# Master-equation approach to deterministic chaos

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A class of exact master equations descriptive of a Markovian process is obtained, starting from the Perron-Frobenius equation for a chaotic dynamical system. The conditions that must be satisfied by the initial probability density for the validity of the master equation are derived. The approach employs projection operator techniques and provides one with a dynamical prescription for carrying out coarse-graining in a systematic manner.

## I. INTRODUCTION

Owing to the exponential divergence of nearby trajectories, predictions on individual histories lose their significance in chaotic dynamical systems, beyond a time of the order of the inverse of Lyapounov exponent or of the Kolmogorov entropy.[1,2] Yet it is well known that despite its intrinsic randomness, chaotic behavior displays also some remarkable large-scale regularities. The question therefore naturally arises of whether such long-term properties can be forecasted in a reliable manner. Our purpose in the present paper is to explore the possibilities afforded by mapping chaotic dynamics into a well-defined stochastic process described by a master equation.

The statistical properties of dynamical systems exhibiting instability of motion in general, and of chaotic attractors in particular, have been the subject of extensive investigations. In most studies emphasis is placed on ergodic properties, and in particular on the existence and main features of an invariant probability density.[2-4] However, some results concerning time-dependent probabilistic behavior, especially for intermittent systems, are also available.[5-7] In the present paper attention is focused on the possibility to cast the dynamics of a chaotic system in the form of a master equation, describing the evolution of an initial *nonequilibrium* distribution toward the invariant "equilibrium" form. This question can be viewed as the extension to dissipative dynamical systems of the familiar question of statistical mechanics, how to derive kinetic equations for reduced distributions starting from a "fine-grained" description based on the Liouville equation. We give a partial answer to this question by showing how the unstable character of the dynamics allows one to perform coarse-graining in a systematic manner. Some algorithms for generating well-defined stochastic processes of varying complexity by this method are provided.

In Sec. II we use the concept of Markov partition to obtain the master equation descriptive of a class of chaotic attractors. The conditions that must be fulfilled for the process to be Markovian are formulated. Special emphasis is placed on the restrictions imposed on the initial probability density. In Sec. III the conditions derived in Sec. II are verified, both analytically and numerically, on simple examples. Section IV is devoted to the properties of the master equation, whereas in Sec. V the main conclusions are summarized.

## II. MARKOV PARTITIONS AND MASTER EQUATION

In this section we focus on chaotic dynamical systems defined by one-dimensional iterative maps[1,2,8]

$$x_{n+1}=f(x_n,\lambda) \tag{1}$$

in a parameter range in which there is a chaotic attractor possessing a nonsingular invariant probability $\rho_\infty(x)$. It is well known that under appropriate conditions involving time scale separation a wide class of continuous time dynamical systems (flows) reduce to mappings of the form of (1). In the analysis carried out in the present section the specific form of $f(x,\lambda)$ need not be specified. In most of the illustrations of the subsequent sections, however, we will use two simple examples, provided, respectively, by the tent map and the logistic map in the fully chaotic region:

$$x_{n+1}=\begin{cases}2x_n, & 0\le x\le\frac{1}{2}\\ 2-2x_n, & \frac{1}{2}\le x\le 1\end{cases} \qquad \rho_\infty(x)=1 \tag{2a}$$

and

$$x_{n+1}=4x_n(1-x_n),\quad 0\le x\le 1 \qquad \rho_\infty(x)=\frac{1}{\pi[x(1-x)]^{1/2}}\ . \tag{2b}$$

If the dynamics satisfies the mixing property one can define a time-dependent distribution $\rho_n(x)$ evolving according to the Perron-Frobenius equation[8]

$$\rho_n(x)=U\rho_{n-1}(x)\ , \tag{3}$$

 

where the action of the evolution operator $U$ on any positive normalized function $r(x)$ is given by

$$Ur(x)=\int_a^b dy\,\delta[x-f(y)]r(y)$$
$$=\sum_\alpha \frac{1}{|f'(f_\alpha^{-1}x)|}r(f_\alpha^{-1}x)\ . \tag{4}$$

In this latter equation the index $\alpha$ runs over the various branches of the inverse mapping $f^{-1}$. For unimodal mappings, to which we will restrict ourselves for simplicity in the subsequent sections, the number of these branches is equal to two. The invariant probability $\rho_\infty(x)$ to which we alluded earlier in this section is a stationary solution of Eq. (3), $\rho_\infty(x)=U\rho_\infty(x)$.

