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Long-term climatic variability and chaotic dynamics

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ABSTRACT

The effect of a periodic forcing on the mean ocean temperature – sea ice system is analyzed in a region of parameters in which this latter system gives rise to a homoclinic bifurcation. The possibility of complex, nonperiodic behavior as well as of long period solutions is established both analytically and by numerical simulations. The implications of the results in the glaciation problem are briefly discussed.

1. Introduction

It is well-known that the variance spectra of the global ice volume of the last million years contain a limited number of pronounced peaks, related to the average periodicities of the earth's orbital variations (see for instance, Berger (1981)). This observation, which suggests that quaternary glaciations should have a marked cyclic character, has prompted many investigators to view long-term climatic variability as a sustained self-oscillation of the limit cycle type. Such self-oscillations were shown to arise quite naturally from the coupling between mean ocean temperature and sea ice (Saltzman et al., 1981, 1982), or mean surface temperature and continental ice systems (Källén et al., 1979). Moreover, under the presence of a weak external periodic forcing, a phase locking could take place, enabling the limit cycle oscillator to adopt the frequency of the forcing or a multiple thereof, while maintaining a fixed phase lag with it (Nicolis, 1984a).

Now, a closer scrutiny of the global ice volume record reveals the presence of a broad band structure, indicating that an appreciable amount of randomness is superimposed on the preferred peaks. Usually, this broad-band component is discarded or attributed to extrinsic noise. In this paper, we suggest that, far from being a spurious effect, it actually indicates that climatic variability can be viewed through deterministic aperiodic dynamics described by a chaotic attractor.

The analysis of the oxygen isotope ratio of a deep sea core sediment following some new methods developed in the theory of dynamical systems, has recently produced evidence of chaotic dynamics associated with climatic variability of the last million years (Nicolis and Nicolis, 1984, 1986). Interesting models giving rise to nonperiodic behavior have also become available (Le Treut and Ghil, 1983; Saltzman et al., 1984). However, the presence of 3 variables (temperature, continental ice extent and bedrock deformation, or temperature, continental and marine ice extent), of several parameters and of the astronomical forcings makes it difficult to disentangle from these models the basic mechanisms responsible for aperiodic behavior. The examination of these mechanisms is the principal goal of the present article.

In Section 2, we consider the Saltzman model oscillator (Saltzman et al., 1981, 1982) near a homoclinic bifurcation and couple it to a single weak external periodic forcing. An analytical study performed in Section 3 suggests a threshold value relating the forcing amplitude to 3 other parameters, beyond which the system should present complex bifurcations, including the onset of non-periodic behavior. Numerical simulations, reported in Section 4, corroborate these conjectures. A family of chaotic attractors is shown to exist, and the power spectrum constructed in a particular case shows both a broad-band structure as well as a preferred peak. The main conclusions and the discussion of the perspectives

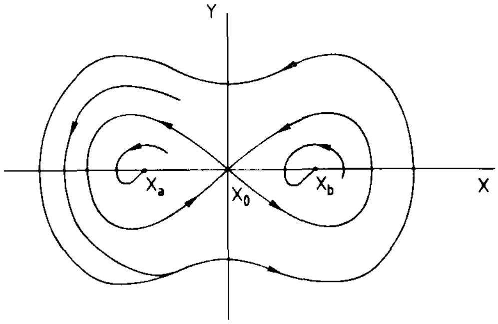

Fig. 1. Phase-space portrait of a 2-variable system involving a pair of homoclinic orbits.

suggested from our results are briefly presented in Section 5.

2. A periodically forced oscillator near a homoclinic birfurcation

A periodically forced oscillator in the vicinity of a Hopf bifurcation studied in previous work by the author (Nicolis, 1984a) cannot account for aperiodic solutions. On the other hand, in the theory of dynamical systems, one shows that such a behavior can arise near parameter values for which the system admits very special orbits known as *homoclinic orbits* (Guckenheimer and Holmes, 1983). Fig. 1 depicts a typical phase space portrait of a 2-variable dynamical system involving a pair of homoclinic orbits. We observe two stable fixed points $(X_a, 0)$ and $(X_b, 0)$ around which the system performs damped oscillations. An intermediate unstable point $(X_0 = 0, 0)$ gives rise to a pair of unstable and a pair of stable trajectories known as separatrices, which eventually merge to form the double-loop structure. These are precisely our homoclinic trajectories, which can also be viewed as infinite period orbits. Further away in phase space, the system admits a large-amplitude stable periodic solution of finite period.

