Physicalia Mag. 24 (2002) 1 pp 17-28

PROBLEMS WITH JEANS' TREATMENT OF SELF-GRAVITATION

Vladimir M. Čadež[1] and Frank Verheest[2]

[1] *Belgian Institute for Space Aeronomy (IASB-BIRA)*
Avenue Circulaire 3, 1180 Brussels
[2] *Sterrenkundig Observatorium, Universiteit Gent*
Krijgslaan 281/S9, 9000 Gent

Linear perturbation analysis of a self-gravitating cloud dates as far back as 1902, when Jeans obtained the instability criterion for harmonic waves whose wavelength exceeds some critical value, now known as the Jeans length. Since Newtonian gravitation in extended mass systems precludes truly homogeneous equilibria, except for very special configurations with specific flow patterns, Jeans' treatment needs to be applied with great care. We discuss one-dimensional or spherically symmetric equilibria in non-magnetized clouds without flows, and introduce plausible density and temperature profiles, in the hope that these could be locally uniform over distances larger than the Jeans length. Numerical computations show that while acceptable density profiles can be obtained, the temperature profiles are then too non-uniform. This indicates that the Jeans procedure cannot be mathematically justified for the considered basic states, and that the problem of the equilibrium and stability of a self-gravitating cloud remains wide open, a full century after Jeans.

1 Introduction

Linear perturbation analysis of a self-gravitating cloud dates as far back as 1902 when Jeans obtained the instability criterion for harmonic waves whose wavelength λ exceeds some critical value λ_J. The latter is known as the Jeans length, defined through

$$\lambda^2 \geq \lambda_J^2 \equiv \frac{\pi c_s^2}{G\rho}, \tag{1}$$

where c_s^2 is the thermal speed squared, G is the gravitational constant and ρ is the gas density. His study was based upon the assumption that the unperturbed gaseous cloud is initially uniform [1], and then the instability is non-oscillatory and purely growing. However, Newtonian gravitation in extended mass systems precludes truly homogeneous equilibria [2], except for very special configurations with specific flow patterns. Disciples of Jeans have skirted around this difficulty by considering local perturbations, with wavelengths small compared to the inhomogeneity scale lengths. This might be an acceptable way of treating an intrinsically complicated problem, were it not that in most cases the internal consistency cannot be tested, because knowledge about the equilibrium and its associated inhomogeneity scale lengths is lacking. This procedure has been called the "Jeans swindle", although this qualification is too harsh, as shall be argued below for specific examples.

Since then, numerous authors (see e.g. [3]–[8] for a non-exhaustive list) made further investigations of the Jeans instability criterion under more complex physical conditions. Nevertheless, models involving media that are uniform either as a whole or in some directions only have to be treated with great caution, the conclusions on instability criteria in particular, because a self-gravitating fluid can be considered uniform only locally. Over larger distances, self-gravitation makes the medium nonuniform, which can be neither avoided nor neglected, as was pointed out in the case of purely neutral gases [9]–[12].

Recently, the Jeans instability has been revisited from a totally different side, in the context of dusty plasma physics. Cosmic dust is a well-known and common constituent of many heliospheric and astrophysical media [13]–[14]. Prime examples in the solar system are circumsolar dust rings, noctilucent clouds, cometary comae and tails, and rings of the Jovian planets. Among astrophysical applications interstellar dust clouds come to mind. Dust grains may be charged or neutral, depending on the nearby sources of radiation like stars and/or the presence of charged particles as in the solar wind. Such a charging of dust grains in a plasma environment was first mentioned by Spitzer [15], and is now believed to play an essential role in many astrophysical and heliospheric plasmas (see e.g. [16]–[20] for further reviews on this subject). The combination of charged dust and plasma is referred to as a dusty plasma. Since the dust grains are typically micron-sized, and can have up to thousands of electron charges, the charge-to-mass ratios of these new plasma constituents are very much smaller than for normal ions. As a consequence, a whole range of new modes can occur in dusty plasmas at very low frequencies. In addition, the presence of these very heavy, charged dust grains might require considering also self-gravitational effects besides the usual electromagnetic ones. Hence, traditional Jeans collapse criteria known from neutral gases have to be modified, depending on the type of plasma wave modes investigated [21]–[29].

