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Chaotic dynamics, Markov processes and climate predictability

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ABSTRACT

A systematic method for mapping deterministic chaos in general, and atmospheric and climate dynamics in particular, into a Markovian process is developed. The possibility of using the Markovian description for carrying out long-term statistical predictions is suggested, and illustrated on simple models and on data pertaining to sea surface temperature anomalies which can be reduced to one-dimensional maps.

1. Introduction

The difficulty of carrying out long-term predictions of the evolution of the atmosphere and climate is a problem of obvious concern. It is nowadays recognized that it can be traced back to two major elements (Lorenz, 1984a, 1987).

(i) Operationally, in defining the state of the atmosphere or climate a number of *errors* are involved. First, there is the inherent uncertainty associated with the finite resolution of the measuring process, as a result of which a variety of "subgrid" elements are dismissed; second, the numerical values used are rounded off to a prescribed (finite) number of digits; and third, whatever the system of equations used at one stage or the other to study the atmosphere might be, it will necessarily involve some approximations and should for this reason be regarded as a model rather than as an exact image of reality.

(ii) The principal atmospheric and climatic variables undergo *complex dynamics*, typical features of which are aperiodicity in time, irregular patterning in space, and rapid amplification of small errors of the kind mentioned above. It follows that two states, that are operationally indistinguishable and supposed to describe equally well the initial state of the atmosphere, will follow completely different histories beyond a certain lapse of time, which may be adequately referred to as *predictability time*. Present estimates from model systems of weather prediction of varying complexity give error doubling times of a few days. Significantly, the growth rate seems to depend very little on the detailed nature of the error, provided that the amplitude of the latter is small enough.

There is increasing awareness that *deterministic chaos* provides a natural prototype of the complexity of atmospheric and climatic dynamics (Lorenz, 1984b). Not only does it give rise to aperiodicity in space and time, but it also displays sensitivity to initial conditions, reflected by the exponential divergence of nearby trajectories. The rate of this divergence is an intrinsic property of the dynamical system and is related to its Lyapounov exponents or, more precisely, to a combination thereof known as Kolmogorov entropy (Guckenheimer and Holmes, 1983). This is to be compared with the statement about growth rate of small errors made above; it suggests that the limits of atmospheric predictability should be intrinsic and related to the inverse of the Kolmogorov entropy or of the Lyapounov exponents of the dynamics underlying atmospheric variability. At present, there exist algorithms enabling one to estimate these quantities from time series data, independent of any modelling. Their application to some key atmospheric variables has recently been reported (Fraedrich, 1987a; Keppenne and Nicolis, 1989; Nicolis and Keppenne, 1989).

Obviously, there is an imperative need to extend our prediction capabilities of the atmosphere and climate beyond the error doubling time or the inverse of the Kolmogorov entropy. Our principal goal in the present paper is to suggest a modest step toward such an objective.

The basic idea is to give up the prediction of an individual realization and resort to a statistical description which, although necessarily coarser than the full description based on the primitive equations, remains an exact image of the underlying dynamics. To arrive at such a description we argue as follows. Since as mentioned above, the description of the atmosphere necessarily involves round-off errors, we stipulate that an atmospheric or climatic state is represented by a cell of finite volume in the phase space, the space spanned by the full set of variables determining the system's dynamics. This "coarse graining" or "lumping" is a non-local operation. As pointed out by Lorenz (1984a), it maps the deterministic description, based on the full set of the primitive equations, into a complex random process. We suggest that it is possible to substantiate further Lorenz's conjecture by identifying conditions under which the partitioning of the phase space leads to a clear-cut stochastic process, like, for instance, a first-order Markov chain. Having achieved this, one may then approach the predictability problem via the properties of the chain which, in view of the Markovian property, can be determined for an arbitrarily long period of time.

In Section 2, we collect the basic prerequisites that must be met if the partitioning of the state space of a system is to give rise to a Markov process. In Section 3, we show that these conditions can be satisfied by certain classes of dynamical systems giving rise to deterministic chaos and by certain classes of partitions transforming in an adequate manner under the dynamics. As an illustration, an explicit form of the stochastic evolution equation is constructed for the particular class of systems described by one-dimensional iterative maps of the form

$$X_{n+1} = f(X_n, \lambda), \tag{1}$$

X being the state variable, f the evolution law and λ a parameter. We subsequently illustrate these ideas (Section 4) on two problems of interest in atmospheric physics and climate: low-order atmospheric models and data pertaining to the El Niño southern oscillation phenomenon. In both cases, we show that the system of interest can be mapped under certain conditions, to a one-dimensional dynamics and to a Markov chain description. The main conclusions are drawn in Section 5.

2. Prerequisites for a lumped variable process to be Markovian

Let $X_1(t), \ldots, X_n(t)$ be the time series of a set of n continuous variables describing the evolution of the atmosphere. For convenience, the shorthand vector notation $\boldsymbol{X}(t) = \{X_1(t), \ldots, X_n(t)\}$ will frequently be adopted. Since, as pointed out in the *Introduction*, an experiment of interest in atmospheric physics can never deal with point states, the natural way to represent the evolution of $\boldsymbol{X}(t)$ is (a), to discretize time t in N equidistant values $t_1, \ldots, t_N$, $\Delta t = t_{i+1} - t_i$ being an adequately chosen sampling time; and (b), to divide the phase space Γ spanned by $X_1, \ldots, X_n$ into ν non-overlapping cells $C_1, \ldots, C_\nu$ and observe the rules governing how these cells are visited in the course of time (see Fig. 1).