As an illustration of the phase space portrait of Fig. 1, consider Saltzman's oscillator (Saltzman et al., 1981, 1982; Nicolis, 1984a). One has in dimensionless variables,

$$\frac{d\eta}{dt} = \theta - \eta,$$

$$\frac{d\eta}{dt} = b\theta - a\eta - \eta^2\theta, \tag{1}$$

where η and θ are, respectively, the (suitably scaled) deviations of the sine of latitude of sea ice extent and of the mean ocean temperature from a reference state. Eqs. (1) may admit up to 3 steady state solutions $(\theta_s = \eta_s = 0)$ and $(\theta_s' = \eta_s' = \pm(b-a)^{1/2})$. The linearized equations around the trivial state $(\theta_s = \eta_s = 0)$ are simply

$$\frac{d\delta\eta}{dt} = \delta\theta - \delta\eta,$$

$$\frac{d\delta\theta}{dt} = b\delta\theta - a\delta\eta,$$

They admit solutions which depend on time exponentially, with an exponent given by the *characteristic equation*

$$\omega^2 - \omega(b-1) + a - b = 0. \tag{2}$$

The uncertainties associated with the specific values of the parameters a and b suggest that one considers all possible dynamical behaviors allowed by the model eqs. (1). When $a > b$ and $b = 1$, this equation admits a pair of purely imaginary solutions. For $b > 1$ and $a > b$, the real part becomes positive. This is the range of Hopf bifurcation which was discussed extensively in previous work (Nicolis, 1984a). On the other hand, for $a < b$, the characteristic equation admits 2 real solutions of opposite sign. The reference state then behaves as a saddle, just like state $(X_0 = 0, 0)$ in Fig. 1. Both kinds of regimes merge for $a = b$, $b = 1$ for which values both roots of the characteristic equation vanish simultaneously. The climatic significance of the various ranges of parameter values is discussed in Section 4.

We now set

$$\theta - \eta = \xi. \tag{3}$$

Substituting into eqs. (1), we obtain

$$\frac{d\eta}{dt} = \xi,$$

$$\frac{d\xi}{dt} = (b-1)\xi + (b-a)\eta - \eta^3 - \eta^2\xi. \tag{4}$$

Near the degenerate situation in which both characteristic roots vanish simultaneously, one has

$$b - 1 = \varepsilon_1 \ll 1,$$
$$b - a = \varepsilon_2 \ll 1. \tag{5}$$

It can be verified that for $\varepsilon_1 = 0.8\varepsilon_2$, eqs. (4)

generate the phase portrait of Fig. 1. As a matter of fact eqs. (4) are no less than the *normal form* of any dynamical system displaying sufficient symmetry for quadratic terms to vanish, and operating near the degenerate situation in which both characteristic roots vanish simultaneously (Guckenheimer and Holmes, 1983). The validity of our conclusions will therefore extend far beyond the specific model of eqs. (1).

As long as one deals only with a two-variable system of the type of eqs. (1) or (4), the complex behavior associated with the infinite period homoclinic orbit is singular in nature, since the slightest change of the parameters will drive the system to one of the stable states X_a or X_b or to the finite period solution surrounding these states (cf. Fig. 1). We now study the coupling between the above-defined system and a weak periodic forcing simulating, for instance, the variation of the solar influx due to the earth's orbital variations. To simplify as much as possible, we consider an additive periodic forcing acting on the second equation (4) alone:

$$\frac{d\eta}{dt} = \xi,$$

$$\frac{d\xi}{dt} = \varepsilon_1 \xi + \varepsilon_2 \eta - \eta^3 - \eta^2 \xi + p \sin \omega t, \tag{6}$$

where p and ω are the amplitude and the frequency of the forcing, respectively (see Section 4 for numerical examples of climatic relevance). The behavior of this 4-parameter system, reported in Section 3 will turn out to be completely different from that of the unforced system.

