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A collisional kinetic model of the polar wind

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Abstract. The effects of Coulomb collisions in the transition region between the collision-dominated regime at low altitudes and the collisionless regime at high altitudes have been taken into account to study the escape of H^+ ions in the polar wind. A specialized spectral method is used to solve the steady state Fokker-Planck equation describing the diffusion and upward bulk motion of H^+ ions through a background of O^+ ions. The H^+ ion velocity distribution function is determined as a function of altitude for realistic initial conditions in the whole velocity space. We emphasize the importance for a correct choice of the boundary conditions at $\mathbf{v} = 0$ to obtain regular mathematical solutions. The results of this kinetic model are compared with results of earlier polar wind models based on different approximations of the plasma transport equations.

1. Introduction

Near the poles, H^+ and He^+ ions drift outward along the magnetic field lines connected to the tail of the magnetosphere and escape from the Earth's high-altitude ionosphere with supersonic speeds. This continuous outflow is called the polar wind. The polar wind has been studied for approximately 25 years with different theoretical models and to lesser extent with satellite measurements (cf. the review article of *Ganguli* [1996]).

In the present paper, we investigate the polar wind H^+ ion distribution as a function of altitude by solving the Fokker-Planck (FP) transport equation. We are concerned with the velocity distribution function (VDF) of H^+ moving in a background plasma composed of electrons and oxygen ions. The background plasma is assumed to be isothermal and Maxwellian in the exobase transition region wherein the H^+ ions change from collision-dominated at the bottom of this transition layer to collisionless at the top. The H^+ velocity distribution function is given as a boundary condition at the top of the transition region where the Coulomb collisions between the thermal ions can be neglected. At this altitude and higher, the velocity distribution function (VDF) of the light ions is highly anisotropic since most of the ions are accelerated upward by the ambipolar diffusion electric field (see *Lemaire and Scherer* [1973] for a discussion of the role of the ambipolar electric field in the polar wind), and due to the absence of collisions in the exosphere above this altitude no particle are backscattered toward the Earth. We have developed a formalism based on an expansion in "speed" polynomials to solve the kinetic FP equation which is explained in the third section. The regularity conditions that the solution must satisfy everywhere even at the boundaries of the transition region are described in the fourth section. Additional boundary conditions are imposed in the fifth section. The aim of this study is to investigate the transformation of the H^+ velocity distribution function in this transition region between the low-altitude collision-dominated regime and the collisionless regime at high altitudes.

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2. Fokker-Planck Equation

In the absence of inelastic collisions, source, and loss terms, the kinetic transport equation of the αth particle species is

$$\frac{\partial f_\alpha(\mathbf{r},\mathbf{v},t)}{\partial t} + (\mathbf{v}.\nabla_\mathbf{r}) f_\alpha(\mathbf{r},\mathbf{v},t) + (\mathbf{a}.\nabla_\mathbf{v}) f_\alpha(\mathbf{r},\mathbf{v},t) = \left(\frac{df}{dt}\right)_c \quad (1)$$

where $f_\alpha(\mathbf{r},\mathbf{v},t)$ is the velocity distribution function of the particles, $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity vectors respectively, $\mathbf{a}$ is the acceleration due to external forces and t is the time. The term on the right-hand side $(df/dt)_c$ represents the effects of the Coulomb collisions. It can be written [*Spitzer*, 1956, *Hinton*, 1983]

$$\left(\frac{df}{dt}\right)_c = -\frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{A} f_\alpha(\mathbf{r},\mathbf{v},t) - \frac{1}{2}\frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{D} f_\alpha(\mathbf{r},\mathbf{v},t))\right] \quad (2)$$

where $\mathbf{A}$ is the dynamic friction vector

$$\mathbf{A} = -4\pi \sum_\beta \frac{Z_\alpha^2 Z_\beta^2 e^4 \ln\Lambda}{m_\alpha^2}\left(1 + \frac{m_\alpha}{m_\beta}\right) \times \int d\mathbf{v}' f_\beta(\mathbf{v}') \frac{(\mathbf{v}-\mathbf{v}')}{(v-v')^3} \quad (3)$$

and $\mathbf{D}$ is the velocity diffusion tensor

$$\mathbf{D} = 4\pi \sum_\beta \frac{Z_\alpha^2 Z_\beta^2 e^4 \ln\Lambda}{m_\alpha^2} \times \int d\mathbf{v}' f_\beta(\mathbf{v}') \left(\frac{\mathbf{I}}{v-v'} - \frac{(\mathbf{v}-\mathbf{v}')(\mathbf{v}-\mathbf{v}')}{(v-v')^3} \right). \quad (4)$$

In (3) and (4), $\ln\Lambda$ is the usual Coulomb parameter containing the screening of the Coulomb field: $\ln\Lambda \sim 15$. Each term of the sum represents the effects of the collisions of the particle species α with the β particles (α included).

Without losing generality, one can reduce as usual the number of phase space dimensions from six to three, by assuming azimuthal symmetry around the vertical axis which is parallel to the magnetic field direction, to the gravitational force, and to the electric force. This simplifies the numerical problem without adulteration of the main physical aspects. No perpendicular electric field is assumed. The velocity distribution is a function of the radial distance r, the velocity v, and $\mu = \cos\theta$, where θ is the angle between the velocity vector and the magnetic field direction.