There is one obvious quantity that may be constructed in the above defined context. Suppose that we were able to conduct a large number of experiments (real or numerical), each starting with slightly different initial conditions but subjected otherwise to identical constraints. At a given instant t_i, we count the number of times a realization is found in cell α and divide by the number of all the realizations. We arrive in this way at the probability for being in C_α at time t_i,

$$P(X_\alpha, t_i) = \mathrm{prob}(X_1 \in C_\alpha, X_2 \in C_\alpha, \ldots, X_n \in C_\alpha; t_i). \tag{2}$$

In a stationary ergodic process, P will reach a fixed value in the limit $t_i \to t_N$ (provided t_N is sufficiently large), which will represent the number of visits of a single realization in C_α, divided by the total number of states reached in the interval $(0, t_N)$.

By extending the argument developed above, one may construct from the original time series, the probability $P(\boldsymbol{X}_\alpha, t_\alpha; \boldsymbol{X}_\beta, t_\beta)$ to reach state β following state α; the probability $P(\boldsymbol{X}_\alpha, t_\alpha; \boldsymbol{X}_\beta, t_\beta;$

X_γ, t_γ) to visit successively states α, β and γ; and so forth.

By definition, the conditional probability $W_{\alpha\beta} = W(X_\beta | X_\alpha)$ that X_β will occur in the next state given that X_α has just occurred, is

$$W(X_\beta | X_\alpha) = \frac{P(X_\alpha, t_\alpha; X_\beta, t_\beta)}{P(X_\alpha, t_\alpha)}. \tag{3}$$

The process which consists in following the values of $X_1, \ldots, X_\nu$ in the course of time will be a first-order Markov if $W(X_\beta | X_\alpha)$ determines all higher-order probability distributions (Feller, 1968):

$$\begin{aligned} P(X_\alpha, X_\beta, X_\gamma) &= P(X_\alpha, X_\beta)\, W(X_\gamma | X_\beta) \\ &= P(X_\alpha)\, W(X_\beta | X_\alpha)\, W(X_\gamma | X_\beta), \\ &\quad \ldots, \text{etc.} \end{aligned} \tag{4}$$

If eq. (4) is incompatible with the time series data, either the particular subdivision of the state space into $C_1, \ldots, C_\nu$ is inadequate, or the process is higher-order Markov. For instance, defining the conditional probability of the event X_γ given the doublet (X_α, X_β) by

$$W(X_\gamma | X_\alpha, X_\beta) = \frac{P(X_\alpha, X_\beta, X_\gamma)}{P(X_\alpha, X_\beta)}, \tag{5}$$

the process will be second-order Markov if the following equality is satisfied:

$$\begin{aligned} &P(X_\alpha, X_\beta, X_\gamma, X_\delta) \\ &\quad = P(X_\alpha, X_\beta, X_\gamma)\, W(X_\delta | X_\beta, X_\gamma) \\ &\quad = P(X_\alpha, X_\beta)\, W(X_\gamma | X_\alpha, X_\beta)\, W(X_\delta | X_\beta, X_\gamma). \end{aligned} \tag{6}$$

The generalization to a third- or higher-order Markov process is straightforward. Notice that the above properties refer to the entire set of the variables $\{X(t) = X_1(t), \ldots, X_n(t)\}$. In general, the individual variables $X_i(t)$ will define a non-Markovian process.

Suppose now that the state space has been judiciously chosen so that eq. (4) is satisfied. A classical result of probability theory is that $W_{\alpha\beta}$ satisfies the Champan-Kolmogorov equation (Feller, 1968), from which a *master equation* governing the time evolution of the one-state probability $P(X_\alpha, t)$ can be deduced (Nicolis and Prigogine, 1977). This latter equation reads:

$$P(X_\alpha, t_{i+1}) = \sum_{\alpha'} W_{\alpha'\alpha}\, P(X_{\alpha'}, t_i). \tag{7}$$

Eq. (7) is of the general form

$$\boldsymbol{P}(X, t_{i+1}) = \boldsymbol{P}(X, t_i)\, \boldsymbol{W}, \tag{8}$$

where $\boldsymbol{P}(X)$ denotes the probability vector, i.e., a row vector whose elements are $\boldsymbol{P}(X_1), \ldots, \boldsymbol{P}(X_\nu)$ and $\boldsymbol{W}$ is a linear operator ($\nu \times \nu$ matrix) whose events can be constructed from the time series. The formal solution

$$\boldsymbol{P}(X, t_n) = \boldsymbol{P}(X, t_0)\, \boldsymbol{W}^n \tag{9}$$

can in principle be computed, at least numerically. It provides us with the means to estimate, from first principles, the statistical state of the system at time t_n on the sole basis of the time series (assumed stationary) and of the initial state. Notice that the Markovian property, eq. (4) is essential for this result. An improper choice of state variables and/or an improper division of state space may generate a process which is incompatible with the Markovian requirement. For such a process, then $W_{\alpha'\alpha}$ would *not* suffice to make statistical predictions.

In atmospheric physics and climatology, there exists a rich literature on Markov chains and predictability based on statistical arguments. A first line of approach is illustrated by the work of Fraedrich and Müller (1982). These authors report tests of Markov property for observational data pertaining to daily sunshine measurements, by dividing them into "clear" and "cloudy" events. They correctly point out that such considerations must be integral parts of a good model building strategy.