Equations (3) and (4) fail to define a physically meaningful stochastic process, owing to the singular character of the "transition probability" $\delta[x-f(y)]$. We shall cope with this difficulty by mapping the "fine-grained" description afforded by (3) into a "coarse-grained" description. Although quite familiar from statistical mechanics,[9-11] coarse-graining is usually carried out in a phenomenological manner. Here we show that under appropriate conditions it can lead to *an exact image* of the dynamics, in the form of a regular stochastic process of the Markov type obeying to a closed-form master equation.

The starting point is to partition the state space of our dynamical system into a set of $N$ nonoverlapping cells $C_i$ $(i=1,\dots,N)$. We shall come shortly to the properties that must be satisfied by this partition, but for the time being we introduce the (noninvertible) projection operator $\pi$ through

$$\pi r(x)=\left[\int_{C_1} r\,dx,\dots,\int_{C_N} r\,dx\right] \tag{5}$$

and cast Eq. (3) into the form

$$\mathbf{P}_n=\pi U\rho_{n-1}(x)\ , \tag{6}$$

where the probability vector $\mathbf{P}_n$ is defined by

$$\mathbf{P}_n=\pi\rho_n\equiv[P_n(1),\dots,P_n(N)]\ , \tag{7a}$$

with

$$P_n(i)=\int_{C_i} dx\,\rho_n(x)\ . \tag{7b}$$

Using Eq. (3) one can thus write the formal solution of (6) as

$$\mathbf{P}_n=\pi U^n\rho_0(x)\ . \tag{8}$$

In principle, Eq. (8) allows one to evaluate $\mathbf{P}_n$ for any given $n$ starting from an initial distribution $\rho_0$. Our objective, however, is to obtain a closed-form autonomous equation for this quantity, which should provide a more transparent picture of the system's dynamics. For instance, it would allow one to characterize the nature of the process, to sort out some general trends (like an $H$-theorem), and to set up, if necessary, suitable approximation schemes. There are two difficulties with which this program is confronted: first, owing to the unstable character of the dynamics, the initial partition is continuously being refined; and second, the initial probability $\rho_0(x)$ in Eq. (8) remains fine grained and belongs therefore to a different class of functions than $\mathbf{P}_n$.

Our first step will be to *restrict* the admissible $\rho_0(x)$'s to the class of functions taking (like $\mathbf{P}_n$) a constant value in each of the cells of the partition. Actually, since we dispose of a nonsingular invariant probability $\rho_\infty(x)$, it will be sufficient to assume stepwise initial conditions of the form

$$\rho_0(x)=\rho_\infty(x)\sum_{i=1}^N \alpha_i\varphi_i \tag{9a}$$

where $\varphi_i$ is the characteristic function of cell $C_i$,

$$\varphi_i(x)=1 \quad \text{if } x\in C_i\ ,$$
$$\varphi_i(x)=0 \quad \text{if } x\notin C_i\ . \tag{9b}$$

Initial conditions of the form of (9a) can be understood as perturbations of the invariant probability $\rho_\infty(x)$ which, in view of the finite resolution involved in a physical experiment, cannot be described in a degree of detail going below the size of any of the partition cells.

To see the nature of the expansion coefficients in Eq. (9a) we multiply both sides of (9a) by $\varphi_k$ and integrate over $x$:

$$\int_a^b \varphi_k(x)\rho_0(x)dx=\int_a^b \varphi_k^2(x)\rho_\infty(x)dx\,\alpha_k$$

or

$$\alpha_k=\frac{\int_{C_k}\rho_0(x)dx}{\int_{C_k}\rho_\infty(x)dx}\ .$$

Defining

$$P_\infty(k)=\int_{C_k}\rho_\infty(x)dx\ ,$$
$$P_0(k)=\int_{C_k}\rho_0(x)dx\ , \tag{10a}$$

we can therefore rewrite expansion (9a) as

$$\rho_0(x)=\rho_\infty(x)\sum_{i=1}^N \frac{\varphi_i}{P_\infty(i)}P_0(i)$$
$$\equiv\rho_\infty(x)\boldsymbol{\varphi}^+\cdot\mathbf{P}_0\ , \tag{10b}$$

where $\varphi^+$ is the column vector,

$$\boldsymbol{\varphi}^+=\left[\frac{\varphi_1}{P_\infty(1)}\ \cdots\ \frac{\varphi_N}{P_\infty(N)}\right]\ . \tag{10c}$$