The acceleration of the particles of mass m_α and charge Z_α depends on the electric force $Z_\alpha e\mathbf{E}$, on the gravitational force $m_\alpha \mathbf{g}$ and on the Lorentz force:

$$\mathbf{a} = \left(\frac{Z_\alpha e \mathbf{E}}{m_\alpha} + \mathbf{g} \right) + \frac{Z_\alpha e}{m_\alpha}(\mathbf{v} \times \mathbf{B}). \quad (5)$$

Expressed in the (r, v, μ) coordinates, the acceleration term in (1) becomes

$$\mathbf{a}.\nabla_\mathbf{v} f = a(r)\left(\mu \frac{\partial f}{\partial v} + \frac{(1-\mu^2)}{v}\frac{\partial f}{\partial \mu} \right) + \frac{3}{2}\frac{v}{r}(1-\mu^2)\frac{\partial f}{\partial \mu}. \quad (6)$$

where we have considered that the cross section of the vertical magnetic flux tubes varies like r^3 with the radial distance, i.e. like the inverse of the magnetic field intensity for a dipole.

In spherical coordinates, the advection term of (1) becomes

$$\mathbf{v}(\mathbf{r}).\nabla_\mathbf{r} f = v\mu \frac{\partial f}{\partial r} \quad (7)$$

Defining a dimensionless velocity y by

$$y = \sqrt{m_\alpha / 2kT_\alpha}\, v = v/c, \quad (8)$$

the left hand side of the Fokker-Planck equation becomes

$$\mathcal{D}f = \frac{\partial f}{\partial t} + c y \mu \frac{\partial f}{\partial r} + \frac{a(r)}{c}\left(\mu \frac{\partial f}{\partial y} + \frac{(1-\mu^2)}{y}\frac{\partial f}{\partial \mu} \right) + \frac{3}{2} c \frac{y}{r}(1-\mu^2)\frac{\partial f}{\partial \mu}. \quad (9)$$

Using multiple scattering due to cumulative binary Coulomb collisions [*Hinton*, 1983], the collision term between the incident ions (α) and the target ions (β) can also be written

$$\left(\frac{df}{dt}\right)_c = \sum_\beta 4\pi \frac{Z_\alpha^2 Z_\beta^2 e^4 \ln\Lambda}{m_\alpha^2} \times \left[-\frac{\partial}{\partial v_i}\left(f_\alpha(\mathbf{r},\mathbf{v},t) \frac{\partial h_\beta}{\partial v_i} \right) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(f_\alpha(\mathbf{r},\mathbf{v},t) \frac{\partial^2 g_\beta}{\partial v_i \partial v_j} \right) \right] \quad (10)$$

where

$$h_\beta = \left(1 + \frac{m_\alpha}{m_\beta}\right) \int \frac{f_\beta(\mathbf{r},\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|} d\mathbf{v}' \quad (11)$$

$$g_\beta = \int f_\beta(\mathbf{r},\mathbf{v}')|\mathbf{v}-\mathbf{v}'| d\mathbf{v}' \quad (12)$$

We restrict this first study to the transport equation of the H^+ ions, considered as a minor species in a background composed of oxygen ions and electrons. We assume that the background plasma is in isothermal hydrostatic equilibrium. Indeed, their thermal velocity is much smaller than their escape velocities and their escape flux is relatively small.

Therefore the O^+ ions have a Maxwellian velocity distribution characterized by zero bulk velocity $\mathbf{u} = 0$ and a temperature T_β which is independent of r:

$$f_\beta(\mathbf{r},\mathbf{v}') = n_\beta \left(\frac{m_\beta}{2\pi k T_\beta} \right)^{3/2} \exp(-x'^2) \exp[-q_\beta(r)] \quad (13)$$

where $x' = \sqrt{m_\beta/(2kT_\beta)}\, v'$ and $\exp[-q_\beta(r)]$ determines the hydrostatic distribution of the O^+ ion density versus altitude, n_β is the number density at the bottom of the transition layer where $q_\beta = 0$.

Since H^+ ions remain minor constituent in the transition region we consider only the $H^+ - O^+$ collisions and assume that the velocity distribution function of the background O^+ ions is almost unperturbed by the outflow of the minor H^+ ions. The collisions $H^+ - H^+$ can be neglected in a first approximation. In a future study, this assumption will be relaxed. The collisions of H^+ ions with the thermal electrons can also be neglected since the rate of change of H^+ ions momentum as a result of collisions with electrons is $\sqrt{m_{O^+}/m_{e^-}}$ time smaller than for $O^+ - H^+$ collisions.

The quasi-neutrality condition for a plasma in static equilibrium in the gravitational field mainly composed by O^+ ions implies that the electric field is close to the Pannekoek-Rosseland field

$$\mathbf{E}(r) = -\frac{(m_{O^+} - m_{e^-})}{2e}\mathbf{g}(r). \quad (14)$$

When $f_\beta(v')$ is a Maxwellian distribution, the integrals (11) and (12) can be calculated analytically:

$$h_\beta(x) = n_\beta\sqrt{\frac{m_\beta}{2kT_\beta}}\exp(-q_\beta(r)) \times \left(1+\frac{m_\alpha}{m_\beta}\right)\frac{\mathrm{erf}(x)}{x} \quad (15)$$

$$g_\beta(x) = n_\beta\sqrt{\frac{2kT_\beta}{m_\beta}}\exp(-q_\beta(r))\left[\mathrm{erf}(x)\left(x+\frac{1}{2x}\right) + \frac{1}{\sqrt{\pi}}\exp(-x^2)\right] \quad (16)$$

where $\mathrm{erf}(x)$ is the error function and $x = \sqrt{m_\beta/(2kT_\beta)}v$.