Following the pioneering work of Charney and DeVore (1979) and the evidence of bimodality in the dynamics of long wavelength atmospheric waves (see for instance Benzi and Speranza, 1987 for a review), a number of discrete Markov chain models for the transition between persistent patterns of atmospheric circulation have been developed (De Swart and Grasman, 1987; Mo and Ghil, 1987). The principal ideas are as follows. Persistent states are regarded as co-existing planetary flow regimes of an underlying dynamical system obtained typically by a finite-mode truncation of the primitive equations or of the vorticity equation. By counting the number of changes between these regimes during a long time interval, a transition probability matrix is constructed, and it is asserted (Mo and Ghil, 1987) that this information together with the initial probabilities defines a Markov chain. In simplified analyses (De Swart and Grasman, 1987), the co-existing regimes are actually reduced to simultaneously stable steady states, and the transitions between such co-existing attractors are attributed to stochastic perturbations. Again, it is taken for granted that such transitions define a Markov chain.

Now, as pointed out in connection with eq. (9), the Markov property generally fails when the variables and phase space partitioning are arbitrarily chosen. In particular, the lumping of an initially continuous, multivariate system into a discrete system involving a small number of states destroys the Markov property. The main originality of our approach is to provide some algorithms for carrying out such a reduction in a consistent manner.

To this end, we shall now go one step further in our analysis and investigate the validity of the Markovian property (eqs. (4) or (6)) from the standpoint of the dynamics. This will lead us (Section 3) to some conditions that must be satisfied both by the dynamics and by the phase space partition, as well as to an explicit form of the conditional probability matrix W. In Section 4, these results will be illustrated on some simple examples from atmospheric and climate dynamics.

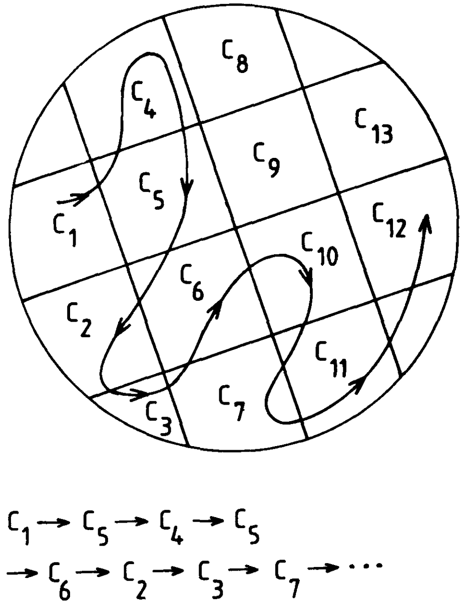

Fig. 1. Time evolution of a phase-space trajectory viewed as a sequence of transitions between the cells of a "coarse-graining" partition.

3. Master equation and deterministic chaos

Let us denote by X_n the value of the state vector X at time $t = t_n$ (cf. beginning of Section 2). The evolution equations of X_n can be written in the general form (cf. eq. (1))

$$X_{n+1} = f(X_n, \lambda), \tag{10}$$

The function f which maps the values of the state variables X at time n to those at time $n+1$ is determined by the dynamical laws such as the primitive equations, the vorticity equation, and so forth. It will be assumed (Lorenz, 1984b) that any realistic model of the atmosphere cast in the form of eq. (10) gives rise to chaotic dynamics displaying sensitivity to initial conditions and strong ergodic properties. One particular type of ergodic property of interest in our discussion is the existence of a *probability density* $\rho_n(X)$, which is (a), a non-singular function of phase space coordinates X and is normalizable to unity, $\int \mathrm{d}X\, \rho_n(X) = 1$; (b) tends to a well-defined invariant density as $n \to \infty$. In the theory of dynamical systems it is shown that eq. (10) induce a closed form evolution equation for ρ_n, known as the *Perron–Frobenius equation* (Guckenheimer and Holmes, 1983). This equation expresses the following idea. The probability of being around state X at time n equals the probability of being at time $n-1$ around the states that transform into X under the effect of the mapping f. More quantitatively, denoting by U the evolution operator of the probability density, one obtains:

$$\rho_n(X) = U\rho_{n-1}(X) = \rho_{n-1}(f^{-1}(X)). \tag{11}$$

Thus, the action of the evolution operator U on the probability density ρ_{n-1} amounts to evaluating ρ_{n-1} at the pre-image of the trajectory point X, f^{-1} being the inverse of the mapping defined in eq. (10).

As explained in detail in Section 2, because of the sensitivity to initial conditions, we do not want to appeal explicitly, as one is forced to do in eq. (11), to the individual trajectories of the underlying dynamical system. Rather, we want to map the initial "fine grained" dynamics based on eqs. (10)–(11) into a "coarse-grained" one involving solely transitions between the cells C_i of a suitable partition reflecting the finite resolution of the observations (Fig. 1). We substantiate this idea by stipulating that instead of the density $\rho_n(\boldsymbol{X})$, we are interested in the probability

$$\boldsymbol{P}_n = \{P_n(1), \ldots, P_n(\nu)\}$$

of the system being in a particular cell of the partition. Clearly, $\boldsymbol{P}_n$ must be related to ρ_n by some sort of averaging. There is no unique way of performing such an averaging, although all are expressed in the generic form

$$\boldsymbol{P}_n = \boldsymbol{E}\rho_n(\boldsymbol{X}) = \sum_{\rho=1}^{\nu} \frac{1}{\mu(C_i)} \int_{C_i} g(\boldsymbol{X})\, \rho_n(\boldsymbol{X})\, \mathrm{d}\boldsymbol{X} \boldsymbol{1}_{C_i}. \tag{12}$$

Here $\boldsymbol{1}_{C_i}$ is the characteristic function of the cell C_i ($\boldsymbol{1}_{C_j} = 0$ if $C_j \neq C_i$, $\boldsymbol{1}_{C_j} = 1$ if $C_j = C_i$), $g(\boldsymbol{X})$ is a weighting factor, and $\mu(C_i) = \int_{C_i} g(\boldsymbol{X})\, \mathrm{d}\boldsymbol{X}$ is the weighted volume of C_i. In most applications of dynamical systems theory, $g(\boldsymbol{X})$ is taken equal to $\rho_\infty(\boldsymbol{X})$, the time-independent solution of eq. (11) (invariant density). Actually, $g(\boldsymbol{X})$ can be any density function related to $\rho_\infty(\boldsymbol{X})$ through a sufficiently smooth transformation.