There is thus a definite need for a careful reexamination of stationary equilibria in gaseous and/or plasma clouds, without assuming a uniform density to start with. Contrary to what the title might infer, we shall not discuss perturbations or instabilities. Even so, it is clear that this is a vast subject, and we will only be able to give some recent ideas and insights in what follows, without any claim of being exhaustive or complete.

2 Cartesian Geometry – 1D Approach

The main purpose of this paper is to consider realistic self-gravitating nonuniform clouds, in order to find out whether or not such clouds could, at least locally, be quasi-uniform over a linear extent exceeding the Jeans length λ_J that would then justify his treatment and conclusions on the onset of the gravitational instability given by Eq (1).

We will look at a self-gravitating, isothermal plasma cloud described by single-fluid magnetohydrodynamic (MHD) equations, because a multispecies description as required in dusty plasmas is much too complicated. Results for neutral gases are included by turning off the charges where occurring. The starting point is the basic set of MHD equations of continuity, induction, momentum and energy, augmented by the gravitational Poisson's equation,

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0,\\
&\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B}),\\
&\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}+\frac{1}{\rho}\nabla p+\nabla\phi=\frac{1}{\mu_0\rho}(\nabla\times\mathbf{B})\times\mathbf{B},\\
&\frac{\partial p}{\partial t}+\mathbf{v}\cdot\nabla p=c_s^2\left(\frac{\partial \rho}{\partial t}+\mathbf{v}\cdot\nabla\rho\right),\\
&\nabla^2\phi=4\pi G\rho.
\end{aligned}
\tag{2}
$$

Here ρ, $\mathbf{v}$ and p represent the fluid mass density, velocity and pressure, respectively, $\mathbf{B}$ is the magnetic field and ϕ the gravitational potential.

For the stationary (static) state we find first of all that the Poisson equation for the equilibrium gravitational potential,

$$\nabla^2\phi=4\pi G\rho, \tag{3}$$

precludes any truly homogeneous equilibrium, except in those configurations where $\nabla^2\phi$ can be constant. This will turn out the case in certain axisymmetric equilibria with azimuthal flows, but these fall outside the scope of our present discussions.

However, we will start with as simple as possible a configuration that is compatible with the basic equations, and introduce some plausible physical assumptions to fix the ideas:

(i) The plasma is treated as a perfect gas, so that c_s is a true constant;

(ii) The equilibrium magnetic field lines are straight, so that $\mathbf{B}=B\mathbf{e}_x$. Then Gauss's law $\nabla\cdot\mathbf{B}=\partial B/\partial x=0$ indicates that the strength of the magnetic field B cannot depend on x, and hence $(\mathbf{B}\cdot\nabla)\mathbf{B}=\mathbf{0}$;

(iii) Furthermore, the magnetic field is assumed to vary in such a way that the ratio β of the plasma pressure to the magnetic pressure $B^2/2\mu_0$ remains constant.

In this model the Alfvén speed V_A is constant, and the magnetic field and plasma pressure are stronger in regions with a higher plasma density ρ. This is rather realistic and may occur in highly conductive plasmas with frozen-in magnetic fields. The magnetohydrostatic equilibrium balance equation,

$$\nabla p+\rho\nabla\phi=\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}, \tag{4}$$

then reduces to

$$\left(1+\frac{1}{\beta}\right)\nabla_\perp p+\rho\nabla_\perp\phi=\mathbf{0}, \tag{5}$$

where the subscript $\perp$ refers to the directions across the equilibrium field. Together with (3), this yields a single equation for ρ_0,

$$\nabla_\perp^2\ln\rho+\frac{8\pi G}{V_A^2+2c_s^2}\rho=0. \tag{6}$$

This shows that the solutions depend upon y and z through $(y \sin \alpha + z \cos \alpha)/H$, and the typical scale length H is defined by $H^2 = (V_A^2 + 2c_s^2)/4\pi G\rho_0$. Here ρ_0 is the density in the center of the cloud. For the discussion we take $\alpha = 0$ and proceed with density profiles $\rho(z)$ having an extremum at $z = 0$, because the gravitational force vanishes at the center of the cloud. Whether such a configuration is stable against perturbations has to be investigated afterwards, also in the lateral directions that are now assumed to be homogeneous.