3. Analytic study

We perform the following scaling of variables and parameters (Guckenheimer and Holmes, 1983; Baesens and Nicolis, 1983):

$$\eta = x\mu, \qquad \xi = y\mu^2,$$

$$\varepsilon_1 = \gamma_1 \mu^2, \qquad \varepsilon_2 = \gamma_2 \mu^2, \qquad p = q\mu^4,$$

$$t = \tau\mu^{-1}, \qquad \omega = \Omega\mu, \qquad \mu \ll 1. \tag{7}$$

Eqs. (6) become:

$$\frac{dx}{d\tau} = y,$$

$$\frac{dy}{d\tau} = \gamma_2 x - x^3 + \mu(\gamma_1 y - x^2 y + q \sin \Omega\tau). \tag{8}$$

These equations can be viewed as the perturbations (for μ small) of a reference system described by

$$\frac{dx_0}{d\tau} = y_0,$$

$$\frac{dy_0}{d\tau} = \gamma_2 x_0 - x_0^3. \tag{9}$$

Remarkably, this is a conservative system known as Duffing's oscillator (Andronov et al., 1971; Guckenheimer and Holmes, 1983). It derives from the Hamiltonian

$$H = \frac{y_0^2}{2} - \gamma_2 \frac{x_0^2}{2} + \frac{x_0^4}{4}, \tag{10}$$

which keeps a constant value (analogous to total energy in mechanics) along the dynamics. This allows us to compute exactly the phase space portrait, shown in Fig. 2 for $\gamma_2 > 0$. For all non-vanishing values of H, we obtain a continuum of periodic trajectories. On the other hand, for $H = H$ (saddle) $= 0$, we obtain

$$y_0 = \pm\left(\gamma_2 x_0^2 - \frac{x_0^4}{2}\right)^{1/2}, \tag{11}$$

which describes orbits in the form of two closed loops extending symmetrically from $x_0 = 0$ to $x_0 = \pm(2\gamma_2)^{1/2}$. These are precisely the homoclinic orbits. The time dependence of the variables x_0, y_0 on these orbits can be found by substituting (11) into the first eq. (9). We obtain

$$\frac{dx_0}{d\tau} = \pm\left(\gamma_2 x_0^2 - \frac{x_0^4}{2}\right)^{1/2},$$

or

$$\int_{x_0}^{\pm(2\gamma_2)^{1/2}} dx_0 \left(\gamma_2 x_0^4 - \frac{x_0^4}{2}\right)^{-1/2} = \pm(\tau - \tau_0),$$

or finally

$$x_0(\tau) = \frac{(2\gamma_2)^{1/2}}{\mathrm{ch}\{(2\gamma_2)^{1/2}(\tau - \tau_0)\}} \tag{12a}$$

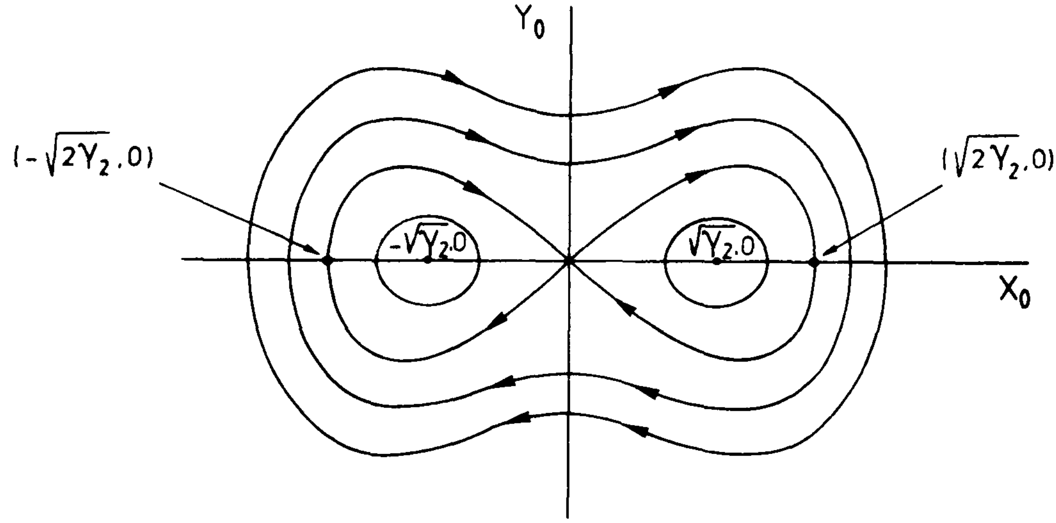

Fig. 2. Phase-space portrait of Duffing's oscillator for $\gamma_2 > 0$.

and consequently (cf eq. (11))

$$y_0(\tau) = \pm x_0(\tau)\left(\gamma_2 - \frac{x_0^2(\tau)}{2}\right)^{1/2}. \quad (12b)$$

As it turns out, x_0, y_0 can also be computed for the continuum of periodic orbits lying on both sides of the homoclinic ones. The calculation is more involved as it requires the use of elliptic integrals, and will not be reproduced here (Guckenheimer and Holmes, 1983).