In order to obtain an equation of evolution for the probability $\boldsymbol{P}_n$, we operate on both sides of (11) by the averaging operator $\boldsymbol{E}$ defined in eq. (12):

$$\boldsymbol{P}_n = \boldsymbol{E}\rho_n = \boldsymbol{E}U\rho_{n-1}(\boldsymbol{X}). \tag{13}$$

Iterating eq. (11) n times and assuming that the initial condition $\rho_0(\boldsymbol{X})$ is coarse grained, $\rho_0(\boldsymbol{X}) = \boldsymbol{E}\rho_0$, we can further write eq. (13) as

$$\boldsymbol{P}_n = \boldsymbol{E}U^n \rho_0(\boldsymbol{X}) = \boldsymbol{E}U^n \boldsymbol{E}\boldsymbol{P}_0. \tag{14}$$

This relation allows one to deduce the instantaneous value of the coarse grained probability $\boldsymbol{P}_n$ from its initial value $\boldsymbol{P}_0$ through the action of the operator $\boldsymbol{E}U^n\boldsymbol{E}$. Now, according to eqs. (7)–(9), if this action is to define a Markov process, the operator $\boldsymbol{E}U^n\boldsymbol{E}$ should be the nth power of the operator describing the first step of the process, leading from $\boldsymbol{P}_0$ to $\boldsymbol{P}_1$. This latter operator is obviously equal to $\boldsymbol{E}U\boldsymbol{E}$. In other words, one should have the condition

$$\boldsymbol{E}U^n\boldsymbol{E} = (\boldsymbol{E}U\boldsymbol{E})^n. \tag{15}$$

It can be shown (Nicolis and Nicolis, 1988) that this condition is equivalent to

$$\frac{1}{P_\infty(i)}\,\mu(C_i \cap C_j^{-n}) = \sum_{k=1}^{\nu} \frac{1}{P_\infty(i)}\,\mu(C_i \cap C_k^{-(n-1)}) \times \frac{1}{P_\infty(k)}\,\mu(C_k \cap C_j^{-1}), \tag{16a}$$

where $P_\infty(k)$ is the invariant probability of being in cell k,

$$P_\infty(k) = \lim_{n\to\infty} P_n(k) = \int_{C_i} \rho_\infty(\boldsymbol{X})\,\mathrm{d}\boldsymbol{X}. \tag{16b}$$

$A \cap B$ denotes the intersection of the sets A and B, and C_j^{-n} is the union of all sets of phase space which are found within cell C_j after n time units. Explicit manipulations involving these quantities are illustrated later on in this section.

Supposing now that the validity of eqs. (15) or (16) can be justified, the evolution equation for $\boldsymbol{P}_n$ (eq. (14)) will reduce to the form

$$\boldsymbol{P}_n = \boldsymbol{P}_{n-1}\boldsymbol{W}$$

or more explicitly

$$P_n(i) = \sum_{j=1}^{\nu} W_{ji} P_{n-1}(j) \tag{17a}$$

with

$$W_{ij} = \frac{1}{P_\infty(i)}\,\mu(C_i \cap C_j^{-1}). \tag{17b}$$

Eqs. (17) provide us with the connection we were looking for between the Markovian description and the underlying dynamics.

Let us comment on the validity of conditions (15) or (16). Clearly, these conditions have a bearing both on the dynamics and on the partitions. We have already emphasized that a necessary condition for the arguments developed in this section to hold is that the dynamics be chaotic. As for the partitions, a necessary, though by no means sufficient condition is obviously that under the effect of the dynamics, the subdivision of state space into the cells C_i is preserved: the boundaries of the cells are transformed onto themselves, or alternatively a given cell is transformed into a union of other cells of the same partition.

Phase space partitions play a central role in the theory of dynamical systems, since they potentially allow for a reduced, symbolic description of (continuous) orbits of various kinds in terms of a finite number of symbols. In this perspective, a well-studied class of systems are *hyperbolic* dynamical systems, i.e., systems for which sensitivity to initial conditions holds not only on average but also for almost everywhere in phase space. For such systems, a privileged family of partitions are the *Markov partitions* (Bowen, 1977; Sinai, 1976), whose cells are delimited by the intersection of stable and unstable manifolds emanating from suitably chosen hyperbolic fixed points. Obviously, this property accounts automatically for the above-mentioned requirement of invariance of the cell boundaries under the dynamics. Furthermore, it turns out that it allows for a symbolic description that is (in the limit of infinitely long times) in *one-to-one correspondence* with the properties of the continuous phase space orbits (Collet and Eckmann, 1980).

It should be emphasized that the mere choice of Markov partition does not guarantee the validity of eqs. (15)–(16). Indeed, the definition of Markov partition is purely topological: it specifies nothing about the probability of being in a particular cell, nor about the rate of transitions between cells, both of which play the central role in our formulation. In the context of the present work, a further limitation of the Bowen-Sinai theory is that, as a rule, the dynamical systems encountered in atmospheric science need not be hyperbolic almost everywhere: a typical chaotic attractor of a dissipative dynamical system would display sensitivity to initial conditions only in some average sense.