Even though the stationary state is uniform along the equilibrium field, we would like to stress that this is not using the Jeans swindle, because the present choice is fully compatible with all equilibrium constraints. It might well turn out that such a state is Jeans unstable in the lateral directions, but that is quite different from starting with a completely uniform equilibrium. Proceeding then with equivalent one-dimensional equilibrium variations, the solutions are

$$\rho(z) = \rho_0 \operatorname{sech}^2(z/H), \qquad B(z) = B_0 \operatorname{sech}(z/H), \tag{7}$$

and the central field strength B_0 is given through $B_0^2 = \mu_0 V_A^2 \rho_0$ as V_A is constant.

In order to correlate this with Jeans' original treatment, we reduce the discussion to a simplified non-magnetic case, as shown in Figure 1. The isothermal density distribution (7) in the figure is a reference profile for a better visualization of effects arising from a temperature nonuniformity modeled by

$$\frac{T}{T_0} = \frac{1}{1 + z^m/\lambda_J^m}. \tag{8}$$

The behaviour of the profile (8) at $z = 0$ is assumed to be the same as that for the density

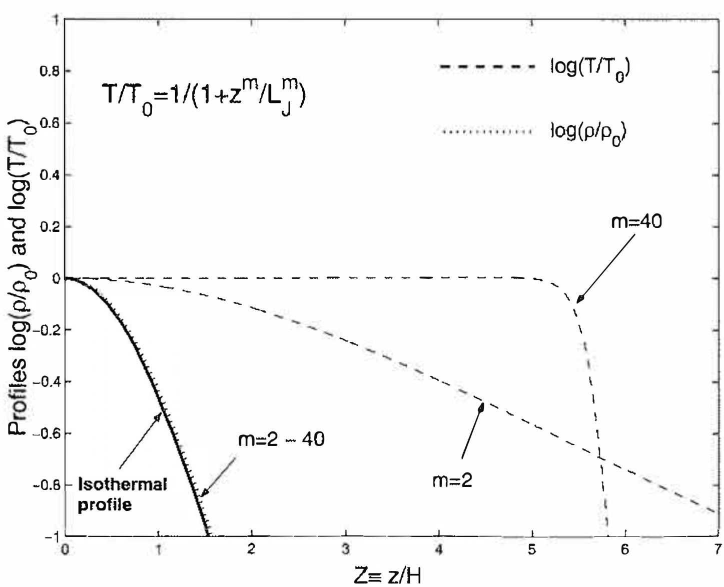

Figure 1: Influence of the indicated gas temperature profile on the density distribution.

ρ. For physical reasons, T has an extremum at $z = 0$. Consequently, the coefficient m in Eq (8) will be taken a positive and even integer, thus $m \equiv 2n \geq 2$.

Figure 1 indicates that density distributions are very little affected by temperature nonuniformities (for $40 \geq m \geq 2$), and that the relative density change exceeds -1 (the condition for a quasi-uniformity) in the log scale already for $z/H \leq 2$. On the other hand, the Jeans length scaled to H is $L_J \equiv \lambda_J/H \simeq 5.74$ meaning that also the nonisothermal basic states are not quasi-uniform over the length L_J. Hence, the standard Jeans procedure based upon the assumption of a uniform medium is not mathematically justified.

3 Spherically Symmetric Gaseous Cloud

The problem of achieving conditions for eventual gravitational instability encountered in the one dimensional approach utilizing Cartesian coordinates may now be reconsidered assuming a more realistic spherical geometry instead. In what follows, we shall assume a spherically symmetric system with all variables dependent on the radial coordinate r only. Magnetic fields are excluded as the presence of any magnetic field $\mathbf{B}$ ultimately violates the assumption of spherical symmetry. This results from the Gauss law $\nabla \cdot \mathbf{B} = 0$, saying that the magnetic field lines have no sources/sinks while the spherical symmetry would require an isotropic radial in/outflow of field lines.