Let us now return to the full problem, eqs. (8). Clearly, going from eqs. (9) to eqs. (8), amounts to inquiring how the phase space structure and, in particular, the infinite period homoclinic orbits of Fig. 2 are perturbed by both "dissipative" terms $\gamma_1 y - x^2 y$ and by the periodic forcing.

Let us first formulate this problem analytically. Setting $x = x_0 + \mu u$, $y = y_0 + \mu v$, we obtain the following equations for the perturbations u, v:

$$\frac{du}{d\tau} = v,$$

$$\frac{dv}{d\tau} = (\gamma_2 - 3x_0^2)u + \gamma_1 y_0 - x_0^2 y_0 + q \sin \Omega\tau. \quad (13)$$

The general solution of this problem (notice that the coefficients are time-dependent through x_0, y_0) is the sum of a particular solution (u_p, v_p) and of the general solution of the homogeneous system:

$$\frac{du_h}{d\tau} = v_h,$$

$$\frac{dv_h}{d\tau} = (\gamma_2 - 3x_0^2)u_h. \quad (14)$$

It is easily verified by inspection that this latter system admits solutions of the form

$$u_h = Ay_0,$$

$$v_h = A(\gamma_2 x_0 - x_0^3), \quad (15)$$

where the constant A is undetermined at this stage.

We now come back to eqs. (13). To find u_p, v_p, we must invert the operator

$$L = \begin{pmatrix} \dfrac{d}{dt} & -1 \\ -(\gamma_2 - 3x_0^2) & \dfrac{d}{dt} \end{pmatrix},$$

which, by eqs. (14)–(15), has a nontrivial null-space that is to say, a set of non-vanishing eigenfunctions corresponding to a zero eigenvalue. In order to avoid the presence of singular terms arising from this inversion, we must therefore make sure that the inhomogeneous part of eq.

(13) be orthogonal to the null solution of the adjoint of (14):

$$\frac{du^*}{d\tau} = -(\gamma_2 - 3x_0^2)v^*,$$

$$\frac{dv^*}{d\tau} = -u^*, \tag{16}$$

which has the form

$$u^* = -(\gamma_2 x_0 - x_0^3),$$

$$v^* = y_0. \tag{17}$$

The solvability condition for (13) therefore reads:

$$\frac{1}{T}\int_{-T/2}^{T/2} d\tau(u^*, v^*)\begin{pmatrix} 0 \\ \gamma_1 y_0 - x_0^2 y_0 + q \sin \Omega\tau \end{pmatrix} = 0. \tag{18}$$

We have defined a scalar product which includes an average over a time interval T. If the period of the reference solution (x_0, y_0) is not rationally related to $2\pi/\Omega$, then the limit $T \to \infty$ should be taken in eq. (18) (Baesens and Nicolis, 1983). This is the case of the homoclinic trajectory, for which eq. (18) takes the following explicit form, known also as the *Melnikov integral* (Guckenheimer and Holmes, 1983):

$$\Delta = \int_{-\infty}^{\infty} d\tau\, y_0[\gamma_1 y_0 - x_0^2 y_0 + q \sin \Omega\tau] = 0. \tag{19}$$

This relation allows us to identify a critical forcing amplitude, q_c beyond which the homoclinic orbit is destroyed by the periodic perturbation. One obtains, after a lengthy calculation:

$$q_c = \tfrac{1}{3}\gamma_2^{\frac{3}{2}}(\gamma_1 - \tfrac{4}{5}\gamma_2)\,\frac{\mathrm{ch}\dfrac{\Omega\pi}{(2\gamma_2)^{\frac{1}{2}}} + 1}{\mathrm{ch}\dfrac{\Omega\pi}{2(2\gamma_2)^{\frac{1}{2}}}}. \tag{20}$$

In the theory of dynamical systems, one shows that for $q > q_c$, a variety of complex non-periodic behaviors may arise (Guckenheimer and Holmes, 1983). Note that if $\gamma_1 = \frac{4}{5}\gamma_2$, the system operates exactly on the homoclinic orbit and $q_c = 0$.