Substituting (10b) into Eq. (8) we obtain

$$P_n=\pi U^n\rho_\infty(x)\boldsymbol{\varphi}^+\cdot\mathbf{P}_0\ . \tag{11}$$

If this equation is to define a Markov process, $\mathbf{P}_n$ should satisfy a *master equation* (forward Kolmogorov equation) of the form

$$\mathbf{P}_n=\mathbf{W}\mathbf{P}_{n-1} \tag{12a}$$

in which the transition probability matrix $\mathbf{W}$ is given by

$$\mathbf{W}=\pi U\rho_\infty(x)\boldsymbol{\varphi}^+\ . \tag{12b}$$

Comparing (12a) and (12b) and (11) we arrive at the compatibility condition

$$\pi U^n \rho_\infty(x)\varphi^+ = [\pi U \rho_\infty(x)\varphi^+]^n \ .$$

Actually, by the principle of induction it will be sufficient to show that

$$\pi U^n \rho_\infty(x)\varphi^+ = \pi U^{n-1}\rho_\infty(x)\varphi^+ \cdot \pi U \rho_\infty(x)\varphi^+ , \quad n \geq 2 \ . \tag{13}$$

To see the implications of this condition we first evaluate the transition matrix $\mathbf{W}$. Taking notice of (4), (5), (10c) and of the fact that $U$ commutes with the invariant probability $\rho_\infty(x)$, we can write the matrix elements of $\mathbf{W}$ as

$$\begin{aligned} W_{ij} &= \int_{C_j} dx\, \rho_\infty(x) \frac{1}{\mu_i} \sum_\alpha \frac{1}{|f'(f_\alpha^{-1}x)|}\varphi_i(f_\alpha^{-1}x) \\ &= \sum_\alpha \frac{1}{P_\infty(i)} \int_{C_{j,\alpha}^{-1}} d\xi\, \rho_\infty(\xi)\varphi_i(\xi) \\ &= \frac{1}{P_\infty(i)} \sum_\alpha \mu(C_i \cap C_{j,\alpha}^{-1}) \equiv \frac{1}{P_\infty(i)}\mu(C_i \cap C_j^{-1}) \ , \end{aligned} \tag{14}$$

where $C_{j,\alpha}^{-1}$ is the pre-image of cell $C_j$ under the effect of branch $\alpha$ of the inverse mapping $f^{-1}$ and $\mu(C)$ denotes the measure of set $C$. The compatibility condition, Eq. (13), now becomes

$$\begin{aligned} &\frac{1}{P_\infty(i)}\mu(C_i \cap C_j^{-n}) \\ &\quad = \frac{1}{P_\infty(i)} \sum_k \mu(C_i \cap C_k^{-(n-1)}) \frac{1}{P_\infty(k)}\mu(C_k \cap C_j^{-1}) \ , \end{aligned} \tag{15}$$

where the indices $i,j,k$ run over all members of the partition (1 to $N$) and $C_j^{-n}$ is the set of all pre-images of $C_j$ under the effect of $n$ reverse iterations of the mapping. Notice that Eq. (14) guarantees that, whatever the partition might be, $W_{ij}$ is a stochastic matrix. Indeed, evaluating the row sums of $\mathbf{W}$ one finds that

$$\begin{aligned} \sum_j W_{ij} &= \frac{1}{P_\infty(i)} \sum_j \mu(C_i \cap C_j^{-1}) \\ &= \frac{1}{P_\infty(i)}\mu(C_i \cap \sum_j C_j) \\ &= \frac{1}{P_\infty(i)}\mu(C_i) = 1 \ , \end{aligned}$$

where the second equality follows from the definition of the invariant measure.