Despite these limitations, Markov partitions constitute a privileged class, to which we shall repeatedly appeal in the sequel. We leave open the interesting question of existence of other classes of partitions compatible with eq. (15), which constitutes the most general condition guaranteeing the Markovian character of the process.

We now describe an explicit example in which the mapping of the deterministic dynamics to a stochastic process can be studied in detail and an explicit form of Markovian equation can be deduced. Consider the one-variable iterative dynamical system

$$X_{n+1} = f(X_n) = 4\eta X_n(1 - X_n), \quad 0 \leqslant X \leqslant 1. \tag{18}$$

It is well known (Guckenheimer and Holmes, 1983) that for η close to 1, this system admits a chaotic attractor covering the entire unit interval, except for a set of points of measure zero. Moreover, this system admits an infinity of unstable periodic orbits (cycles) of arbitrary long periods whose points transform into each other as

$$\begin{aligned} &\bar{X} = f(\bar{X}) \quad \text{(period one, or fixed point)} \\ &X_2 = f(X_1), \quad X_1 = f(X_2) \quad \text{(period two)} \\ &X_2 = f(X_1), \quad X_3 = f(X_2), \quad X_1 = f(X_3) \\ &\quad \text{(period three)} \end{aligned} \tag{19}$$

etc. . . .

As an example, Figs. 2a, b depict the graphical construction leading to the fixed point $\bar{X}$ and the period two cycle (X_1, X_2). To determine these points analytically, one has merely to solve the following algebraic equations.

(i) *For the fixed point* $\bar{X}$

$$\bar{X} = 4\bar{X}(1 - \bar{X}).$$

The solution of this quadratic equation is

$$\bar{X} = 0.75.$$

(ii) *For the period two cycle* (X_1, X_2)

$$X_2 = 4X_1(1 - X_1)$$

$$X_1 = 4X_2(1 - X_2)$$

or,

$$X_2 = 16X_2(1 - X_2)(1 - 4X_2(1 - X_2)).$$

In addition to $X = 0$ and $X = 1$ this equation admits the solutions

$$X_1 \approx 0.345, \quad X_2 \approx 0.905,$$

constituting the period-two cycle.

Let us now come to the statistical properties of the system of eq. (18) for $\eta = 1$. The invariant density ρ_∞ (see comments after eq. (12)) is (Collet and Eckmann, 1980)

$$\rho_\infty = \frac{1}{\pi[X(1 - X)]^{1/2}},$$

and the cells defined by 0, 1 and the points of an unstable cycle constitute a Markov partition for which the validity of eqs. (15), (16) can be verified both analytically and numerically. As an example, consider again the cycle of order two (Fig. 2b). We have seen that $X_1 = 0.345$,

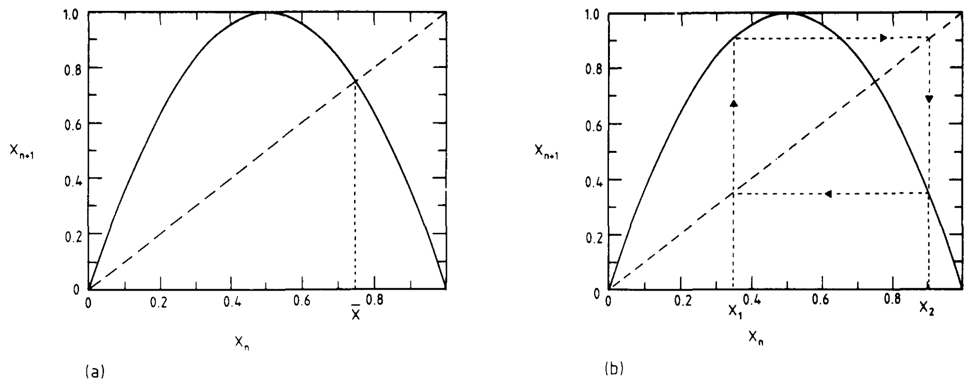

Fig. 2. Markov partitions of the logistic map, eq. (18). (a) Two-cell partition defined by the end points 0, 1 and the fixed point $\bar{X}$. (b) Three-cell partition defined by the end points and the period two cycle (X_1, X_2).

$X_2 = 0.905$. The transition probability $\boldsymbol{W}$, which reduces here to a 3×3 matrix, can be determined analytically from eq. (17b). One first computes $P_\infty(i)$ using the relation between $P_\infty(i)$ and $\rho_\infty(X)$, eq. (16b). The result is:

$$P_\infty(1) = \int_0^{0.345} \frac{\mathrm{d}X}{\pi[X(1-X)]^{1/2}},$$

$$P_\infty(2) = \int_{0.345}^{0.905} \frac{\mathrm{d}X}{\pi[X(1-X)]^{1/2}},$$

$$P_\infty(3) = 1 - [P_\infty(1) + P_\infty(2)].$$

Performing the integration explicitly one finds

$$P_\infty(1) = P_\infty(2) = \tfrac{2}{5}, \quad P_\infty(3) = \tfrac{1}{5}.$$

To compute $\mu(C_i \cap C_j^{-1})$, one has to find the pre-images of the cells of our partition or, equivalently, the pre-images of their boundaries. Because of the quadratic character of the mapping (eq. (18)), each point except 0 and 1 has two pre-images X'_{-1}, X''_{-1} obtained by solving the equation

$$X = 4X_{-1}(1 - X_{-1}).$$

In other words, the pre-image of a given cell consists of two disjoint intervals, whose intersection with the cells of the partition determines the transitions that can be expected if the system is started from this cell. The procedure is illustrated graphically in Fig. 3 for cell C_1.