4. Numerical results: frequency demultiplication and aperiodic behavior

In this section, we study the complex regimes predicted by eq. (20) by means of numerical simulations. The most unambiguous way to characterize the type of regime displayed by a dynamical system is to perform a Poincaré surface of section. Remember that we are dealing with a forced system involving the two variables x and y. Effectively, such a dynamics takes place in a 3-dimensional space, since one can always express the forcing through $q \sin\chi$, $d\chi/dt = \Omega$ thereby introducing its phase χ as a third variable. One can now map the original continuous dynamical system into a discrete time system by following the points at which the trajectories cross (with a slope of prescribed sign) the plane $\cos\chi = C$, corresponding to a given value of the forcing. One obtains in this way, a Poincaré map, (Fig. 3), that is to say, a recurrence relation

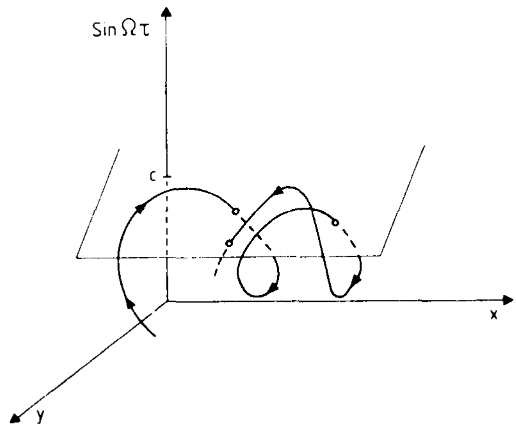

Fig. 3. Schematic representation of a Poincaré map for a forced system. As the system evolves, a representative trajectory cuts a plane of section C at discrete times τ_n. The study of the dynamics on the surface of the section gives valuable information on the qualitative behavior of the initial system.

$$x_{n+1} = f(x_n, y_n),$$
$$y_{n+1} = g(x_n, y_n), \tag{21}$$

where n labels the successive intersections.

Suppose that the trajectories of the continuous time flow tend, as $t \to \infty$, to an asymptotic regime. In the 3-dimensional state space, this regime will be characterized by an invariant object, the attractor. The signature of this object on the surface of the section will obviously be an attractor of the discrete dynamical system, eqs. (8). Conversely, from the existence of an attractor on the surface of the section, we can infer the properties of the underlying continuous time flow.

Let us now comment on the choice of numerical values of the parameters in connection with the properties of both the astronomical forcings and the climatic system itself. We first recall that the dimensionless parameters a and b and the time t are related to the original dimensional parameters and time $\bar{t}$ introduced by Saltzman et al. (1981, 1982) through (Nicolis, 1984a, b)

$$a = \frac{\psi_1 \phi_1}{\phi_2^2}, \qquad b = \frac{\psi_2}{\phi_2}, \qquad t = \bar{t}\phi_2. \tag{22}$$

Here ϕ_1 (K^{-1} yr^{-1}) and ϕ_2 (yr^{-1}) are coefficients related to shelf ice inertia, whereas ψ_1 (K yr^{-1}), ψ_2 (yr^{-1}) contain the effects of release of latent heat of fusion.

In all studies of ice ages reported so far in which Saltzman's model has been utilized, the adopted values of the above parameters correspond to very long characteristic times, such that the self-oscillation period of the unforced system is brought to the range of 100 kyr. An important result of the present analysis is, on the other hand, that a response dominated by slow time scales may be achieved by rather low values of the ice inertia and of the latent heat relaxation time. The reason for this is that for values of a, b close to $a = 1$, $b = 1$, there is a *collective* phenomenon of slowing down arising from the interaction between the various processes, rather than from the high inertia of the processes themselves.

We will now focus our attention on a concrete numerical example. Choosing $\mu = 0.1$, $\gamma_2 = 1$ and $\gamma_1 = 1.01$, we find that the system of eqs. (8) in the absence of forcing may evolve (depending on the initial conditions) to a periodic solution (the continuation of the outer limit cycle of Fig. 1) of period $T \sim 7$. In the variables of eqs. (6), this corresponds to a period of $7\mu^{-1} \sim 70$, and in the original dimensional variables to a period of $70/\phi_2$ yr. Taking $\phi_2^{-1} \sim \psi_2^{-1} \sim 500$ yr, $\psi_1^{-1} \sim 300$ yr, $\phi_1^{-1} \sim 800$ yr, we achieve $a \sim 1$, $b \sim 1$ as desired in our analysis, and a period of self oscillation of about 35 kyr.