For physical reasons, it is plausible to assume that all unperturbed basic state quantities have extrema at the center $r = 0$, meaning that the medium is fairly uniform in the central region. Its nonuniformity will show up with increasing r and we shall consider the medium locally uniform within the domain of r in which both the density $\rho(r)$ and temperature $T(r)$ satisfy the conditions $|\log(\rho(r)/\rho_0)| \leq 1$ and $|\log(T(r)/T_0)| \leq 1$, where $\rho_0 \equiv \rho(0)$ and $T_0 \equiv T(0)$. The standard Jeans procedure can then be justified for those basic states in which the extent of local uniformity exceeds the Jeans length λ_J. The density and temperature profiles of the basic state satisfy Eqs (2) for the hydrostatic equilibrium which in the spherically symmetric case reduce to the following equation [30]:

$$\frac{d^2\epsilon}{dR^2} + \left(\frac{2}{R} + \frac{d\tau}{dR}\right)\frac{d\epsilon}{dR} + \frac{d^2\tau}{dR^2} + \left(\frac{d\tau}{dR}\right)^2 + \frac{2}{R}\frac{d\tau}{dR} + e^{\epsilon-\tau} = 0. \tag{9}$$

Here $R \equiv r/L$, the scaling length L is given through $L^2 = c_s(0)^2/4\pi\gamma G\rho_0$, and

$$\epsilon \equiv \log(\rho(R)/\rho_0), \qquad \tau \equiv \log(T(R)/T_0).$$

The nondimensional radial distance of the Jeans length is then $R_J \equiv \lambda_J/L = 2\pi\gamma$. For a monatomic gas $\gamma = 5/3$ and $R_J \simeq 8$.

Eq (9) can be solved in two ways, either the temperature profile $T(R)$ is initially known while the density $\rho(R)$ is computed or the density profile $\rho(R)$ is given while the temperature $T(R)$ is computed. The known profiles can be obtained from observations or simply specified by some model functions with realistic physical properties.

The relevant initial conditions imposed on ϵ and τ at $R = 0$ are as before

$$\epsilon(0),\ \tau(0) = 0 \qquad \text{and} \qquad \frac{d\epsilon}{dR},\ \frac{d\tau}{dR} = 0. \tag{10}$$

Let us now look at some characteristic cases of both types of solutions.

3.1 Prescribed temperature profiles

If the temperature profile is known and initially prescribed, the simplest choice for $T(R)$ is the assumption of an isothermal basic state with $T(R) = T_0 = \text{const}$. In that case $\tau = 0$ and Eq (9) has an analytic solution for ϵ with an asymptotic behaviour $\epsilon \sim -2\log(R)$ which yields the density profile $\rho(R) \sim R^{-2}$ and consequently a divergent total mass $\mathcal{M} = \int_0^\infty \rho(R)R^2 dR$ of the cloud (see [31] and references therein). Clearly, if the total mass $\mathcal{M}$ is to be finite the density $\rho(R)$ has to decrease faster than R^{-3} at large R. To overcome the problem of infinite total mass, i.e. to change the asymptotic behaviour of ρ, one can either retain the assumption of an isothermal basic state and impose a bound to the radial extent of the cloud i.e. to put the whole cloud inside a sphere of a fixed finite radius and solve the eigenvalue problem as done in [31], or one can relax the assumption of a constant temperature to steepen the asymptotics of the density profile as we do in this paper.

In this sense Figure 2 shows how the density distribution is affected by nonisothermal temperature profiles in which the temperature drops monotonously from T_0 at $R = 0$ to $0.2\ T_0$ when $R \to \infty$:

$$\frac{T}{T_0} = 0.2 + \frac{0.8}{1 + R^m}. \tag{11}$$

The initial conditions in Eq (10) regarding the temperature profile $\tau \equiv log(T/T_0)$ require the power m to be a positive, even integer $m = 2n \geq 2$, in Eq (11) as well as in similar temperature profiles that follow below.

We see in Fig 2 that no matter how steep this transition is i.e. how large m is, the asymptotic slope of $\rho(R)$ follows that of an isothermal ($m = 0$) cloud at large R. In the log-log scale all curves tend to parallel straight lines with the slope -2.

