

J. Lemaire, A. Baraket and J. M. Lesceux

Institute d'Aeronomie Spatiale de Belgique

3 Avenue Circulaire, B-1180 Brussels, Belgium

and B. Shizgal

Department of Chemistry, University of British Columbia

Vancouver, B. C. Canada V6T 1Y6

The quasi-neutrality (QN) equation is usually solved by an iterative procedure to obtain the electrostatic potential distribution in plasmas. We present, first, a new numerical method to obtain the same result more efficiently. This method is based on the numerical integration of the differential form of the QN equation. In the case when the QN approximation fails to be a valid approximation of Poisson's equation (PE), the latter equation needs to be solved. A robust numerical method to solve PE, with boundary conditions at two different altitudes, has also been proposed. This method consists in integrating numerically by the Quadrature Discretization Method (QDM) a differential form of PE. For boundary conditions corresponding to the QN solution the result coincides with the QN solution.

I. INTRODUCTION

Photoionization of the components of the terrestrial atmosphere occurs at altitudes greater than approximately 80km, which is the lower level of the ionosphere. At very high altitudes (1000km-2000km), only the lighter constituents such as protons and ionized helium exist together with a population of free electrons. The kinetic theory of this plasma under the influence of a gravitational field and the (self consistent) electrostatic field has been the subject of intense interest among both kinetic theorists and planetary scientists.¹⁻⁷

The calculation of the transport properties of this multicomponent plasma and its stability against escape from the planet has been considered by many authors.^{2,4-6} The escape of protons and alpha particles from the topside ionosphere (approximately 1500 km) is referred to as the "polar wind"^{2,3} in analogy with the supersonic expansion of the solar atmosphere which is known as the solar wind.⁶⁻⁸ A major concern with regard the escape of protons and alpha particles from the terrestrial ionosphere is the determination of the self-consistent electrostatic field that plays an important role in the determination of the distribution functions of the ions and electrons.

The quasi-neutrality condition is given by,

$$\sum_i Z_i n_i(h) = 0 \quad (1)$$

where Z_i and $n_i(h)$ are the charge and number density profile of the i th constituent, and h is the altitude. Equation (1) is generally used to calculate the electrostatic potential distribution $\phi_E(h)$, in geophysical and astrophysical plasmas.⁷ Indeed, the gravitational force acting with different strengths on the ions and on the electrons produces a charge separation E-field tending to preserve local and global charge neutrality of the plasma.

Equation (1) is a zero order approximation to the solution of Poisson's equation for the electrostatic potential

$$\nabla^2 \phi_E(h) = - (e/\epsilon_0) \sum_i Z_i n_i(h) \quad (2)$$

where e is the electronic charge. In all cases when the Debye length is much smaller than the characteristic scale length of the system, the electric potential distribution, $\phi_E(h)$, obtained by solving Eq. (1) iteratively is a satisfactory zero order solution of Poisson's equation, Eq. (2).

However, at large radial distances, where the geophysical plasma density tends to zero (i.e. where the Debye length becomes arbitrary large) Poisson's equation, which is a non-linear differential equation, must be solved subject to appropriate boundary conditions in order to determine the correct solution. Similarly, at the interface between two regions filled with plasma of different temperatures, densities, electrostatic double layers occur, and the quasi-neutrality approximation gives non-realistic solutions of Eq.(2). The numerical solution of Eq.(2) is generally difficult, since the right hand side of Poisson's equation is very small. It is given by the difference of the electron and ion densities which are very large and almost exactly equal. Truncation errors lead in this case to numerically unstable solutions.

We have found an original mathematical method to avoid this numerical difficulty. This new method, based on the numerical integration of a higher order differential equation derived from Eq.(2), gives accurate and numerically stable solutions for $\phi_E(h)$. This method is applicable to the case when the Debye length is small compared to the scale length of variation of ϕ_E . It also gives accurate results in the case when the quasi-neutrality condition, Eq.(1) is a satisfactory first order solution. This method has been applied to the calculation of the ambipolar electric potential distribution in the Earth's ionosphere. Also, a Quadrature Differential Method (QDM) has been used to calculate the solution of the differential equations for boundary conditions given at two different altitudes, and the solutions obtained were compared with the other methods of solution.