Analytically, we find the following two pre-images of the endpoint $X_1 = 0.345$:

X'_{-1} : 0.095

X''_{-1} : 0.905.

The intersections $C_i \cap C_1^{-1}$ can now be found explicitly. Referring once again to Fig. 3, one has:

$$C_1 \cap C_1^{-1} = C_{1,1}^{-1}, \quad C_2 \cap C_1^{-1} = 0,$$

$$C_3 \cap C_1^{-1} = C_{1,2}^{-1}.$$

Consequently,

$$\mu(C_1 \cap C_1^{-1}) = \int_0^{0.095} \frac{\mathrm{d}X}{\pi[X(1-X)]^{1/2}} = \frac{1}{5},$$

$$\mu(C_2 \cap C_1^{-1}) = 0,$$

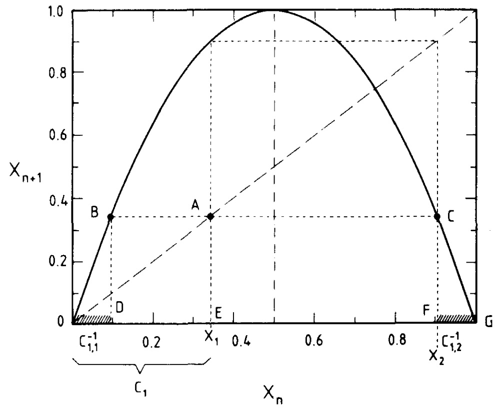

Fig. 3. Illustration of the procedure leading to the computation of $\mu(C_i \cap C_j^{-1})$, for $j = 1$. The cell C_1 is the interval OE $(0 \leqslant X \leqslant X_1)$. The two pre-images of E are the points D and F, and those of the origin O are the origin and the point $G(X = 1)$. Thus the two pre-images $C_{1,1}^{-1}$, $C_{1,2}^{-1}$ of the interval OE are the intervals OD and FG. To calculate, e.g., $\mu(C_1 \cap C_1^{-1})$, one has to determine the intersection of C_1 (i.e., OE) with $C_{1,1}^{-1}$ (i.e., OD) which is OD itself. $\mu(C_1 \cap C_1^{-1})$ is then given by $\mu(C_1 \cap C_1^{-1}) = \int_O^D \rho_\infty(X)\,\mathrm{d}X$.

$$\mu(C_3 \cap C_1^{-1}) = \int_{0.905}^{1} \frac{\mathrm{d}X}{\pi[X(1-X)]^{1/2}} = \frac{1}{5}.$$

Dividing through by $P_\infty(i)$, one obtains using definition (17b):

$W_{11} = \frac{1}{2}$,

$W_{21} = 0$,

$W_{31} = 1$.

Proceeding in the same way for cells C_2 and C_3, one obtains

$$W = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}. \tag{20a}$$

The solution of the master equation

$$P_{n+1}(i) = \sum_{j=1}^{3} W_{ji} P_n(j) \tag{20b}$$

thus describes a series of jumps of varying probability, depending on the states that are connected through the transition. In our present example, using the explicit form of eq. (20a), one arrives at the following "forecasting":

state 1 $\rightarrow$ state 1 or state 2,

state 2 $\rightarrow$ state 2 or state 3,

state 3 $\rightarrow$ state 1.

More generally, as pointed out in Sections 1, 2, the solution of the master equation will allow one to forecast the statistical state of the system successively at $n = 1, 2, \ldots$ etc., on the basis of the initial condition $P_0(j)$.

Now, eqs. (17) or (20b) are linear equations possessing a unique stationary state solution (which for the specific case of the matrix (20a) was just shown to be $P_\infty(1) = P_\infty(2) = \frac{2}{5}$, $P_\infty(3) = \frac{1}{5}$). Furthermore, a basic result of the theory of Markov processes shows that starting from any initial condition, the system converges to this unique stationary solution as $n \rightarrow \infty$. Put differently, small changes in the (probabilistic) initial conditions will give rise to only slight differences in the time evolution. This fact, to be contrasted with the sensitivity to initial conditions of the underlying dynamical system guarantees the viability of the statistical forecasting. Notice, however, that sensitivity to initial conditions has been instrumental in allowing us to cast the dynamics in a Markovian form.

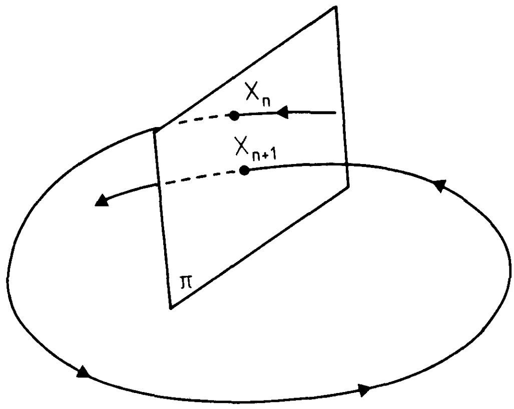

Fig. 4. Schematic representation of a Poincaré surface of a section obtained by successive intersections $\{X_n\}$ of the phase space trajectory with a surface π.