Consequently, any nonuniform temperature profile that approaches a finite $T(R)$ at

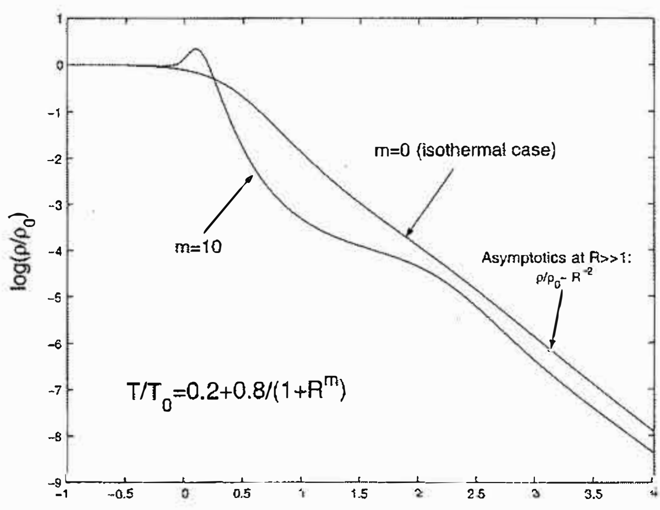

Figure 2: Equilibrium density distributions if the gas temperature decreases monotonously from T_0 at $R = 0$ to $0.2T_0$ at $R \gg 1$.

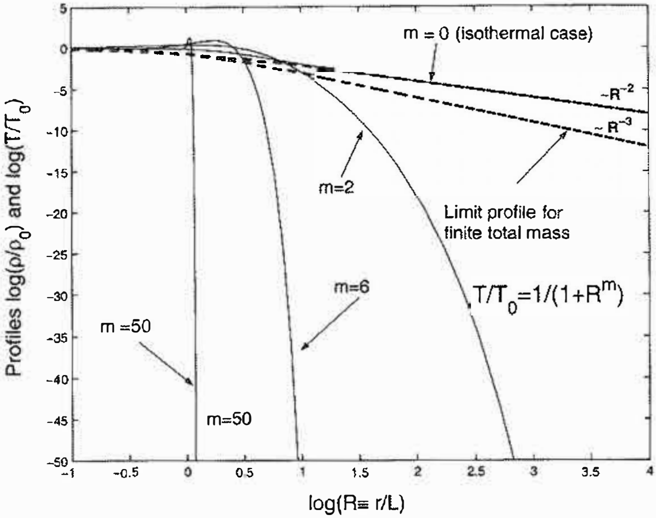

Figure 3: Equilibrium density distributions if the gas temperature decreases monotonously from T_0 at $R = 0$ to 0 at $R \gg 1$.

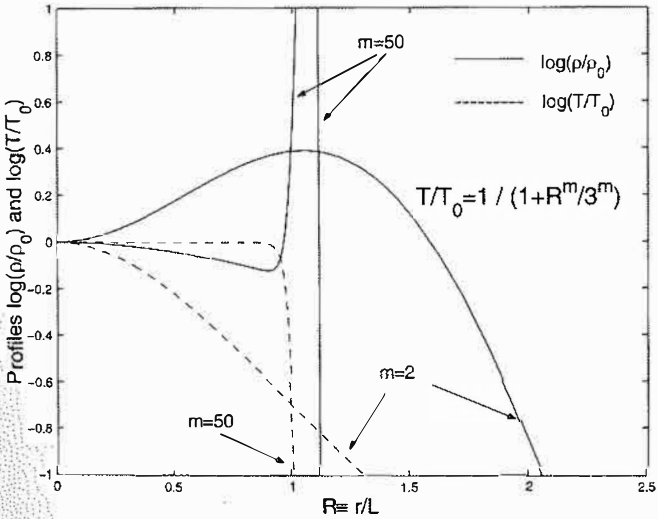

Figure 4: Quasi-uniform domains of density distributions computed from prescribed temperature profiles with different power m.

$R \gg 1$ does not alter the unsuitable asymptotics of the isothermal density distribution leading to a divergent total mass. Such a behaviour is understandable as the basic state with a finite constant temperature behaves asymptotically as isothermal in the $R \gg 1$ limit.

The whole situation changes radically if the temperature vanishes at $R \gg 1$, like if the following profile is assumed for $T(R)$:

$$\frac{T}{T_0} = \frac{1}{1+R^m}. \tag{12}$$

In that case, Fig 3 shows that the density asymptotically falls off with R faster than R^{-3} if $m \geq 2$ which provides for a finite total mass of the cloud. Moreover, in the log-log scale the density distribution is not a linear function of the radius meaning that ρ decreases with R faster than any power of R.