II. ION DENSITY DISTRIBUTIONS

The ion and electron density distributions in planetary ionospheres do not depend only on the gravitational potential ϕ_g given by, $\phi_g(h) = -GM/(R+h)$, but also on the polarisation electric field distribution which is induced in the plasma by the gravitational forces acting on the ions and lighter electrons. Indeed the gravitational force for ions is m_i/m_e times larger than that for the electrons, where m_i and m_e are the ion and electron masses respectively. These forces tend to produce a charge separation which induces the polarisation electric field maintaining local quasi-neutrality in the whole ionosphere.

With the assumption that the atmosphere is isothermal and in hydrostatic equilibrium, the density distribution of particles of charge $Z_j e$ and mass m_j is given by:

$$n_j(h) = n_j(h_0) \exp[(-m_j \phi_g - Z_j e \phi_E)/kT_j] \quad (3)$$

where $\phi_E(h)$ represents the electrostatic potential; T_j are the temperatures of the ions (i) and electrons (e); $n_j(h_0)$ are their number densities at the reference altitude, h_0 , that we shall take equal to 1000 km appropriate to the Earth's ionosphere. This implies that each constituent is in diffusive equilibrium in the gravitational and electrostatic fields: $g = -d\phi_g/dh$ and $E = -d\phi_E/dh$.

The hydrostatic equations for ions and electrons are then given by :

$$kT_i \, dn_i/dh = -n_i m_i \, d\phi_g/dh - Z_i e n_i \, d\phi_E/dh \quad (4)$$

$$kT_e \, dn_e/dh = -n_e m_e \, d\phi_g/dh - Z_e e n_e \, d\phi_E/dh \quad (5)$$

III. ELECTROSTATIC POTENTIAL DISTRIBUTION

We obtain the electrostatic field intensity for which the plasma is locally quasi-neutral ($\sum_j Z_j n_j = 0$) by adding Eqs.(4) and (5) after having multiplied their l.h.s and r.h.s. respectively by Z_i and Z_e .

$$-(\sum_j Z_j n_j m_j) \, d\phi_g/dh - (\sum_j Z_j^2 e n_j) \, d\phi_E/dh = 0 \quad (6)$$

Indeed when the quasi-neutrality condition is verified, the gradients of the particle densities satisfy the relation :

$\sum_i Z_i \, dn_i/dh = -Z_e \, dn_e/dh$. Using the definitions of g and E , we find that

$$e E = - \frac{\sum_j Z_j m_j n_j / kT_j}{\sum_j Z_j^2 n_j / kT_j} g \quad (7)$$

This relation is the generalisation for a multi-ionic plasma of the so called Pannekoek-Rosseland (PR) formula which was established by Pannekoek⁹ and Rosseland¹⁰ for a fully ionized hydrogen stellar atmosphere. In this case the PR electrostatic potential is simply

related to the gravitational potential by :

$$\phi_E(h) = -(m_p - m_e) \phi_g(h) / (2e) + \text{constant}.$$

It is easy to verify that under these conditions the total (gravitational + electrostatic) force and potential energy is the same for the electrons and protons. From Eq. (3), the proton density is strictly equal to the electron density at all altitudes if this is true at the reference altitude h_0 , as imposed by the quasi-neutrality condition.

IV. GENERAL SOLUTION OF THE QUASI NEUTRALITY (QN) EQUATION.

For multi-ionic plasmas there is no such simple formula, and the method then used to determine the value of ϕ_E at a given altitude h_1 , is to iterate until that value is found such that the r.h.s. of the quasi-neutrality Eq. (1) is equal to zero with an arbitrarily chosen precision.

The QN equation is a non-linear function of ϕ_E . Common iteration procedures are the bracketing or bisection methods, which can be quite time-consuming in certain cases; furthermore, these numerical methods fail sometimes to converge toward the expected solution. But in general, when the initial value of $\phi_E(h_1)$ in the iteration procedure is close enough to the solution, convergence is generally rather fast. This numerical method, well documented in Press et al.¹¹, has been used for instance by Lemaire and Scherer^{2,6} to determine the electrostatic potential distribution in kinetic polar wind and solar wind models. These numerical methods can become cumbersome and imprecise when first and second order derivatives need to be computed numerically from a series of values of $\phi_E(h_n)$ corresponding to a set of different altitudes h_1, h_2, \dots, h_{N-1} . In the next section we propose a more efficient and faster method to determine the solution of the QN equation.