4. Some applications from atmospheric and climatic dynamics

A multivariate dynamical system evolving continuously in time can be cast in a discrete iterative form of eq. (10) in a number of ways. One possibility is to monitor the successive extrema, $X_n, \ldots$ $(n = 1, 2, \ldots)$ of some of the variables and relate them to their values $X_{n-1}, \ldots$ immediately preceding each of the X_n's. A second possibility is to consider the successive intersections of the phase space trajectory with a surface, referred to as Poincaré surface of section, whose dimension is smaller than the phase space dimension by one unity (Fig. 4). Furthermore, if in this latter case, there is a large separation between time scales, the dynamics on the Poincaré surface can be reduced along a single dominant coordinate leading to a one-dimensional recurrence $X_n = f(X_{n-1})$, as, e.g., in eq. (18).

An arbitrary choice of monitored variables and of Poincaré surface would typically lead to an iterative law possessing no obvious regularity. In many cases, however, one obtains some remarkably simple trends. For instance, it is by now well established that in large classes of systems of interest in physics and chemistry giving rise to deterministic chaos, some universal patterns of one-dimensional iterative dynamics emerge. One of those most frequently occurring, starting either from mathematical models or from experimental data, is the *logistic map* shown in Fig. 2, (cf. eq. (18)). The great advantage of reductions of this kind is that one variable iterative dynamical systems can be studied in a considerable amount of detail (Collet and Eckmann, 1980). One therefore has access to a number of properties of the original multivariate system, whatever the nature of the variables other than X might be. Besides, as we saw in Section 3, a Markovian description of such systems enabling one to make statistical predictions can be set up explicitly.

Coming now to systems of interest in atmospheric physics and climate, one-dimensional maps have been constructed from the multidimensional dynamics in a few instances. No systematic work producing higher dimensional maps has been reported so far. In the following, we therefore show how the statistical description outlined in Section 3 can be implemented on some of the most typical one-dimensional maps available in atmospheric physics.

4.1. *Thermal convection and low-order atmospheric circulation models*

In his classical study of the Boussinesq equations describing turbulent convection beyond the Rayleigh-Bénard instability, Lorenz (1963) performed a three-mode truncation leading to the equations

$$\frac{dx}{dt} = \sigma(-x + y),$$
$$\frac{dy}{dt} = rx - y - xz,$$
$$\frac{dz}{dt} = xy - bz. \qquad (21)$$

Here, x is related to the lowest-order Fourier coefficient of the vertical component of the velocity field; y and z are related to the amplitude of the first two Fourier modes of the deviation of the temperature field from the linear profile; σ is the Prandtl number; r the Rayleigh number divided by its critical value of onset of thermal convection, and b a positive parameter related to the geometry and size of the convection cells.

By plotting the successive maxima of z versus their preceding values, Lorenz discovered the one-dimensional cusp-like map depicted in Fig. 5 in normalized units. It was shown subsequently that this map can be approximated analytically in a very satisfactory manner (Hemmer, 1984):

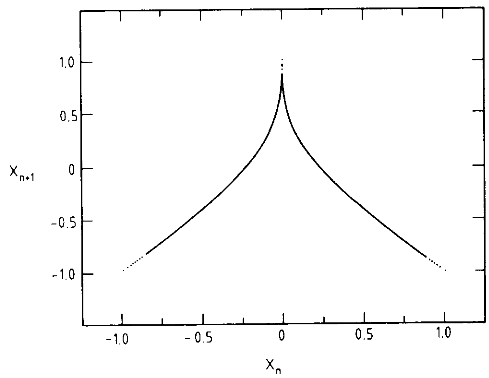

Fig. 5. Cusp-like return map constructed from the maxima of the variable z in eqs. (21), in normalized units.

$$f = 1 - \xi |X|^{\theta}, \quad -1 \leqslant X \leqslant 1. \qquad (22)$$

For the parameter values $\sigma = 10$, $r = 28$, $b = \frac{8}{3}$ for which eqs. (21) give rise to a chaotic attractor, a best fit of ξ and θ gives $\xi = 2$, $\theta = 0.5$.

A similar reduction to one-dimensional map applies to a class of low-order atmospheric circulation models, introduced by Lorenz (1960, 1987) and studied further by Gent and McWilliams (1982).

4.2. *Sea surface temperature anomalies in connection with the El Niño event*

We next turn to an example showing that one-dimensional iterative dynamics can also be deduced directly from experimental data. We will be concerned with the El Niño/southern oscillation phenomenon, recently analyzed from this point of view by Fraedrich (1987, 1988). By observing the rules governing the succession of 120 events of varying intensity I, Fraedrich found that the I_{n+1} versus I_n graph is U-shaped, looking very much like an inverted logistic graph of Fig. 2. He subsequently studied the dynamical behavior deduced from this map in terms of the thermal inertia coefficient, which here plays the role of bifurcation parameter, and derived an estimate of the predictability time of about 1.5 years.

In summary, we have seen three instances in which certain appropriately selected variables in atmospheric or climate dynamics evolve in time in discrete steps according to a one-dimension nonlinear iterative law of the form represented in Fig. 2 or Fig. 5. Keeping in mind the analysis of Section 3, it is now an easy matter to accomplish on these systems the program set forth in the present paper, namely carry out statistical predictions in a consistent manner. Consider the thermal convection problem. Using the analytical representation of the cusp map given by eq. (22), one can compute numerically the unstable cycles of the system and construct in this way a Markov partition of the interval in which X is varying. Let us take, for instance, the simplest case of a two-cell partition provided by the end points of the interval and the fixed point $\bar{X}$. Using the explicit form of the cusp map (eq. (22)) and the first eq. (19), we can determine $\bar{X}$ from the relation