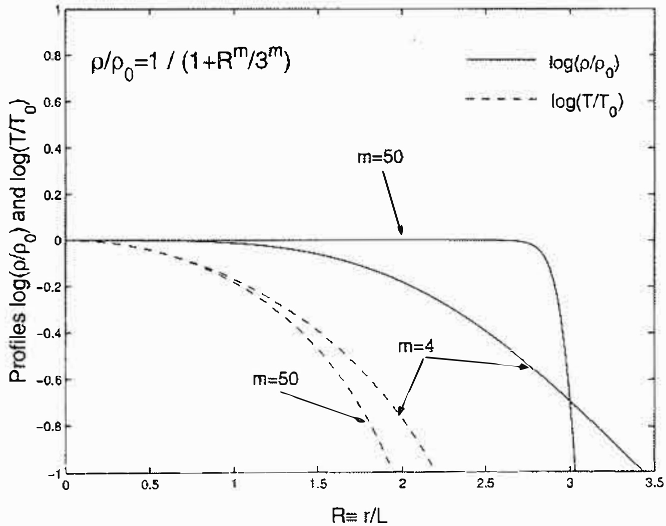

Figure 5: Quasi-uniform domains of temperature distributions computed from prescribed density profiles with different power m.

This leads to the conclusion that a self-gravitating cloud of a infinite extent can have a finite total mass only if its temperature tends to zero with R faster than R^{-1} i.e. for $m \geq 2$.

Having resolved the problem of finite total mass of the cloud we can now look for a subgroup of basic states that yield quasi-uniform domains of maximal extent in the central region which are defined by inequalities $|\log(\rho/\rho_0)| \leq 1$ and $|\log(T/T_0)| \leq 1$. Figure 4 thus shows plots of $\log(\rho/\rho_0)$ and $\log(T/T_0)$ within the interval ± 1 as functions of R if the temperature profile is given by

$$\frac{T}{T_0} = \frac{1}{1+(R/3)^m}. \tag{13}$$

The resulting conclusion from Figure 4 is that the largest quasi-uniform domain exists for $m = 2$ and R slightly above 2. In this case, both the temperature and density are quasi-uniform, i.e. they do not vary by more then $1/e$ of their central values (at $R = 0$). At larger m, the quasi-uniform domain shrinks to smaller R, so that if $m = 50$ it is confined approximately within $R \leq 1$. This finally means that the Jeans length $R_J \simeq 8$

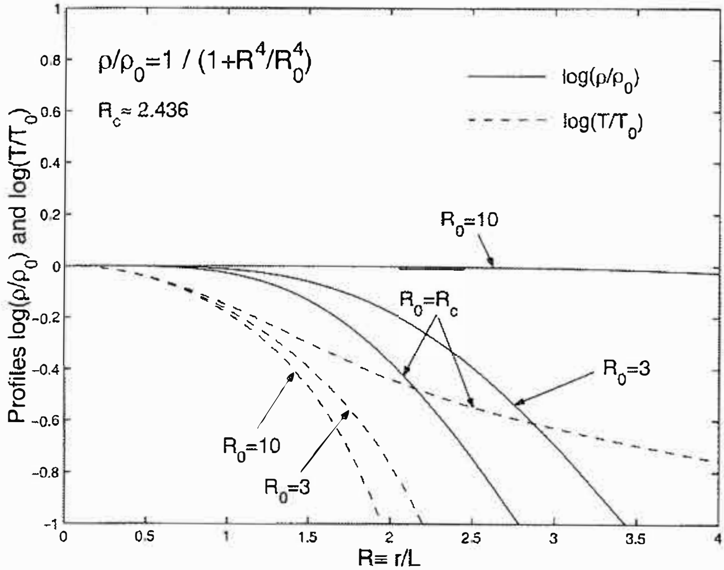

Figure 6: Quasi-uniform domains of temperature distributions computed from prescribed density profiles with different R_0 and $m = 4$.

significantly exceeds the size of the domain in which the basic state can be considered quasi-uniform and the classical Jeans treatment of the gravitational instability is not applicable in this case.