Defining

$$\mathcal{F}(x) = \frac{\mathrm{erf}(x)}{x} - \frac{2}{\sqrt{\pi}}\exp(-x^2) \quad (17)$$

the collision term (2) can be written

$$\left(\frac{df}{dt}\right)_c = \frac{n_\beta c_0}{m_\alpha^2}\exp(-q_\beta(r))\left(\frac{m_\beta}{2kT_\beta}\right)^{3/2} \times \left[\frac{1}{x^2}\frac{\partial}{\partial x}\left\{\mathcal{F}(x)\left(2\frac{m_\alpha}{m_\beta}xf+\frac{\partial f}{\partial x}\right)\right\} + \frac{1}{x^3}\frac{\partial g_\beta}{\partial x}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu}\right] \quad (18)$$

with $c_0 = Z_\alpha^2 Z_\beta^2 e^4 \ln\Lambda$.

With the definitions of y and x, it can be verified that

$$y = \sqrt{\frac{m_{\mathrm{H}^+}T_{\mathrm{O}^+}}{m_{\mathrm{O}^+}T_{\mathrm{H}^+}}}x = \frac{w_{\mathrm{O}^+}}{w_{\mathrm{H}^+}}x, \quad (19)$$

Using these variables the Fokker-Planck, (1) becomes

$$\frac{\partial f}{\partial t} + w_{\mathrm{H}^+}\, y\mu\frac{\partial f}{\partial r} + \frac{a(r)}{w_{\mathrm{H}^+}}\left(\mu\frac{\partial f}{\partial y} + \frac{(1-\mu^2)}{y}\frac{\partial f}{\partial \mu}\right) + \frac{3}{2}w_{\mathrm{H}^+}\frac{y}{r}(1-\mu^2)\frac{\partial f}{\partial \mu}$$
$$= \frac{c_0}{m_{\mathrm{H}^+}^2}n_{\mathrm{O}^+}\exp(-q_{\mathrm{O}^+}(r))\left(\frac{m_{\mathrm{O}^+}}{2kT_{\mathrm{O}^+}}\right)^{3/2}\left(\frac{w_{\mathrm{O}^+}}{w_{\mathrm{H}^+}}\right)^4 \times \left[\frac{1}{y^2}\frac{\partial}{\partial y}\left\{\left(2yf\frac{T_{\mathrm{H}^+}}{T_{\mathrm{O}^+}}+\frac{\partial f}{\partial y}\right)\mathcal{F}\left(\frac{w_{\mathrm{H}^+}}{w_{\mathrm{O}^+}}y\right)\right\} + \frac{1}{y^3}\frac{\partial g}{\partial y}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu}\right]. \quad (20)$$

This equation is similar to that derived in the article of *Shizgal et al.* [1986]. It can also be reduced to the Fokker-Planck equation used in *Lie-Svendsen and Rees* [1996].

3. Method of Resolution

We have used a specialized spectral method to solve the partial differential equation (20) in (r, y, μ) and obtain steady state solutions for the H^+ ion velocity distribution function.

First, we expand the solution in Legendre polynomials with respect to μ or $\cos\theta$ [*Canuto et al.*, 1988]:

$$f(r,y,\mu) = \sum_{l=0}^{n-1} f_l(r,y)P_l(\mu). \quad (21)$$

Calculating

$$\int_{-1}^{1}\left[\mathcal{D}f = \left(\frac{df}{dt}\right)_c\right]P_{l'}(\mu)d\mu \quad (22)$$

and taking into account the orthogonality properties of the Legendre polynomials:

$$\int_{-1}^{1} P_l(\mu)P_{l'}(\mu)d\mu = \frac{2}{2l+1}\delta_{ll'}, \quad (23)$$

the equation is decomposed into n partial differential equations where the n unknowns are $f_l(r,y)$:

$$w_{\mathrm{H}^+}y\left(\beta_1(l)\frac{\partial f_{l-1}}{\partial r} + \beta_2(l)\frac{\partial f_{l+1}}{\partial r}\right) + \frac{a(r)}{w_{\mathrm{H}^+}}\left(\beta_1(l)\frac{\partial f_{l-1}}{\partial y} + \beta_2(l)\frac{\partial f_{l+1}}{\partial y}\right) + \left(\frac{a(r)}{yw_{\mathrm{H}^+}} + \frac{3}{2}\frac{w_{\mathrm{H}^+}y}{r}\right)\left(\beta_3(l)f_{l-1} + \beta_4(l)f_{l+1}\right)$$
$$= C_0\left[\frac{1}{y^2}\frac{\partial}{\partial y}\left\{\left(2yf_l\frac{T_{\mathrm{H}^+}}{T_{\mathrm{O}^+}} + \frac{\partial f_l}{\partial y}\right)\mathcal{F}\left(\frac{w_{\mathrm{H}^+}}{w_{\mathrm{O}^+}}y\right)\right\}\frac{2}{2l+1} + \frac{1}{y^3}\frac{\partial g}{\partial y}\left(\frac{-2l(l+1)}{2l+1}f_l\right)\right] \quad (24)$$

with $C_0 = \frac{n_{\mathrm{O}^+}c_0}{m_{\mathrm{H}^+}^2}\exp[-q_{\mathrm{O}^+}(r)]\left(\frac{m_{\mathrm{O}^+}}{2kT_{\mathrm{O}^+}}\right)^{3/2}\left(\frac{w_{\mathrm{O}^+}}{w_{\mathrm{H}^+}}\right)^4$

$$\beta_1(l) = \frac{2l}{(2l+1)(2l-1)} \quad (25)$$

$$\beta_2(l) = \frac{2(l+1)}{(2l+1)(2l+3)} \quad (26)$$

$$\beta_3(l) = \frac{-2(l-1)l}{(2l+1)(2l-1)} \quad (27)$$

$$\beta_4(l) = \frac{2(l+1)(l+2)}{(2l+1)(2l+3)}. \quad (28)$$

The unknowns $f_0, ..., f_{n-1}$ are functions of the altitude (r) and of the velocity of the particles (y). To determine the dependence of f_l on y, we introduce another spectral decomposition. The partial derivatives with respect to the y variable are expanded using the quadrature differential method (QDM) or discrete ordinate (DO) method outlined by *Shizgal* [1981], *Shizgal and Blackmore* [1984], *Mansell et al.* [1993], and *Shizgal and Chen* [1996; 1997].