We now consider the effect of the forcing. Owing to the multiplicity of attractors and to the difficulty to locate, *a priori*, the chaotic regime, we have first considered, essentially for "economic" reasons, a high-frequency forcing ($\Omega = 3$). This enabled us to explore the parameter space and to study the effect of varying the initial conditions.

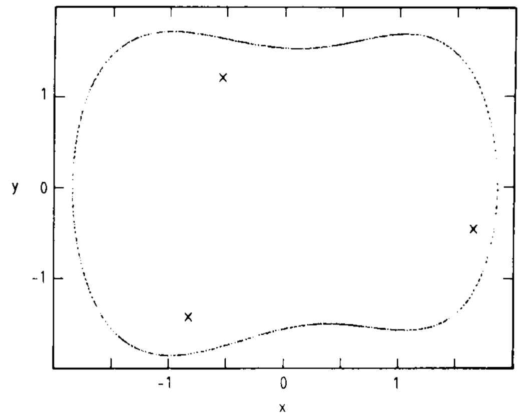

Fig. 4. Poincaré map of the forced system eqs. (8) for $\gamma_2 = 1$, $\gamma_1 = 1.01$, $\mu = 0.1$, $\Omega = 3$ and $q = q_c + 0.2$, where $q_c = 1.96$. Depending on the initial conditions, the system evolves either to a quasi-periodic attractor (closed curve) or to a periodic one, with a period 3 times as large as the forcing period (points and crosses).

Fig. 4 depicts the Poincaré surface of section for values of q near the threshold $q_c = 1.96$, of eq. (20). We observe the coexistence of 2 attractors: a periodic one, with a period 3 times larger than the forcing period, and a quasi-periodic one, represented in the full phase space by a 2-dimensional toroidal surface.

By increasing q further from q_c, one observes the coexistence of period-3 and period-2 attractors. As a matter of fact, there exist 2 attractors of a given period (Fig. 5a, b, points and crosses), differing by their phase relative to the phase of the forcing (Fig. 6a, b). For still larger values of q, the periodic attractors disappear and a manifestly non-periodic stable regime dominates (Fig. 7). We conjecture that we are in the presence of a chaotic attractor. More complex dynamical regimes which can be qualified as *intermittent* are also observed. For instance, after spending some time on a seemingly chaotic regime, the system jumps on a period-3 regime. In view of this latter regime, one may naturally raise the question of whether a number of global changes of the climatic system, like for instance the onset of glaciations during the quaternary era, was the manifestation of a "metastable" intermittent behavior.

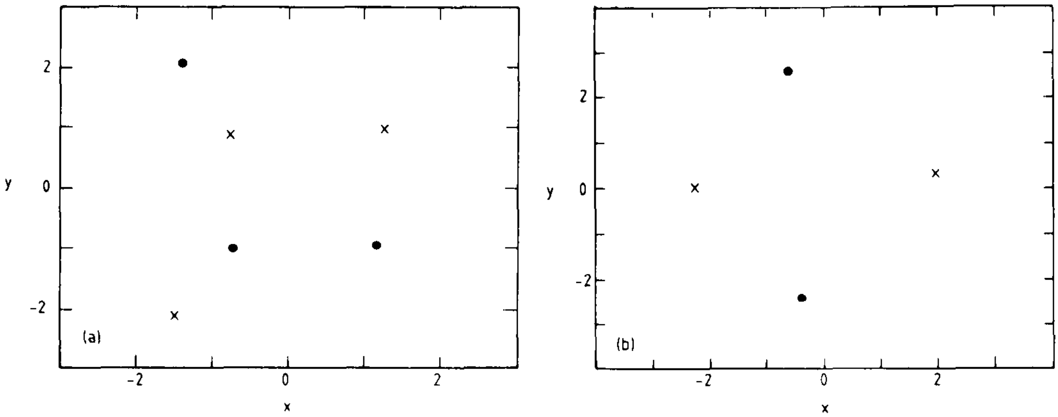

Fig. 5. Poincaré maps of the forced system eqs. (8) for the same parameter values as in Fig. 4 but with $q = 12q_c$. Depending on the initial conditions, the system evolves to one of the 4 coexisting attractors: (a) 2 different period-3 attractors; (b) 2 different period-2 attractors.