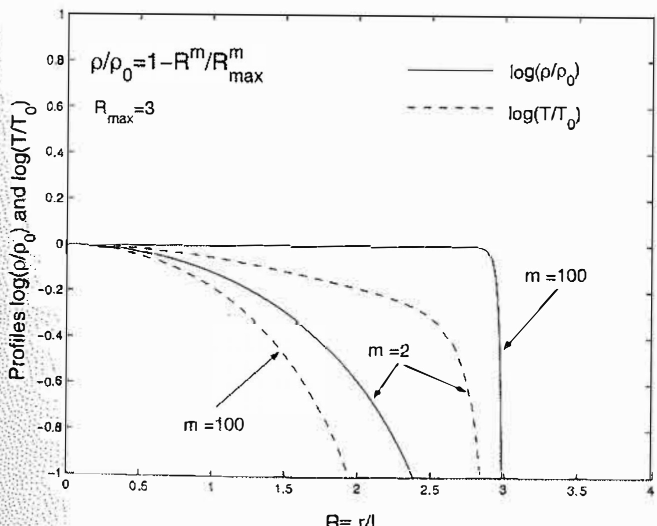

Figure 7: Quasi-uniform domains of temperature distributions computed from prescribed density profiles ($m = 2$ and $m = 100$) limited to the domain $R \leq R_{max} = 3$.

3.2 Prescribed density profiles

Another approach is to prescribe a suitable density profile and then compute the corresponding temperature distribution by solving Eq (9) for τ, with initial conditions $\tau(0) = 0$ and $d\tau/dR = 0$ at $R = 0$.

Thus we first assume a cloud of infinite extent the density profile of which falls of with R faster than R^{-3} (so as to keep the total mass $\mathcal{M}$ finite) and consider the following model distribution in Eq (9):

$$\frac{\rho}{\rho_0} = \frac{1}{1+(R/R_0)^m} \qquad (m \geq 4). \tag{14}$$

The resulting temperature profile for $R_0 = 3$ is presented in Figure 5 together with ρ given by Eq (14) in the log scale in the interval of ± 1. The quasi-uniform domains are seen to range between $R \simeq 2.4$ if $m = 3$ and $R \simeq 2$ if $m = 50$ which is small in comparison with $R_J \simeq 8$. The standard Jeans procedure leading to gravitational instability is thus again inapplicable. A possible way out might now be in taking a larger R_0 in Eq (14) which expands the domain of quasi-uniformity of density ρ. The resulting effects are shown in Figure 6 where R_0 varies between $R_0 = R_c = 2.436...$ and $R_0 = 10$. If $R < R_c$ the computed temperature profile diverges at $R \gg 1$ which is physically unrealistic. Figure 6 indicates the influence of R_0 on the temperature profile. Contrary to the density profile, the domain of quasi-uniformity of temperature shrinks if R_0 rises. Thus, while the e-fold limit for ρ/ρ_0 is far beyond $R = 4$ if $R_0 = 10$, this limit does not exceed $R = 2$ for T/T_0. Consequently, the considered basic states cannot have the temperature and density profiles that are both quasi-uniform in a domain of sufficiently large extent ($R > R_J \simeq 8$) as to justify the standard Jeans procedure.

Let us now consider a density profile of finite extent as done in [31] where $T = T_0$ is assumed constant. Such a density profile can be modeled by

$$\frac{\rho}{\rho_0} = 1 - \frac{R^m}{R_{max}^m}, \tag{15}$$

which yields solutions to Eq (9) for $\tau \equiv \log(T/T_0)$ as plotted in Figure 7 for $R \leq R_{max} = 3$. The parameter R_{max} defines the outer boundary of the gaseous sphere where the gas density drops to zero. We see that the resulting temperature is not constant, it monotonously decreases inside the quasi-uniform region if $m \geq 2$. Not plotted numerical computations show that the effect of R_{max} on the solution is analogous to that of R_0 in Figure 6. If R_{max} is increased, the domain of quasi-uniform density expands while that of the temperature shrinks and closely approaches the dotted curve $m = 100$ for any $m \geq 2$ if R_{max} is sufficiently large. Consequently, the extent of the quasi-uniform domain remains too small in comparison with R_J, meaning that the Jeans procedure cannot be mathematically justified for the considered basic states.