V. SOLUTION OF QN EQUATION BY NUMERICAL INTEGRATION

Instead of solving the algebraic equation (1) as described above, we found that it is easier and faster to obtain the solution of the QN equation by a different method. This new procedure consists of differentiating analytically the QN Eq.(1), and then integrating numerically the first order differential equation obtained with an appropriate boundary condition at h_0 . With Eq.(3), we find that this differential equation becomes:

$$A(\phi_E) \, d\phi_E/dh = B(\phi_E) \quad (8)$$

where A and B are non-linear functions of ϕ_E :

$$A = (e^2/\epsilon_0) \sum_j [Z_j^2 N_j(h)/kT_j] \exp[-Z_j e \phi_E/kT_j] \quad (9)$$

$$B = -(\epsilon/\epsilon_0) \sum_j [Z_j N_j(h) m_j g/kT_j] \exp[-Z_j e \phi_E/kT_j] \quad (10)$$

with

$$N_j(h) = n_j(h_0) \exp[-m_j \phi_E/kT_j] \quad (11)$$

The boundary condition at h_0 is imposed to be $\phi_E = 0$, in order to satisfy the QN at the reference altitude. The solution of Eq.(8) has been calculated for $n_{H^+}(h_0) = n_e(h_0) = 800 \text{ cm}^{-3}$; $T_{H^+} = T_e = 3000 \text{ K}$ at $h_0 = 1000 \text{ km}$ altitude; $h_{N-1} = 3000 \text{ km}$ ($N = 28$).

The solid line in Fig.1 shows the distribution of ϕ_E as a function of the altitude; it corresponds to the solution of the QN equation (1) evaluated by any of the algebraic methods described in section IV. The symbols (*) in Fig.1 correspond to the values of $\phi_E(h_n)$ at the 28 quadrature points (h_n) which coincide with the zeros of the Legendre Polynomial of 28th order. The Quadrature Discretization Method (QDM) proposed by Shizgal and Blackmore¹² is one of many available numerical methods to solve the differential equation (8). Of course, a standard Runge-Kutta or Hamming algorithm can also be used to perform this numerical integration, in this case. The numerical results obtained by any of these algorithms coincide exactly with the solution of Eq.(1) obtained usually by solving this algebraic equation.

The values of the electric field, $E = -d\phi_E/dh$, are given by $-B/A$, with A and B defined as functions of h in Eqs.(8) and (9). It can be seen that $E(h)$ is not obtained here by numerical differentiation, and consequently the results with this method are more accurate than with the algebraic method outlined in section IV.

Similarly, the values of the second derivative of ϕ_E , which is proportional to the electric charge density, is easier to calculate by differentiating B/A analytically than by differentiating $\phi_E(h_n)$ numerically as indicated in the previous section. The values of the polarisation electric field is generally small in planetary ionospheres: $E = 10^{-8}-10^{-7} \text{ V/m}$. The solid line in Fig.2 shows electric field distribution evaluated as described in section IV by solving iteratively Eq.(1) and differentiating numerically the $\phi_E(h_n)$. The symbols correspond to the results obtained at the quadrature points with the new resolution method outlined in section V. The agreement is perfect.

The distribution of the excess charge density, $\Delta n/n$, which is responsible for this polarisation electric field can be determined from

$$\Delta n/n = -\epsilon_0 / (ene) (d/dh) (B/A) \quad (13)$$

It can be seen that in the case considered, the value of $\Delta n/n$ is indeed extremely small: $[10^{-16}-10^{-15}]$. This is generally the case in plasmas where the Debye length L_D is small compared to the characteristic scale height (H) of the density distribution. The Debye length (in cm) is given by: $L_D = 7 (T/n)^{1/2}$, where T is the temperature (in K) and n the electron number density (in cm^{-3}). In the terrestrial ionosphere where $T = 3000 \text{ K}$ and $n(h_0) = 800 \text{ cm}^{-3}$, the electron density scale $H = 100 \text{ km}$, and $L_D = 13 \text{ cm}$.

However, along auroral magnetic field lines thin Double Layers (DL) are formed; within these DLs the quasi-neutrality approximation fails to be valid. Within these electrostatic sheaths confined in regions which are only a few Debye lengths thick, Poisson's equation (PE) must be solved instead of its QN approximation.