The integral of a function $G(y)$ continuous on an interval $[a, b]$ where it has a weight function $W(y)$ can be evaluated rather accurately by a weighted sum of values of $G(y)$ at a series of discrete points y_i:

$$\int_a^b W(y)G(y)dy \simeq \sum_{i=0}^{N-1} w_i G(y_i) \quad (29)$$

where y_i are the N roots of the polynomial $Q_N(y)$ of degree N associated to the weight function $W(y)$ defined on $[a, b]$; $G(y_i)$ are the values of the function for the points y_i and w_i are corresponding weights [*Canuto et al.*, 1988; *Press et al.*, 1992].

In this discrete ordinate basis, the derivatives of any continuous function $G(y)$ can be approximated by the following expansion:

$$\left(\frac{\partial G}{\partial y}\right)_{y=y_i} \simeq \sum_{j=0}^{N-1} D_{ij} G(y_j) \quad (30)$$

where D_{ij} are the matrix elements of the derivative operator in the polynomial basis [*Shizgal and Blackmore*, 1984].

This discretisation method has been used in several applications in kinetic theory of nonuniform gases. Equations describing the velocity distribution function in the Milne problem or the transport of neutral hydrogen in the neutral atmosphere-exosphere and involving integrodifferential operators have been successfully solved by *Blackmore and Shizgal* [1985] and *Shizgal and Blackmore* [1985] using this quadrature differential method and "speed polynomials" $Q_N = S_N(y)$ whose weight function is $W(y) = y^2 \exp(-y^2)$. The Coulomb Milne problem [*Lindenfeld and Shizgal*, 1983; *Barrett et al.*, 1992] involves the Fokker-Planck operator of (20).

The Maxwellian function with the same temperature and the same bulk velocity as the background ions is a steady state solution of the Fokker-Planck equation, when the escape flux of H^+ ions is strictly equal to zero. However, since in the polar wind the upward flux of H^+ is not equal to zero, the isotropic Maxwellian velocity distribution function is not a solution corresponding to the polar wind case.

To absorb the exponential variation of f for VDFs near thermal equilibrium, it is convenient to introduce the function $f' = f \exp(y^2)$ and to solve the equation for the f' function. At thermal equilibrium, f' is a constant in the whole velocity space.

The transport equation as well as its solution is expanded in $n = 10$ Legendre polynomials $P_l(\mu)$ for the variable μ. The coefficients $f_l'(r, y)$ in (21) are developed in linear combination of $N = 10$ speed polynomials $S_s(y)$:

$$f(r, y, \mu) = \exp(-y^2)\left(\sum_{l=0}^{n-1}\sum_{s=0}^{N-1} a_{ls}(r) P_l(\mu) S_s(y)\right). \quad (31)$$

where the matrix elements $a_{ls}(r)$ are the coefficients of the expansion whose dependence on altitude is sought. The speed polynomials form a basis of orthonormal functions:

$$\int_0^\infty y^2 \exp(-y^2) S_s(y) S_r(y) dy = \delta_{sr}. \quad (32)$$

The dependence of a_{ls} on r is determined by an iterative Runge-Kutta method based on an algorithm described by *Brankin et al.* [1991]. Boundary conditions are needed. They will be imposed at the top of the transition layer and not at the bottom of the transition region as it is common practice. Before we discuss these boundary conditions, we must first introduce regularity conditions that any solution of (20) must satisfy everywhere included at r_0.

4. Regularity Conditions

A study of the behavior of the Fokker-Planck equation (20) for $y \to 0$ and $\mu \to 0$ shows that the velocity distribution function of the H^+ ions must satisfy the following regularity conditions:

$$\frac{1}{y}\frac{\partial g}{\partial y}\left[\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f(0,y,\mu)}{\partial \mu}\right] \to 0 \quad (33)$$

when $y \to 0$ for any μ values

$$\left\{\frac{\partial}{\partial y}\left[\left(\frac{\partial f(0,y,\mu)}{\partial y} + \frac{T_{H^+}}{T_{O^+}} 2y f(0,y,\mu)\right)\mathcal{F}\right]\right\} \to 0 \quad (34)$$

when $y \to 0$ for any μ values

$$\frac{a(r)}{w_{H^+}}\frac{1}{y}\frac{\partial f}{\partial \mu} + \frac{3}{2} w_{H^+} \frac{y}{r}\frac{\partial f}{\partial \mu} - C_0\left\{\frac{1}{y^2}\frac{\partial}{\partial y}\left[\left(2yf\frac{T_{H^+}}{T_{O^+}} + \frac{\partial f}{\partial y}\right)\mathcal{F}\left(\frac{w_{H^+}}{w_{O^+}}y\right)\right] + \frac{1}{y^3}\frac{\partial g}{\partial y}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu}\right\} \to 0 \quad (35)$$

when $\mu \to 0$ for any y values.