To proceed further we need to specify the nature of the partition $\{C_i\}$. We shall choose from now on a *Markov partition* whose principal property[3,8,9]

$$f(C_k) \cap C_j \neq \phi \Longrightarrow f(C_k) \supset C_j \tag{16}$$

implies that the boundaries between cells are kept invariant by the dynamics. This allows us therefore to define properly the states of the underlying process. Moreover, Markov partitions are well suited for a symbolic description of the orbits of the dynamical system.[8,12] The symbol space is the set of (infinite) sequences involving a number of symbols equal to the number of cells of the partition, in which the dynamics $f$ [Eq. (1)] induces a shift map of sequences. We refer to this dynamical system as *subshift of finite type.* In view of the shadowing lemma[13] one expects that the orbit realized by this process for a *finite* time interval, $n_1 < n < n_2$ gives an approximation to some orbit of the full system.

An important class of Markov partitions are the *generating partitions.* They enjoy the property that a particular realization of the subshift process for $-\infty < n < \infty$ specifies uniquely an orbit of the underlying dynamical system (1). If the successive states of the process are uncorrelated the partition will be referred to as *Bernouilli partition.*[14] In general it is very difficult to construct explicitly a generating partition fulfilling this property.

Ordinarily, the statistical description of unstable dynamical systems is based on the computation of transitions between cells of a generating partition, induced by the dynamics and weighted with the invariant probability $\rho_\infty$. Such a view cannot lead to a well-defined stochastic process: on the one side the underlying dynamics and $\rho_\infty(x)$ remain fine grained; but on the other side the outcome of the individual transitions is integrated over the cells of the partition, thus providing information on cell to cell transitions in terms of coarse-grained quantities. The main originality of our approach is (a) to set up a closed-form description associated to a well-defined stochastic process by dealing solely with coarse-grained quantities, (b) to explore wider classes of partitions than the generating partitions, and (c) to incorporate explicitly in the description the evolution of an initial probability distribution $\mathbf{P}_0$ toward the stationary solution $\mathbf{P}_\infty$. To achieve this we need, however, to define the range of validity of the compatibility condition, Eq. (15). The mere choice of Markov partition does not guarantee this validity. Indeed, the definition of Markov partition is purely topological: it specifies nothing about the invariant measure or the initial condition, both of which play an important role in our formulation. We shall therefore regard Eq. (15) from now on as an additional condition to be imposed on the partition. In Sec. III we prove analytically its validity on some concrete examples and report on the results of numerical experiments in a number of representative situations.

## III. PROOF OF THE COMPATIBILITY CONDITION AND NUMERICAL EXPERIMENTS

We consider the particular class of Markov partitions provided by the endpoints 0 and 1 of the phase space and the (unstable) periodic orbits of (1). We first focus on the two-cell partition of the tent map [Eq. (2a)] involving

$$C_1 = \{x:\ 0 \leq x < \bar{x}\}, \quad C_2 = \{x:\ \bar{x} < x \leq 1\} \ ,$$

where $\bar{x} = \frac{2}{3}$ is the position of the unstable fixed point, and prove Eq. (15) analytically in this case. Notice that $\rho_\infty(x) = 1$ for the tent map.

In the probabilistic formulation induced by the above partition one deals only with two states, which we number by 1 and 2. Since $C_1^{-1}=(0,\frac{1}{3})\cup(\frac{2}{3},1)$ and $C_2^{-1}=(\frac{1}{3},\frac{2}{3})$ one obtains, using Eq. (14), the transition matrix

$$\mathbf{W}=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} . \tag{17}$$

To evaluate $\mu(C_2\cap C_1^{-n})$ we observe that each backward iteration of $C_1$ transfers on $C_2$ half of the length of $C_1^{-(n-1)}$ that was not on $C_2$ to begin with. Since the length (which plays here the role of invariant measure) is conserved by the backward transformation, this leads to the recurrence relation

$$\mu(C_2\cap C_1^{-n})=\tfrac{1}{2}[\tfrac{2}{3}-\mu(C_2\cap C_1^{-(n-1)})] \tag{18a}$$

or, more explicitly,

$$\mu(C_2\cap C_1^{-n})=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{3\cdot 2^{k-1}} . \tag{18b}$$