VI. RESOLUTION OF POISSON'S EQUATION (PE)

Poisson's equation (2) is a second order differential equation for ϕ_E . Its r.h.s. is a small quantity (proportional to the minute excess charge density) which is unfortunately the difference of large number densities. A small round-off error in the value of the potential ϕ_E can lead in some cases to unphysical diverging solutions, as a consequence of numerical instabilities. Therefore, the PE is difficult to integrate and to solve in the case of plasmas where the Debye length is small compared to the other characteristic scale lengths of the system. This is also the main reason why one is inclined to solve the QN approximation of PE instead of PE directly. In this section, we outline a new technique which enabled us to find numerically stable solutions of PE, in cases where other numerical methods tend to fail.

This mathematical technique consists first in differentiating analytically the two sides of PE (2) and then integrating the third order equation obtained numerically with appropriate boundary conditions. It can be seen that this procedure is similar to that followed above when we differentiated the algebraic QN equation to obtain a first order differential equation for ϕ_E . The third order differential equation derived from PE is given by :

$$d^3\phi_E/dh^3 + A(\phi_E) d\phi_E/dh = B(\phi_E) \quad (14)$$

where A and B are the non-linear functions of ϕ_E given by Eqs.(9) and (10).

The three boundary conditions can either be specified at h_0 or at h_{N-1} when Runge-Kutta or Hamon algorithms are used to integrate (14). But when a Quadrature Discretisation Method is used boundary conditions can also be defined at different points (altitudes). This is one of the main advantage and power of this alternative numerical algorithm, as compared to the other two methods.

In the following examples the boundary conditions are determined by: $\phi_E = 0$, at $h = h_0$; $\phi_E = \varphi$, at $h = h_{N-1}$; $d^2\phi_E/dh^2 = \varphi''$ at $h = h_{N-1}$; where φ and φ'' are arbitrary input data for the integration algorithm. When these boundary conditions are chosen to be precisely equal to the corresponding values of the QN solution obtained and discussed in section IV, one obtains almost exactly the same distributions as those deduced above by integrating Eq.(8). The values of the electric potential are so close to that of the QN solution shown in Fig.1 that it is difficult to distinguish them on this graph. The electric field intensity is then also everywhere the same as that of the QN solution shown in Fig.2.

When the boundary condition ϕ_E at h_{N-1} is taken to be either smaller or larger than the value corresponding to the QN solution, electrostatic potential sheaths are formed at h_0 and h_{N-1} . Large electric field intensities are then obtained in thin layers a few Debye lengths thick. However, at some distance from these thin layers the electric field intensity tends to be very small as in the QN case. This is an important property of plasmas in general.

VII. CONCLUSIONS

From this study it is concluded that differentiating analytically the QN equation and integrating it numerically is an easier and more efficient method to obtain the electrostatic potential distribution in a plasma. Therefore this procedure should replace the classical one consisting of solving the algebraic QN Eq.(1) by bracketing or bisection iterative methods.

Although the QN solution is a very good approximation for the small polarisation electric field intensity in plasmas, to calculate the large charge separation electric potential variation occurring in electrostatic double layers (generally a few Debye lengths thick), Poisson's equation (2) must be solved.

Solving Poisson's non-linear second order differential equation is numerically difficult. A numerically stable and reliable solution has been proposed above. It consists in differentiating analytically the PE (2), and integrating numerically the third order Eq. (14) for appropriate boundary conditions. With this new procedure, we have shown that the QN solution is recovered for specific boundary conditions. But this new method enables one to obtain a more general solution corresponding to double layers.

