n^m = \lim_{t\to 0} D_t^n \operatorname{sech}^m(t)$. For $m = 1$, the numbers defined by the expression inside the parentheses in (7.31) are Sloane's A009014.

We can derive another expression for the E_n^m in terms of the $S(n, k)$, directly from the generating function $\operatorname{sech}^m(x)$, as was done in Luo, et al. [7], but it turns out to involve a double sum. In the particular case $m = 1$ however, we can obtain this other expression from our results by combining Eqs. (3.22) and (5.13), and then it reads

$$E_n = \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} (-1)^l l! 2^{k-l} S(k, l). \tag{7.32}$$

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