If these conditions are not satisfied, the collision term on the right-hand side of (20) diverges and tends to ∞ in the limit of $y \to 0$ for any value of $\mu = \cos\theta$. It diverges also for any non-zero value of y when $\mu \to 0$, i.e., when the velocity vector of the particles is in the horizontal direction. When we impose a function $f(r_0, y, \mu)$ which does not satisfy (33), (34), and (35) at the boundary r_0, the numerical integration code does not converge toward a stable numerical solution of the Fokker-Planck equation.

Therefore these regularity conditions are essential to obtain numerically stable solutions of the time independent Fokker-Planck equation. They are consequences of the divergence of the classical Coulomb collision cross section when the relative velocity of the interacting particles tends to zero, i.e., when $y \to 0$. In the limit of $y \to 0$, the acceleration and convection terms in the left-hand side of (20) cannot be balanced by the collision term of the right-hand side, unless (33) and (34) are satisfied.

The conditions (33) and (34) imply that the VDF is locally isotropic and Maxwellian for small velocities, while the condition (35) implies that it is also Maxwellian for all horizontal velocities, i.e., for $\theta = 90°$ or $\mu = 0$.

The condition (35) implies that the distribution of horizontal velocities (i.e. for $\mu = 0$) is a non-displaced Maxwellian function like that of the background O^+ ions. Note that the dispersion of the H^+ ions velocities can be characterized by a temperature T_{H^+}, which is different from that of the O^+ ions.

In the limit $y \to 0$, the condition (33) implies that the VDF is isotropic around the origin in the velocity space, i.e., that its symmetry about the origin is similar to that of the O^+ ion VDF. Indeed, we have assumed that the bulk velocity of the background particles is equal to zero ($\mathbf{u}_{O^+} = 0$) and that it has no temperature anisotropy, i.e., $T_{O^+\parallel} = T_{O^+\perp}$.

The condition (34) implies that the VDF tends to a Maxwellian when $y \to 0$. Note that these regularity conditions have not been taken into account in the polar wind model of *Lie-Svendsen and Rees* [1996]. This may be the reason why their velocity distribution function is not determined in the vicinity of the origin in the velocity space for $v_\parallel \to 0$ and $v_\perp \to 0$.

To determine a velocity distribution function at r_0 with the required properties for $\mu \to 0$ and $y \to 0$, we expand this function in polynomials of μ and y [*Pierrard*, 1997]:

$$f(r_0, y, \mu) = \exp(-y^2) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_{nm} \mu^m y^n \qquad (36)$$

where b_{nm} are constant to be determined by the regularity and the boundary conditions. The values of the H^+ density $n_{H^+}(r_0)$, escape flux $F_{H^+}(r_0)$ and of higher-order moments of the VDF at r_0 determine additional relations that the set $\{b_{nm}\}$ must satisfy. These additional relations will be discussed in more details in the next section. It can be verified that the conditions (33), (34), and (35) require that $b_{nm} = 0$ for $n < 3$ and $m < 3$.

It should be pointed out that the truncated Maxwellian VDF assumed at the exobase of *Lemaire and Scherer* [1970, 1971, 1974] and *Pierrard and Lemaire*'s [1996] exospheric models satisfy the regularity conditions in the whole exosphere when the escape velocity is positive. However, the displaced Maxwellian velocity distribution function implicitely assumed in hydrodynamical models of the polar wind do not satisfy these regularity conditions.

5. Additional Boundary Conditions

As indicated above, the coefficients b_{nm} of the polynomial expansion (36) of $f(r_0, y, \mu)$ must satisfy algebraic relations which are obtained by replacing (36) in the Fokker-Planck equation (20) and by matching terms of the same powers of μ and y in the right- and left-hand sides. Additional relations are imposed on the set of b_{nm} coefficients by forcing a given number of moments of the VDF $f(r_0, y, \mu)$ to be equal to certain fixed values at the upper boundary of the integration domain, i.e., at $r = r_0$.

The moments of the VDF can be calculated by using the polynomial expansion (37):

$$f(r, y, \mu) = \exp(-y^2) \left(\sum_{l=0}^{n-1} \sum_{s=0}^{N-1} a_{ls}(r) P_l(\mu) S_m(y) \right). \qquad (37)$$

For instance the density at any altitude r (i.e., the moment of zeroth order of this VDF) is given by

$$\begin{aligned} n(r) &= 2\pi \int_{-1}^{1} d\mu \int_0^{\infty} y^2 \exp(-y^2) \sum_{l=0}^{n-1} f_l'(r, y) P_l(\mu) dy \\ &= 4\pi \int_0^{\infty} f_0'(r, y) y^2 \exp(-y^2) dy \qquad (38) \end{aligned}$$

since $\int_{-1}^{1} P_0(\mu) P_l(\mu) d\mu = 2\delta_{l0}$ by orthogonality of the Legendre polynomials and since $P_0(\mu) = 1$.

Using the expansion in speed polynomials introduced by *Shizgal and Blackmore* [1984], one obtains

$$\begin{aligned} n(r) = &= 4\pi \int_0^{\infty} \sum_{s=0}^{N-1} a_{0s}(r) S_s(y) y^2 \exp(-y^2) dy \\ &= 2\pi \sqrt[4]{\pi} a_{00}(r) \qquad (39) \end{aligned}$$

since $\int_0^{\infty} S_0(y) S_s(y) y^2 \exp(-y^2) dy = \delta_{s0}$ by orthonormality of the speed polynomials and since $S_0(y) = 2/\sqrt[4]{\pi}$. Assuming $n(r_0)$, the H^+ density at r_0 is for instance equal to $100\,\mathrm{cm}^{-3}$, the value of $a_{00}(r_0)$ is then determined by (39).