A similar argument for $\mu(C_2\cap C_2^{-n})$ leads to

$$\mu(C_2\cap C_2^{-n})=\tfrac{1}{2}[\tfrac{1}{3}-\mu(C_2\cap C_2^{-(n-1)})] \tag{19a}$$

or, more explicitly,

$$\mu(C_2\cap C_2^{-n})=\sum_{k=0}^{n}\frac{(-1)^{k}}{6\cdot 2^{k}} . \tag{19b}$$

As for the other elements, they are immediately deduced from the property that the invariant measure is conserved by the backward transformation:

$$\mu(C_1\cap C_1^{-n})=\tfrac{2}{3}-\mu(C_2\cap C_1^{-n}) , \tag{20a}$$

$$\mu(C_1\cap C_2^{-n})=\tfrac{1}{3}-\mu(C_2\cap C_2^{-n}) . \tag{20b}$$

Substitution of (18)–(20) into Eq. (15) leads to an identity. We have therefore proven that the master equation, Eq. (12a), with a transition matrix given by (17) describes a Markovian process. The latter is *the image of the exact dynamics* [Eqs. (1) and (3)]. In other words, the coarse-graining operator defined by the projection of the probability mass on the cells of the partition commutes with the dynamical evolution operator, and this allows one to propagate an initially given coarse distribution for all subsequent times according to a well-defined stochastic game.

The class of partitions defined by the points $k/2^{m-1}$ ($k=0,\ldots,2^{m-1}$) is of special interest. For $m=2$ one obtains the well-known two-cell left-right partition; the four-cell partition corresponding to $m=3$ is defined by the middle point $x=\frac{1}{2}$ and its two pre-images $x=\frac{1}{4}$ and $x=\frac{3}{4}$, and so forth. The first few transition probability matrices are

$$\mathbf{W}=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \text{ (two-cell partition) ,}$$

$$\mathbf{W}=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \text{ (four-cell partition) .} \tag{21}$$

Notice that they are *doubly stochastic.*

By construction, these partitions satisfy the Markov condition [Eq. (16)]. They are also generating: the two-cell partition is known to provide a full symbolic description of the orbits,[1,9] and all other partitions constitute refinements of the above. The proof of the compatibility condition [Eq. (15)] is straightforward, noticing that $\mu(C_i\cap C_j^{-n})=\mu_j^{-n}=P_\infty^{-n}(j)$ when $C_i\cap C_j^{-n}\neq\phi$ and that all $P_\infty(j)$'s are equal. Actually, for the two-cell partition one can prove the much stronger Bernouilli property (zeroth-order Markov process).

We now turn to the logistic map, Eq. (2b). We consider the Markov partitions defined by the unstable fixed point ($\bar{x}=0.75$), the cycle of order two ($\bar{x}_1=0.345$, $\bar{x}_2=0.905$), the fixed point $\bar{x}$ and its pre-image $\bar{x}'=0.25$, and the cycle of order four ($\bar{x}_1=0.277$, $\bar{x}_2=0.637$, $\bar{x}_3=0.801$, $\bar{x}_4=0.925$). We also analyze the two generating partitions defined by the middle point ($x=0.5$) and the middle point along its pre-images ($x_1=0.15$, $x_2=0.5$, $x_3=0.85$). Figures 1(a)–1(g) depict these partitions.

In each case, starting from numerically computed values of $\mu(C_i\cap C_j^{-n})$, the compatibility condition (15) was verified up to $n=14$. An alternative test of the Markovian character was conducted by following the system's trajectory for a very long period of time and evaluating numerically the probabilities $P_1(i_0)$, $P_2(i_0,t_0;i_1,t_0+1),\ldots,P_k(i_0,t_0;i_1,t_0+1;\ \ldots,i_{k-1}t_0+k-1)$. The conditions for Markovian process,

$$P_k=P_1(i_0)W_{i_0i_1},\ldots,W_{i_{k-2}i_{k-1}} , \tag{22}$$

where $W_{ij}=P_2(i,j)/P_1(i)$ were subsequently checked using these values, and excellent agreement was found up to $k=7$. The agreement was further corroborated by statistical tests such as the $\chi^2$ test. Interestingly, taking a partition provided by an arbitrary set of points (e.g., a three-cell partition with $x_1=0.1$, $x_2=0.65$) one is led to a complex process which does not satisfy the test of Markov process of order up to 7.