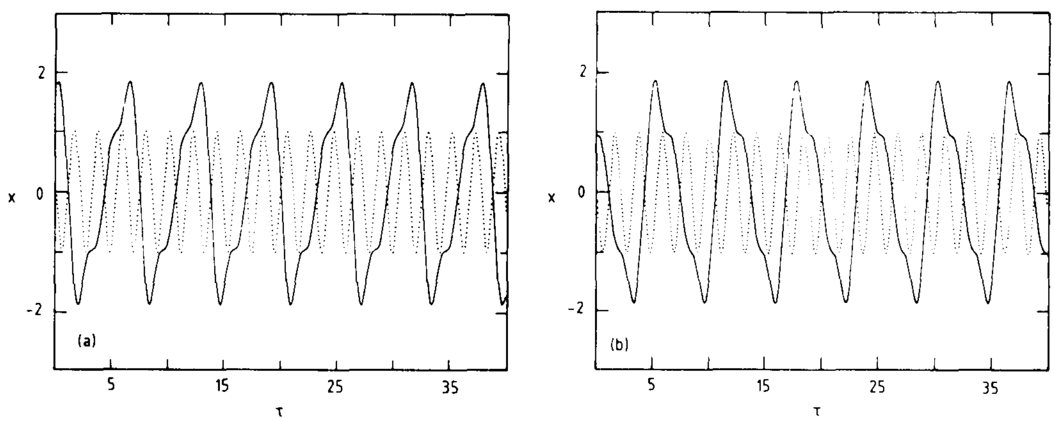

Fig. 6. Time dependence of variable x of the 2 period-3 solutions ((a) and (b)) for the same parameter values as in Fig. 4 but with $q = 13.5q_c$. The dotted line represents the forcing with amplitude normalized to unity. Notice that the 2 solutions exhibit different phase-shifts relative to the forcing.

Let us now consider a choice of forcing of climatic significance. For $\Omega = 0.3$ and for values of characteristic times reported above, one has a period of about 100 kyr, which is known to be the period of one of the astronomical forcings. From eq. (20), we find $q_c = 0.148$. Fig. 8a reports the result of a simulation carried out for $q = 7.4$ which corresponds to a rather realistic amplitude of $p \sim 7 \cdot 10^{-4}$ in eq. (6). We are again in the presence of a chaotic attractor. This is further corroborated by the time-dependence of the variables (Figs. 8b, c) as well as by the power spectrum (Fig. 8d). It is noteworthy that the latter exhibits an important broad-band component, a dominant (though noisy) peak at the forcing frequency, and a lower and more noisy peak close to the unforced system self-oscillation frequency (equal to about 3 times the forcing frequency). The similarity with paleoclimatic spectra is rather striking, considering the simplicity of the model. Especially significant is, we believe, the fact that there is no need of resonance between system and external forcing. In a sense, chaotic dynamics is tantamount to an infinity of

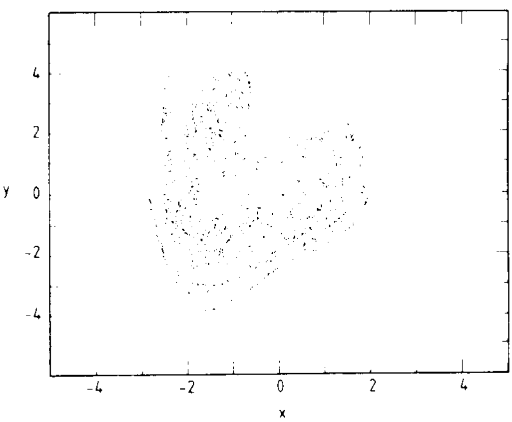

Fig. 7. Non-periodic attractor of system (8) for the parameter values of Fig. 4 except $q = 20q_c$.

resonances, from which the continuous background of spectrum is built.

5. Discussion

We have identified a mechanism of climatic variability in which the broad-band structure of the spectrum of climatic data is attributed to the presence of a non-periodic (chaotic) attractor. In as much as the dynamics on a chaotic attractor exhibits a marked sensitivity to initial conditions, our result provides us with a natural interpretation of the intrinsic unpredictability of the climatic system (Lorenz, 1984).