Additional relations are obtained by fixing the value of the polar wind escape flux $F_{H^+}(r_0)$ which is a first order moment of the VDF. According *Hoffman* [1968] and *Moore et al.* [1997], the polar wind flux of H^+ is of the order of $10^8\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ or larger. Let us assume for instance this value as the boundary condition at 2000 km altitude. This boundary condition determines one more algebraic relation between the 10×10 coefficients $b_{nm}(r)$ and the same number of $a_{ls}(r)$:

$$\begin{aligned} F_\parallel(r) &= \frac{4\pi}{3} \sqrt{\frac{2kT_0}{m}} \int_0^{\infty} f_1'(r, y) y^3 \exp(-y^2) dy \qquad (40) \\ &= \frac{8}{3\sqrt{\pi}} \sqrt{\frac{2kT_0}{m}} \left(\sqrt{\frac{4}{\sqrt{\pi}\left(\frac{3\pi}{8} - 1\right)}} a_{11}(r) + \frac{\sqrt[4]{\pi}}{2} a_{10}(r) \right) \end{aligned}$$

since

$$S_1(y) = \sqrt{\frac{4}{\sqrt{\pi}\left(\frac{3\pi}{8}-1\right)}}\left(\frac{\sqrt{\pi}}{2}y-1\right) \tag{41}$$

and $P_1(\mu) = \mu$.

The parallel and perpendicular temperatures at r_0 can be used to obtain two additional linear algebraic relationships between the $\{a_{ls}\}$. In this first study case, we assume that $T_{\parallel H^+}(r_0) = 2400$ K and $T_{\perp H^+}(r_0) = 1500$ K.

The value of higher moments of the H^+ velocity distribution function are used to complete this set of linear equations which enables to determine the values of the same number of coefficients $\{b_{nm}\}$ and $\{a_{ls}\}$. Indeed, by expanding the Legendre polynomials $P_l(\mu)$ and speed polynomials $S_m(y)$ in powers of μ and y, one obtains an expansion similar to (36) where the coefficient of the term $y^n\mu^m$ is a linear combination of the coefficients $\{a_{ls}\}$. The value of $b_{nm}(r)$ is equal to $B_{nm}(\{a_{ls}\})$. Additional relations between the $\{b_{nm}\}$ come from the regularity conditions.

This procedure shows that there is a great flexibility and freedom in the choice of the $f(r_0, y, \mu)$. The number of coefficients $\{b_{nm}\}$ or $\{a_{ls}\}$ which can be determined this way depends on the number of moments of the VDF imposed at the boundary r_0 of the transition region. This number can be increased by using a larger number of moments of the VDF to define the boundary conditions at r_0.

All other coefficients b_{nm} which cannot be determined by this limited set of equations can be set equal to zero: $b_{nm} = 0$ for $n > n_{\max}$ and $m > m_{\max}$. Similar truncations are adopted in Chapman-Enskogg or Grad expansions [*Grad*, 1949; *Chapman and Cowling*, 1970] of the VDF based on the expectation that the series of coefficients b_{nm} tend to zero when n and m become large.

Since we have determined the values of b_{nm} which correspond to that of the functions $B_{nm}(\{a_{ls}\})$ at r_0, we can determine the values of a minimum number of coefficients $a_{ls}(r_0)$ at the reference altitude r_0, by resolving the set of linear equations:

$$B_{nm}(\{a_{ls}\}) = b_{nm}. \tag{42}$$

Because of the limited number of boundary condition, only a limited number of coefficient b_{nm} and a_{ls} can be determined uniquely.

Once all the values of $a_{ls}(r_0)$ are known for $l < n$ and $s < N$, the set of differential equations

$$\frac{df_i'(r, y_i)}{dr} = \mathcal{L}[f_k'(r, y_j)] \tag{43}$$

can be integrated between r_0 and r_1 by a Runge-Kutta numerical method to obtain the values of $a_{ls}(r)$ above or below r_0.

Appropriate initial values for $a_{ls}(r_0)$ or for $b_{nm}(r_0)$ are those which give physically realistic VDFs at high altitudes. We chose to fix the regularity and boundary conditions at the top of the transition region where we know that most of the particles escape. Above this altitude, in the exosphere, the collisions are of minor importance and the well-developed asymmetrical "halo" component of the VDF predominates in the velocity space over the isotropic and Maxwellian "core" component close to $y = 0$.

Our choice of the VDF at r_0 was guided by the results obtained by the Monte Carlo simulations of the polar wind by *Barakat et al.* [1995]. The velocity distribution that we adopted at the top of the integration interval was also inspired by those usually adopted in exospheric models [*Lemaire and Scherer*, 1970, 1973; *Pierrard and Lemaire*, 1996]. In these models, the VDF generally have (1) an isotropic Maxwellian core around the origin in the velocity space and (2) a highly asymmetrical halo population of suprathermal escaping particles. The high-altitude boundary conditions are essential in the resolution of the Fokker-Planck equation. They determine how much particles can escape out of the transition region. The flux of particle at the top of the transition region will be maximum if the magnetic flux tubes are "open" and when nonlocal collisions in the exosphere may be completely neglected. However, when magnetic flux tubes are not opened and/or if distant collisions contribute to backscatter a fraction of the "escaping" particles, the net upward flux is reduced. Whatever value we assume for the flux of escaping par-

Table 1. Values of the Moments of the VDF of H^+ Ions at Different Altitudes in the Transition Region.