$$1 - 2\bar{X}^{1/2} - \bar{X} = 0.$$

The solution of this quadratic equation belonging to the interval $(-1, 1)$ is $\bar{X} \approx 0.17$. The question now arises as to whether the two-cell partition $\{C_i\!: \ -1 \leqslant X \leqslant 0.17;\ C_2\!: \ 0.17 \leqslant X \leqslant 1\}$ satisfies the Markovian conditions, eqs. (4), (15) and (16). The validity of these conditions cannot be proven analytically in this case. However, using a generalized χ^2 test (see, e.g., Billingsley, 1961) we have checked numerically that they are indeed satisfied. This guarantees that the probability vector $\boldsymbol{P} = (P(1), P(2))$ evolves according to eq. (17a). The explicit form of the transition probability matrix and of the invariant probability are found to be

$$\boldsymbol{W} = \begin{pmatrix} 0.6 & 0.4 \\ 1 & 0 \end{pmatrix} \tag{23a}$$

$$P_\infty(1) = 0.7, \quad P_\infty(2) = 0.3. \tag{23b}$$

Using this information and eq. (17a), one may now predict the probability that starting with a certain (probabilistic) initial condition, the maximum of the variable X will exceed or will be below the threshold value $\bar{X}$ at any particular moment. Similar predictions can be made for the variable of low-order atmospheric models that casts the dynamics in a one-dimensional iterative form, provided that the appropriate parameters ξ and θ in eq. (22) have been identified.

Let us now consider the U-shaped map introduced by Fraedrich in connection with the El Niño/southern oscillation anomaly. Adopting the same parameter values as those considered by Fraedrich in his qualitative discussion, we can write the I_{n+1} versus I_n relation as (Fraedrich, 1987):

$$I_{n+1} = 4 - 4I_n + I_n^2. \tag{24}$$

Introducing the new variable

$$Y = -\tfrac{1}{4}I + 1, \tag{25}$$

it is an easy matter to show that eq. (24) reduces to eq. (18) with $\eta = 1$. This suggests that there is a systematic way of classifying the intensities of the various El Niño events by using the subdivision of the state space provided by the Markov partitions. Naturally, this implies that a continuous state variable, say the sea surface temperature (SST), must be used instead of the integer-valued intensities involved in the original data set. In any case, one expects that there should be an important advantage of this classification, as compared to the more obvious one, consisting of dividing the state space into equal parts (like, e.g., the "no", "very weak", "weak" and "moderate-strong" subdivision of Fraedrich). Indeed, in the former case, we deal with a Markovian process satisfying a master equation (eq. (17a)), whose solution gives us the tool to make statistical predictions for an arbitrarily long time interval. On the other hand, in the latter case, one cannot assess the character of the process; as a result, the transition probability between two states is not sufficient for predicting the system's evolution beyond one unit of time.

As pointed out earlier for a given continuous variable evolving chaotically, there is no unique way to generate a Markov chain by partitioning the state space into discrete cells. For instance, the number of cells involved in the partition is not fixed. Here, in order to establish the connection with Fraedrich's classification, we discuss the particular example of a four-cell partition of the logistic map (eq. (18)). As shown in Nicolis and Nicolis (1988), such a partition can be constructed by using the end points 0 and 1, the middle point 0.5, and its two pre-images 0.15 and 0.85. Using the transformation provided by eq. (25), one can map this partition into the following one for Fraedrich's U-map:

$C_1: 0 \leqslant I \leqslant 0.6$ no event

$C_2: 0.6 \leqslant I \leqslant 2$ very weak event

$C_3: 2 \leqslant I \leqslant 3.4$ weak event

$C_4: 3.4 \leqslant I \leqslant 4$ moderate to strong event.

With this convention and eq. (17a), and knowing the particular type of the last event that has occurred, we can compute the probability of evolution to any of the four types of our classification at a given time. The transition matrix itself is given by Nicolis and Nicolis (1988):

$$W = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}. \tag{26}$$

This leads to the following type of prediction: starting with, e.g., state C_1 (no event), the system with equal probability will either remain in C_1 or evolve to C_2 (very weak event). Notice the relative persistence of states C_1 and C_3 as compared to C_2 and C_4.

5. Concluding remarks

In this paper, we suggested a statistical approach to the long-term predictions of complex systems undergoing chaotic dynamics. We outlined a general algorithm for casting deterministic chaos into a Markovian process described by a master equation and illustrated the theory by some examples of interest in atmospheric and climate dynamics.

Although in all of the examples treated explicitly, the multi-variable time evolution could be reduced to a one-dimensional iterative map, the general idea is by no means limited to this latter class of dynamical systems. Its applicability rests, however, on the possibility of constructing appropriate phase-space partitions satisfying eqs. (15) or (16). For one-dimensional maps, this has been done explicitly here. Two and higher-dimensional maps constitute a major open problem. One can argue that Markov partitions again constitute a privileged class likely to satisfy conditions (15) and (16). In the absence of a general analytical procedure, such partitions should be constructed numerically. Some results in this direction have been reported recently for two-dimensional iterative maps (Franceschini and Zironi, 1985). In the future, it is important to examine from a similar point of view atmospheric and climatic data as well as models of different levels of complexity.

Inevitably, carrying out statistical predictions for arbitrarily long times is made at the expense of a loss of information on the instantaneous state of the system. In our approach, this corresponds to the finite size of the cells of the partition, which obviously provides us with a "coarse" description as compared to the knowledge of the full trajectory. On the other hand, in atmospheric and climate dynamics, such a knowledge is elusive anyway, owing to the exponential growth of errors. Statistical elements therefore inevitably come into play. We have suggested one way to analyze them in a systematic manner.

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