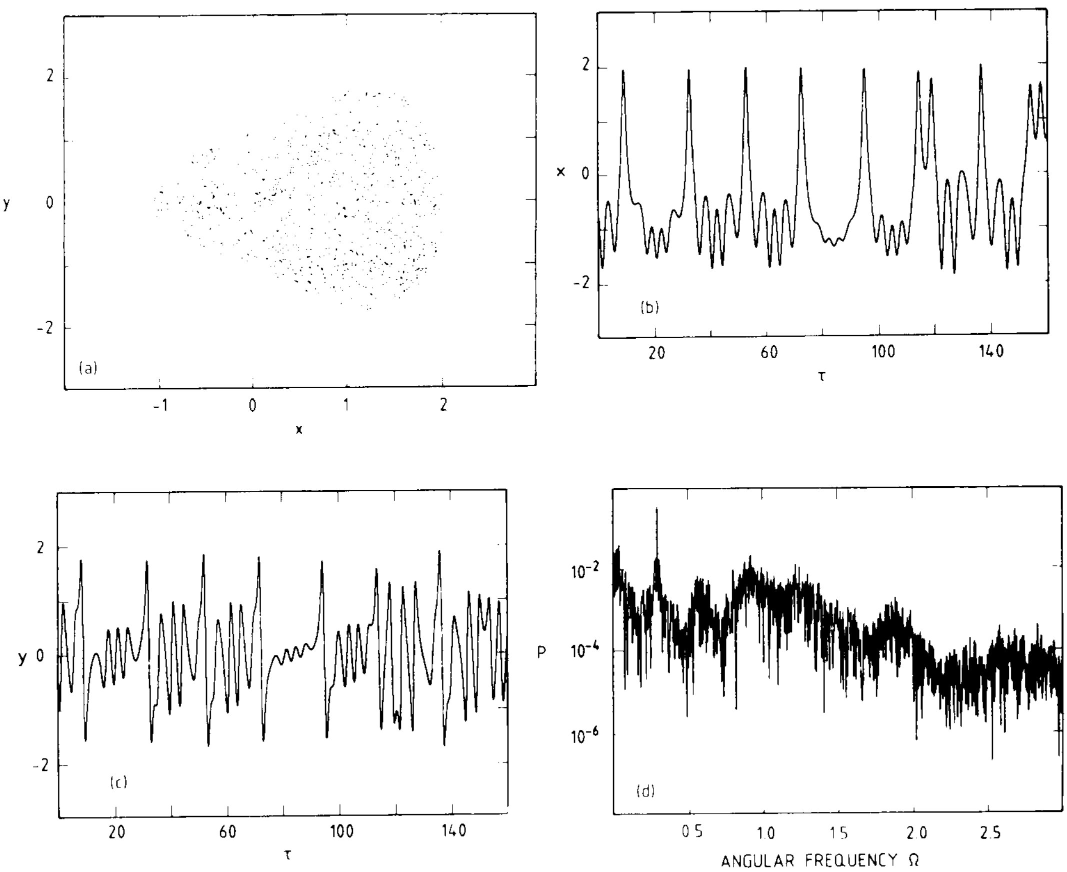

Fig. 8. Numerical integration of eqs. (8) for the same parameter values as in Fig. 4 but $\Omega = 0.3$ and $q = 50q_c$, where now $q_c = 0.148$. (a) Poincaré map; (b) time evolution of variable x; (c) time evolution of variable y; (d) power spectrum of variable x.

In this paper, we limited ourselves to a *minimal model*, since our purpose was to identify the principal mechanisms responsible for aperiodic behavior. Being a 3-dimensional dynamical system (2 internal variables and the forcing) our model cannot obviously reproduce the fractal dimensionality of 3.1 of the climatic attractor suggested by the analysis of the V28238 core carried out recently (Nicolis and Nicolis, 1984, 1986). Still, it could provide a skeleton on which further quantitative improvements could be made. For instance, keeping in mind the constraints imposed by the analytical results, one can tune the parameters to get an optimal agreement with the record. A more realistic possibility is to add an additional external forcing compatible with the earth's orbital variations. A second possibility would be to consider 2 oscillators (simulating the 2 hemispheres), each near a homoclinic bifurcation, coupled through a weak energy transfer across the equator. Some results far from the regime of homoclinic bifurcation have already been obtained for this latter model (Nicolis, 1984b), indicating the existence of complex dynamical behavior.

6. Acknowledgements

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