	Altitude, km 1000	1500	2000
Density n_{H^+}, cm^{-3}	1590	350	100
Flux F_{H^+}, cm^{-2}s^{-1}	1.5×10^8	1.2×10^8	10^8
Bulk velocity u_{H^+}, km s^{-1}	0.9	3.4	10
Parallel temperature $T_{\parallel H^+}$, K	1120	1348	2400
Perpendicular temperature $T_{\perp H^+}$, K	1014	1060	1500
Parallel energy flux $\epsilon_{\parallel H^+}$, ergs cm^{-2}s^{-1}	3.65×10^{-8}	2.55×10^{-8}	1.70×10^{-10}
Perpendicular energy flux $\epsilon_{\perp H^+}$, ergs cm^{-2}s^{-1}	7.1×10^{-8}	3.88×10^{-9}	4.75×10^{-9}
Knudsen number l_{H^+}/H_{O^+}	5	0.5	0.1
Oxygen density n_{O^+}	10^4	2.7×10^3	2×10^2

ticles at the top of the transition region, the solution of the Fokker-Planck equation tends to an almost isotropic maxwellian at low altitude where the collisions become more and more important.

The results presented below are obtained for the VDF illustrated in of Figure 1a at the altitude 2000 km corresponding to the top of the interval of integration. Column 2 in Table 1 shows the density, flux, bulk velocity, parallel and perpendicular temperatures, and flux of energies carried by the H^+ ions at the top of the transition region, taken as reference altitude r_0.

Once the coefficients $a_{ls}(r)$ of the polynomial expansion (36) are obtained at all altitudes by solving (43), one has an analytic representation of the VDF at the quadrature points in the velocity space. The moments of the VDF can then also be calculated anywhere below r_0 by expressions similar to (39) and (41). All other moments of the VDF are determined by linear combinations of the $a_{ls}(r)$ coefficients and can be calculated at any altitude below r_0. A study of the positivity and convergence of the velocity distribution function similar to that of *Leblanc and Hubert* [1997] will be presented in a separate article.

The solutions of the Fokker Planck equation in the transition region between 2000 and 1000 km is presented in the following section.

6. Results and Discussion

The values of the moments of the VDF are given in Table 1 at 2000, 1500, and at the bottom of the integration interval, i.e., 1000 km. The moments of the velocity distribution function are not very affected by the number of polynomials taken in the expansion except if $n < 8$ or $N < 8$. The differences of the value of the moments for simulations with 10, 16, and 20 polynomials are lower than 5%. The oxygen density is also given in Table 1. These ions are everywhere more abundant than the H^+ ions diffusing through the transition layer, i.e., that the H^+H^+ collisions could be neglected compared to the H^+O^+ collisions at least in a first approximation. The Knudsen number for H^+ ions is also given. It can be seen that it changes from a small value at the bottom of the transition region to a large one at the top.

Figure 1 illustrates isocontours of the VDF in the plane $v_{\parallel}, v_{\perp}$ at these same altitudes for these boundary conditions. The Figure 1a illustrates the VDF adopted as boundary condition at the top where the collision frequency of H^+ ions with the O^+ ions is significantly reduced and where the Knudsen number of a thermal H^+ ion is equal to 5.

This velocity distribution function which is similar to that obtained by *Barakat et al.* [1995] using a Monte Carlo simulation, has a "lima-bean" shape. It is also similar to VDF obtained by *Lie-Svendsen and Rees* [1996] at 2500 km altitude by using a finite difference method to solve the Fokker-Planck equation (1). *Lie-Svendsen and Rees* [1996] used boundary conditions at the top as well as at the bottom of the transition region where they assumed that the VDF is a displaced Maxwellian for all upward moving particles. A similar displacement of the VDF is assumed in standard hydrodynamical polar wind models [*Banks and Holzer*, 1969; *Schunk*, 1975]. At the top of the transition region, they assumed that the VDF is empty for all downward moving particles. This low-altitude boundary condition implies that the tail of the distribution is not enriched with suprathermal particles, i.e., that the VDF decreases exponentially.

Although not apparent in Figure 1a, the VDF that we assume at the top has a small nondisplaced isotropic Maxwellian component centred around the origin ($v_{\parallel} = 0$, $v_{\perp} = 0$, i.e., for $y = 0$). This core component cor-

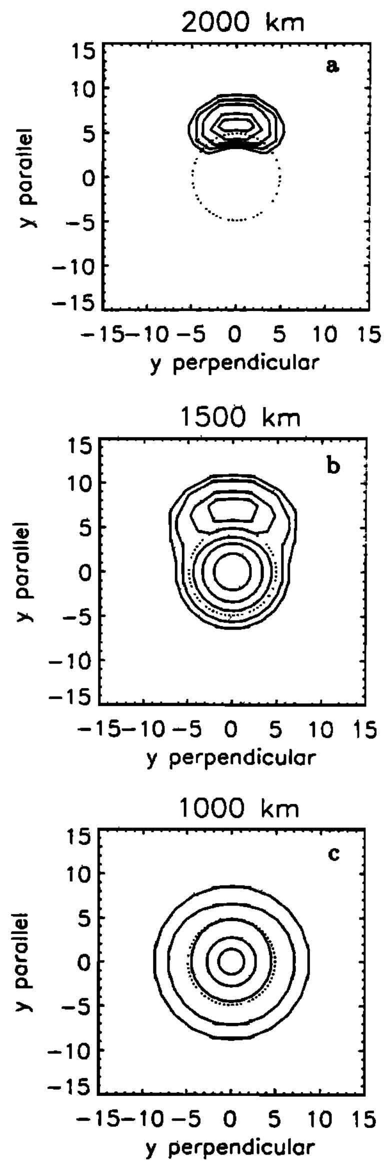

Figure 1. Contour levels of the H^+ velocity distribution function at (a) the top (2000 km), (b) 1500 km, and (c) at the bottom (1000 km) of the transition region.