An interesting question is whether the Markovian character subsists when some of the cells of the above partitions are lumped two by two and the transitions between the states of the resulting (coarser) partition are studied. For partitions of the type $\bar{J}_3$ or $\bar{J}_5$, lumping of the first two cells and of the first two and last two cells, respectively (see Fig. 1), keeps the process first order Markov. On the other hand, lumping of two cells in partitions of the type $J_3$ or $J_5$ induces a *second-order* Markov process. In other words, there exist partitions involving the same number of cells ($J_2$ and the lumped $J_3$ or $J_{4,g}$ and the lumped $J_5$) exhibiting quite different properties as far as memory and range of correlations are concerned.

Let us sketch the proof of this property for partition $J_3$ in which the last two cells are lumped. We call 1,2,3 the states of $J_3$ and I,II those of the lumped partition. Using the property

$$\hat{P}(\ldots,\mathrm{II},\ldots)=P(\ldots,2,\ldots)+P(\ldots,3,\ldots) ,$$

we obtain the transition matrix $\hat{\mathbf{W}}$ of the lumped chain in the form

$$\hat{\mathbf{W}}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix}.$$

We next proceed to construct higher-order conditional probabilities.[15] Thus,

$$\hat{W}_{ABC}=\frac{\sum_{\{\alpha,\beta,\gamma\}\subset\{2,3\}} P(\alpha\beta\gamma)}{\sum_{\{\alpha,\beta\}\subset\{2,3\}} P(\alpha)W_{\alpha\beta}}, \tag{23a}$$

where $ABC$ denote I or II, $\alpha\beta\gamma$ denote 1, 2, or 3, and the sum acts over states that are 2 or 3. Similarly,

$$\hat{W}_{ABCD}=\frac{\sum_{\{\alpha,\beta,\gamma,\delta\}\subset\{2,3\}} P(\alpha\beta\gamma\delta)}{\sum_{\{\alpha,\beta\}\subset\{2,3\}} P(\alpha)W_{\alpha\beta}}. \tag{23b}$$

If the process is to be second order Markov, the following conditions must be satisfied:

$$\hat{W}_{ABCD}=W_{ABC}W_{BCD} \tag{24}$$

or, using (23a) and (23b) and the Markov property of the process in $J_3$,

$$\sum_{\{\alpha,\beta,\gamma,\delta\}\subset\{2,3\}} P(\alpha)W_{\alpha\beta}W_{\beta\gamma}W_{\gamma\delta}=\frac{\left[\sum_{\{\alpha,\beta,\gamma\}\subset\{2,3\}} P(\alpha)W_{\alpha\beta}W_{\beta\gamma}\right]\left[\sum_{\{\beta,\gamma,\delta\}\subset\{2,3\}} P(\beta)W_{\beta\gamma}W_{\gamma\delta}\right]}{\sum_{\{\beta,\gamma\}\subset\{2,3\}} P(\beta)W_{\beta\gamma}}. \tag{25}$$

Using the invariant probability vector $\mathbf{P}=(0.4,0.4,0.2)$ and the explicit form of the transition matrix

$$\mathbf{W}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\\ 1 & 0 & 0\end{bmatrix}, \tag{26}$$

one can verify the validity of (24). Furthermore, one sees that $\hat{W}_{\mathrm{IIII}}$ is zero whereas $\hat{W}_{\mathrm{III}}$ is finite. This rules out the possibility that the process reduces to a first-order Markov chain. It should be pointed out that this result depends on the specific structure of the transition matrix $\mathbf{W}$. In general, the lumping will destroy the Markov character altogether.