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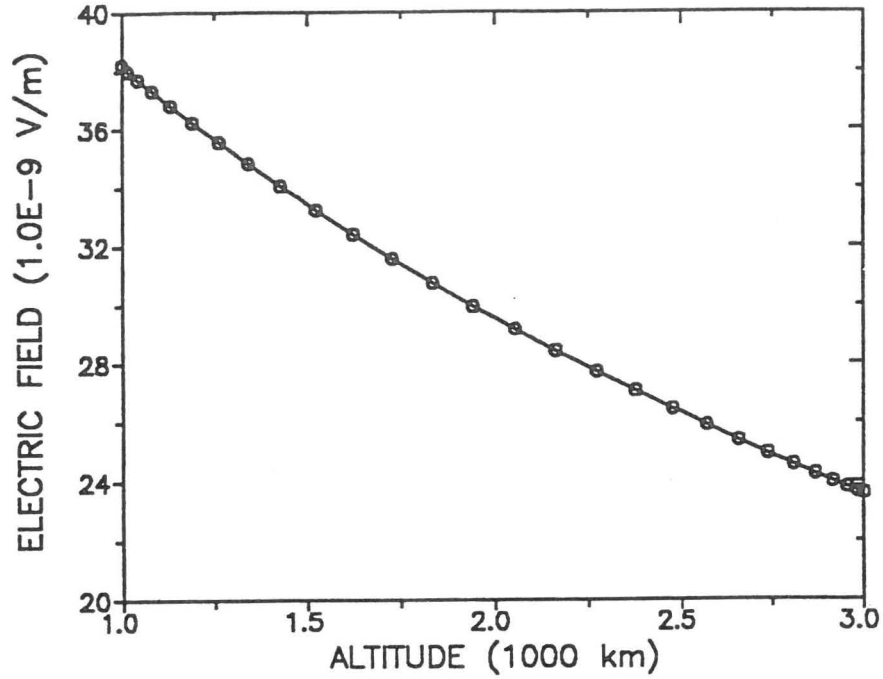


Fig.1. Electrostatic potential distribution corresponding to the quasi-neutrality (QN) solution obtained by solving the algebraic Eq.(1) by an iterative method (solid line); the symbols correspond to the QN solution obtained by integrating the differential Eq.(8) with the Quadrature Discretization Method (QDM); the solution obtained by integrating the 3rd order differential Poisson's Eq.(14) with the QDM is also given by the symbols at 28 quadrature points.

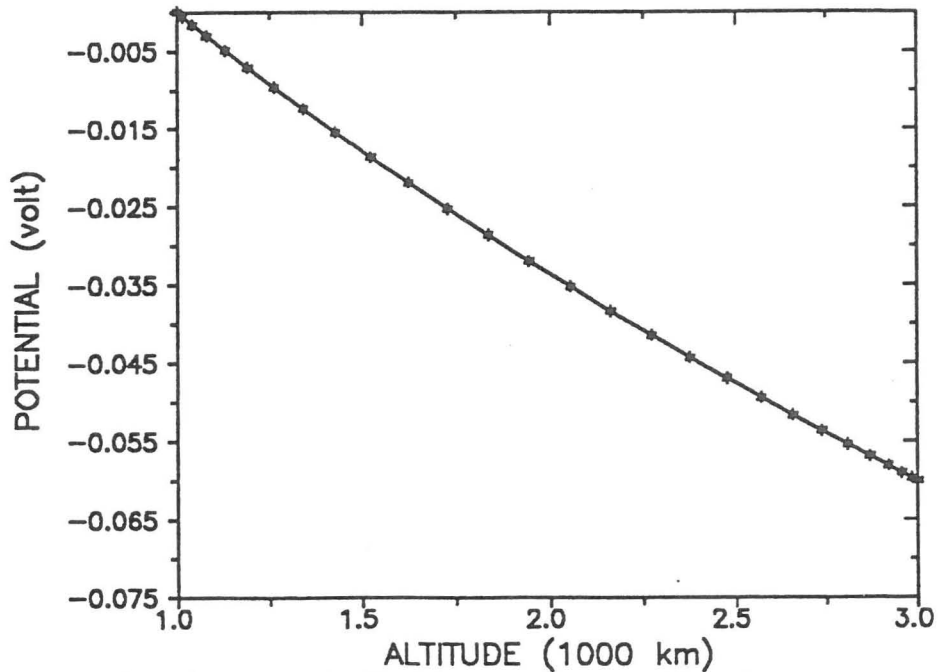


Fig.2. Electric field distribution corresponding to the quasi-neutrality (QN) solution obtained by solving the algebraic Eq.(1) by an iterative method (solid line); the symbols correspond to the QN solution obtained by integrating the differential Eq.(8) with the Quadrature Discretization Method (QDM); the solution obtained by integrating the 3rd order differential Poisson's Eq.(14) with the QDM is also given by the symbols at 28 quadrature points.