responds to a population of particles which does not contribute to the net upward flux of the polar wind ions. It can be seen from Figure 1 that it is only the halo population of particles with a velocity vector in the lima-bean shaped area that contribute to the polar wind ionization flow of H^+ ions as well as to the flow of energy. Note, however, that the core population contributes to the exospheric density and pressure tensor while it do not contribute to the net field-aligned flux of particle parallel to the magnetic field direction.

Figure 1b shows that at intermediate altitudes in the transition region, the asymmetry of the VDF becomes smaller as one penetrates deeper into the collision-dominated region. The almost isotropic and almost Maxwellian core component becomes more prominant at the expense of the highly asymmetric halo component formed of suprathermal particles populating the tail of the VDF.

This figure also shows that the dispersion of the H^+ ions velocities becomes highly anisotropic in the transition region and in the exosphere where the thermalizing effect of collisions becomes less and less effective.

Note also that the field-aligned energy flux $\epsilon_{\parallel H^+}$ is mainly contributed by the halo particles. The flux of energy is directed upward, i.e., the energy is transported from the collision-dominated region to the topside exospheric region where the spread of the particles velocities (i.e., their temperatures) is larger than at the bottom of the transition. Indeed, it can be seen in Table 1 that the parallel and perpendicular temperatures increase with the altitude. It may seem paradoxical that energy may be transported from a cooler region to a hotter one, but it should be noted that the temperature parameter defined above based on the law of perfect gas should not be compared to the temperatures of classical thermodynamics where the system is collision-dominated and the VDF close to displaced Maxwellian distributions.

The field-aligned heat flux $q_{\parallel}$ is defined as the flux energy in a frame of reference moving with the bulk velocity or mean velocity particles:

$$\mathbf{q} = \epsilon - \mathbf{F}\frac{mu^2}{2}. \qquad (44)$$

The heat flux decreases with the altitude as in the Monte-Carlo simulation of *Barakat et al.* [1995].

At the bottom of the transition region the Knudsen number is equal to 0.1 for H^+ ions with velocities equal to the average thermal velocity. It can be seen from Figure 1 that the secondary peak in the VDF corresponding to the halo population has vanished at the expense of the core population centred at the origin in velocity space. At the location of this secondary peak, the slope of the VDF is just slightly smaller. This slight asymmetry determines the upward polar wind flux imposed by the continuity equation. The VDF tends more and more toward an isotropic Maxwellian as one penetrates deeper into the collision-dominated region. Figure 1 nicely illustrates the secondary peak of halo particles emerging from the almost isotropic Maxwellian VDF which is close to that of a barometric equilibrium VDF.

7. Conclusions

The kinetic model presented in this article describes the transformation of the H^+ velocity distribution function in the transition region. At high altitudes, the distribution is highly anisotropic and asymmetric. The light ions are accelerated upward by the polarization electric field produced by the gravitation charge separation of O^+ and e^-. The tips of their velocity vectors are located in a "lima-bean-shaped" region of velocity space. A population of core particles scattered by Coulomb collisions form an isotropic and Maxwellian peak in the VDF which is centered around the origin at $(v_{\parallel} = 0, v_{\perp} = 0)$.

At lower altitudes, the distribution becomes more and more isotropic due to the increasing effects of the Coulomb collisions. The halo population gradually disappears into the main peak forming the core population. This is the consequence of the strong dependence of the Coulomb collision cross section on the relative velocity of the colliding particles. Since this interaction is stronger for small velocities, one expects the VDF to be isotropic and Maxwellian around $v = 0$ (or $y = 0$), while for larger values of v (or y), the asymmetries which develop in the VDF due to their acceleration by the upward directed ambipolar electric field, is not so efficiently destroyed by Coulomb collisions than for particles with subthermal velocities.

The escape flux is due to these suprathermal particles forming the halo population. This implies that in the transition region between the collision-dominated and collisionless region the VDF changes from a singly peak almost isotropic and almost Maxwellian centred around the origin ($v = 0$), into a double-hump distribution with a second lima-bean-shaped maximum. Drifting Maxwellians and bi-Maxwellians generally used to describe the polar wind moment approximation or in hydrodynamic models are quite different from this type of velocity distribution functions. In particular, the drifting Maxwellian is not centered around zero and in this case the low-energy particles contribute to the flux as well as the suprathermal particles. On the contrary, in the case of the new distribution, the flux is mainly contributed by suprathermal particles but not by the core population. The low-energy particles are more affected by collisions with the background plasma which has no bulk velocity by assumption in the present model study.

Our model is based on the resolution of the Fokker-Planck equation and is quite similar to the kinetic model of *Lie-Svendsen and Rees* [1996]. However, they solved the Fokker-Planck equation using a finite difference numerical method, which is different from the spectral described in this paper. In our study, regularity conditions are imposed on the velocity distribution function to obtain a fast convergence toward a stable stationary numerical solution of the Fokker-Planck equation. Results of these both kinetic models are in good agreement with those of the Monte Carlo simulations [*Barghouthi et al.*, 1990; *Barakat et al.*, 1995].

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