## IV. PROPERTIES OF THE MASTER EQUATION

The master equation on the chaotic attractor [Eq. (12a)] is a linear equation which admits a unique stationary normalized solution given by

$$P_\infty(j)=\sum_i W_{ij}P_\infty(i). \tag{27}$$

Moreover, since $\mathbf{W}$ is a stochastic matrix the master equation satisfies an $H$-theorem[11] implying monotonic approach of any initial probability vector toward $\mathbf{P}_\infty$. The general form of the $H$-functional is

$$H_n=\sum_k P_\infty(k)F\left[\frac{P_n(k)}{P_\infty(k)}\right], \tag{28}$$

where $F$ is a convex function of its argument. A familiar choice of $F$ is $F(x)=x\ \ln x$, yielding a quantity identical (up to the sign) to the *relative entropy*.[16] If $P_\infty$ is constant (as in the case of generating partitions or of partitions of the type $\bar{J}$), relative entropy is identical (up to a constant) to information theoretic entropy,

$$S_n=-\sum_j P_n(j)\ln P_n(j). \tag{29}$$

In this case, therefore, the $H$-theorem describes a monotonic evolution of $S$ itself. Notice that $P_n$ need not evolve

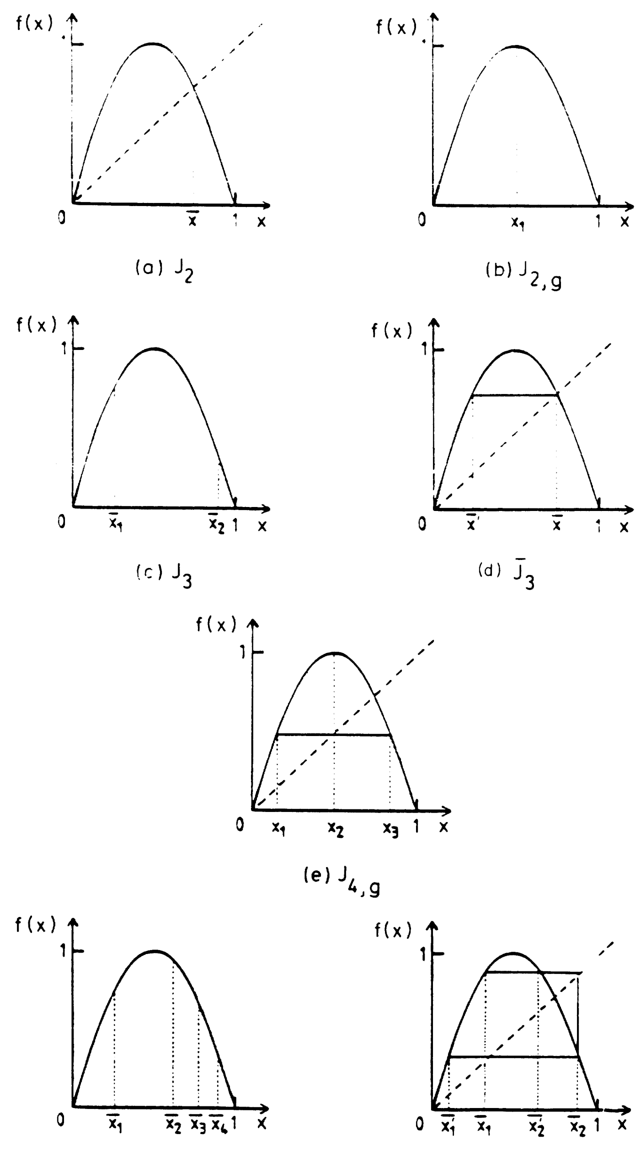


FIG. 1. The seven Markov partitions of the logistic map considered in Secs. III and IV.

monotonically. For instance, for partition $\bar{J}_5$ one finds that the transition matrix $\mathbf{W}$ has two zero eigenvalues, one eigenvalue equal to one, and a pair of complex conjugate eigenvalues equal to $\pm i0.5$.