4 Conclusions

We have discussed in this paper one-dimensional or spherically symmetric equilibria of mostly non-magnetized clouds, and have introduced plausible density and temperature profiles, in the hope that these could be considered locally uniform over distances larger than the Jeans length. Numerical computations show, however, that while acceptable density profiles can be obtained, the temperature profiles are then too non-uniform for the Jeans requirement. This indicates that it is difficult to mathematically justify the Jeans procedure for the basic states considered, so that the problem of the equilibrium and stability of self-gravitating clouds remains wide open, a full century after Jeans' original paper was published.

Acknowledgments

V.M. Čadež acknowledges support by the Belgian Institute for Space Aeronomy and by the Belgian Federal Office for Scientific, Technical and Cultural Affairs. F. Verheest thanks the Bijzonder Onderzoeksfonds (Universiteit Gent) for a research grant.

References

[1] J.H. Jeans, *Astronomy and Cosmogony* (Cambridge University Press, Cambridge, 1929).

[2] A.M. Fridman and V.L. Polyachenko, *Physics of Gravitating Systems I & II* (Springer, New York, 1984).

[3] S. Chandrasekhar, *Astrophys. J.* **119**, 7 (1954).

[4] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, New Jersey, 1987).

[5] T. Nakano, *Proc. Astron. Soc. Japan* **40**, 593 (1988).

[6] M.G. Corona-Galindo and H. Dehnen, *Astrophys. Space Sci.* **153**, 87 (1989).

[7] C.G. Lacey, *Astrophys. J.* **336**, 612 (1989).

[8] C.S. Gehman, C. Adams and R. Watkins, *Astrophys. J.* **472**, 673 (1996).

[9] L. Spitzer, *Physical Processes in Interstellar Medium* (Wiley, New York, 1978), p. 321.

[10] A.P. Boss, *Astrophys. J.* **319**, 149 (1987).

[11] V.M. Čadež, *Astron. Astrophys.* **235**, 242 (1990).

[12] J. Vranješ and V.M. Čadež, *Astrophys. Space Sci.* **164**, 329 (1990).

[13] D.C.B. Whittet, *Dust in the Galactic Environment* (Institute of Physics Publishing, Bristol, 1992).

[14] A. Evans, *The Dusty Universe* (J. Wiley and Sons, Chichester, 1994).

[15] L. Spitzer, *Astrophys. J.* **93**, 369 (1941).

[16] D.A. Mendis and M. Rosenberg, *Annu. Rev. Astron. Astrophys.* **32**, 419 (1994).

[17] P. Bliokh, V. Sinitsin and V. Yaroshenko, *Dusty and Self-gravitational Plasmas in Space* (Dordrecht, Kluwer, 1995).

[18] M. Horányi, *Annu. Rev. Astron. Astrophys.* **34**, 383 (1996).

[19] F. Verheest, *Space Sci. Rev.* **77**, 267 (1996).

[20] F. Verheest, *Waves in Dusty Space Plasmas*, (Kluwer Academic, Dordrecht, 2000).

[21] P.V. Bliokh and V.V. Yaroshenko, *Sov. Astron.* **29**, 330 (1985).

[22] K. Avinash and P.K. Shukla, *Phys.Lett. A* **189**, 470 (1994).

[23] B.P. Pandey, K. Avinash and C.B. Dwivedi, *Phys. Rev. E* **49**, 5599 (1994).

[24] L. Mahanta, B.J. Saikia, B.P. Pandey and S. Bujarbarua, *J. Plasma Phys.* **55**, 401 (1996).

[25] P. Meuris, F. Verheest and G.S. Lakhina, *Planet. Space Sci.* **45**, 449 (1997).

[26] F. Verheest, P. Meuris, R.L. Mace and M.A. Hellberg, *Astrophys. Space Sci.* **254**, 253 (1997).

[27] F. Verheest, M.A. Hellberg and R.L. Mace, *Phys. Plasmas* **6**, 279 (1999).

[28] M. Salimullah and P.K. Shukla, *Phys. Plasmas* **6**, 686 (1999).

[29] F. Verheest, G. Jacobs and V.V. Yaroshenko, *Phys. Plasmas* **7**, 3004 (2000).

[30] V.M. Čadež, *Astron. Astrophys.* **235**, 242 (1990).

[31] P.H. Chavanis, *Astron. Astrophys.* **381**, 340 (2002).