For more general types of partition like those formed by the points on the unstable cycles, $\mathbf{P}_\infty$ is state dependent. Let us analyze the time evolution of $S$ in this case. We have [cf. Eq. (12a)]

$$\Delta S_n = S_n - S_{n-1} ,$$
$$S_n = -\sum_j \sum_i (\mathbf{W}^n)_{ij} P_0(i) \ln \sum_i (\mathbf{W}^n)_{ij} P_0(i) . \tag{30}$$

Suppose that the system is started with probability one from a particular cell $\alpha$ of the partition $P_0(i) = \delta_{i\alpha}^{kr}$. Equation (30) becomes

$$\Delta S_n(\alpha) = -\sum_j (\mathbf{W}^n)_{\alpha j} \ln (\mathbf{W}^n)_{\alpha j} . \tag{31}$$

For a single step ($n=1$) one obtains, upon averaging over all $\alpha$'s using the invariant distribution,

$$S_K = -\sum_\alpha P_\infty(\alpha) W_{\alpha j} \ln W_{\alpha j} > 0 . \tag{32}$$

This quantity is usually regarded as the analog of Kolmogorov entropy, measuring the average amount of information created by the system in one unit of time. We see, however, from Eq. (30) that $S_K$ is not identical to the full balance, $\Delta S_n$ of information entropy: $\Delta S_n$ may not be always positive, reflecting a nonmonotonic evolution of $S_n$ itself. For instance, for partition $J_3$ the transition matrix [Eq. (26)] has one eigenvalue equal to one and a pair of complex conjugate eigenvalues equal to $\pm i0.50$. Starting initially with the probability mass concentrated entirely in a cell yields a variation of $\Delta S_n$ displaying an oscillatory trend before finally approaching to zero. Figure 2 depicts the corresponding variation of total entropy $S_n$. This result means therefore that during the evolution induced by the dynamics on the state space defined by our coarse partition the system may *create as well as compress* information.[17]

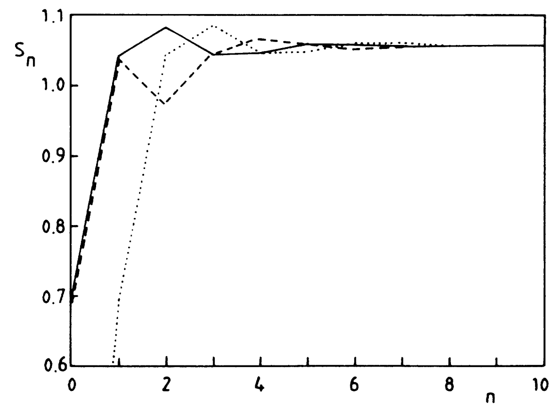


FIG. 2. Nonmonotonic time evolution of information entropy for the Markov process generated by partition $J_3$. Solid, dashed, and dotted lines refer, respectively, to initial conditions on cell 1, 2, and 3.

## V. DISCUSSION

The mapping of chaotic dynamics to a master equation developed in the present paper should constitute a natural tool for describing a variety of complex systems encountered in nature and forecasting their future trends. Our formulation takes fully into account the structure of the attractor [through the presence of the invariant probability $\rho_\infty(x)$ in Eq. (10b)] as well as the possibility of initial conditions far from the steady state [through the presence of $\mathbf{P}_0$ in Eq. (10b)]. Actually, since $\rho_\infty(x)$ is an invariant of the evolution operator $U$ one might equally well introduce a projector $\pi$ containing $\rho_\infty(x)$ as a weighting factor and an initial condition $\rho_0$ independent of $\rho_\infty(x)$. The two formulations are naturally identical for systems like the tent map admitting the Lebesgue measure as invariant measure.

The approach followed in the present paper has many similarities with the theory of nonunitarity transformations for conservative strongly unstable dynamical systems, developed by Prigogine *et al.*[18–21] Our work has been concerned more specifically with coarse-grained probabilities, for which one can define a perfectly regular stochastic process. The main point is that no phenomenological assumptions similar to those made traditionally had to be invoked.[22] In fact, the possibility to formulate the compatibility condition (15) and check its validity in a number of cases provides us with a *dynamical formulation* of coarse-graining.

An interesting question relates to the class of attractors for which the compatibility condition (15) can be expected to hold. In particular, if the dynamics is highly nonuniform as in intermittent systems[23] it is not clear whether a finite partition of state space suffices for securing the Markovian character of the process. In such cases it may be necessary to envisage the generalization of our method to partitions containing an infinite (but countable) number of cells.

Finally, our approach suggests that continuous time dynamical systems (flows) exhibiting chaos may also generate Markovian stochastic processes, defined on partitions of the state space delimited by the unstable manifolds of certain distinguished trajectories.


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