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Reconstruction of the dynamics of the climatic system from time-series data

(climatic variability/quaternary glaciations/chaotic dynamics)

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ABSTRACT **The oxygen isotope record of the last million years, as provided by a deep sea core sediment, is analyzed by a method recently developed in the theory of dynamical systems. The analysis suggests that climatic variability is the manifestation of a chaotic dynamics described by an attractor of fractal dimensionality. A quantitative measure of the limited predictability of the climatic system is provided by the evaluation of the time-correlation function and the largest positive Lyapounov exponent of the system.**

The most conspicuous property of the climatic system is the large number of processes that potentially may take part in an observed phenomenon. A realistic description should thus appeal, in principle, to a large number of variables that are expected to be coupled in a highly nonlinear fashion owing to the feedbacks present in the various mechanisms. Clearly, only a numerical approach can be envisaged within the framework of a description of this kind. As a result, the possibility of obtaining clear-cut information concerning both the long-term behavior of the system and the role of the key parameters would be compromised heavily.

An alternative approach to the modeling of climate, which received considerable attention in the last years, is to start at the outset with simplified models involving a limited number of variables. The equations of evolution of such models can be studied to some extent analytically and, as a result, the role of the parameters in the system's evolution can be followed in a transparent way. Still, a major difficulty confronting the modeler is to identify unambiguously the appropriate set of variables that are likely to play the dominant role in the problem of interest and to assign representative parameter values to their evolution equations.

Typically, the behavior of a complex system like climate is probed by observing a variable considered to be "pertinent" during a certain period of time. Fig. 1 presents an example of such a *time series* that pertains to the main theme of this paper, the problem of climatic variability over the last 1,000,000 years. Specifically, the figure describes the changes of the global ice volume inferred from $^{18}O/^{16}O$ isotope data of deep sea core sediments. The isotopic analysis of the particular core depicted in the figure, which is referred to as the V28-238 core, has been performed by Shackleton and Opdyke (1), whereas the corresponding time series was provided by the data bank of the University of Louvain (2).

At first sight, a time series of the kind depicted in Fig. 1 appears to provide a rather limited amount of information. In particular, one might argue that it is restricted to a "one-dimensional" view of a system that, in reality, contains a large number of interdependent variables. However, it turns out that a time series contains far richer information. It is the

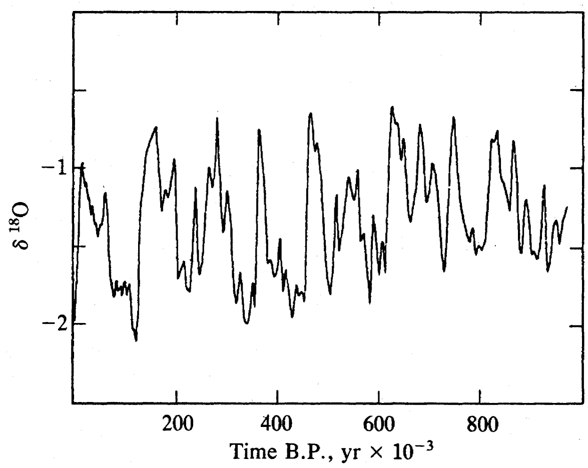

FIG. 1. Oxygen isotope record deduced from the V28-238 core (1). $\delta\,^{18}O = 1000(R_{sample} - R_{standard})/R_{sample}$, where R is the $^{18}O/^{16}O$ ratio.

purpose of this paper to explore a new method of data analysis (3) that will allow us to sort out this information for the problem of climatic variability (4).

In the next section we outline the main steps of construction, on the sole basis of the data, of the *climatic attractor*. This mathematical object, which captures some essential features of climate viewed as a dynamical system, belongs to the class of fractals descriptive of an irregular, chaotic motion. One of the essential features of such motions is their sensitive dependence on small variations of initial conditions, as a result of which it is impossible to predict their future evolution beyond a certain time, on the sole basis of their present state. In the subsequent sections we analyze successively the static properties of this attractor (for instance, its invariant probability distribution) as well as some dynamical properties like time-correlation functions.

Construction of the Climatic Attractor from the Time-Series Data

Let $X_0(t)$ be the time series availabe from the data, and $\{X_k(t)\}$, where $k = 0, 1, ..., n - 1$, the set of variables actually taking part in the dynamics. It is expected that $\{X_k(t)\}$ will satisfy a set of nonlinear differential equations of first order in time. By successive differentiations with respect to time, one can reduce this set to a single (generally highly nonlinear) differential equation of nth order in time for one of these variables. Thus, instead of $X_k(t)$, $k = 0, 1, ..., n - 1$, we may consider $X_0(t)$ and its $n - 1$ successive derivatives $X_0^{(k)}(t)$, with $k = 1, ..., n - 1$, to be the n variables of the problem. Now, both X_0 and its derivatives can be deduced from the single time series [the one for $X_0(t)$] as provided by the data. We see therefore that, as anticipated in the preceding section,

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we have in principle sufficient information at our disposal to go beyond the one-dimensional space of the original time series and to take into account the multidimensional character of the system's dynamics.

Actually, instead of using $X_0(t)$ and $X_0^{(k)}(t)$, it will be easier to work with $X_0(t)$ and the set of variables obtained from it by shifting its values by a fixed lag $\tau = m\Delta t$, where m is an integer, and Δt is the interval between successive samplings. It suffices to choose τ in such a way that the following n variables remain linearly independent:

$$
\begin{array}{l}
X_0(t_1),\ X_0(t_2),\ \ldots\ldots,\ X_0(t_N) \\
X_0(t_1 + \tau),\ X_0(t_2 + \tau),\ \ldots\ldots,\ X_0(t_N + \tau) \\
\vdots \\
X_0[t_1 + (n-1)\tau],\ X_0[t_2 + (n-1)\tau],\ \ldots\ldots,\ X_0[t_N + (n-1)\tau],
\end{array}
\qquad [1]
$$

where N is the number of data points.

We will use the above variables to define a *dynamical system*, and we will embed its evolution in an abstract space spanned by these variables, the *phase space*. The set $\{X_0(t + k\tau)\}$ ($k = 0, 1, \ldots, n - 1$) for different values of t will define the phase-space trajectories. The theory of dynamical systems shows that the structure of these trajectories is conditioned by two basic elements: (*i*) the dimensionality of phase space; in other words, the number of variables present. (*ii*) The existence of *attractors*; that is, of the stable states attained asymptotically in the course of time. An essential feature of these attractors is their dimensionality, d. Thus, a point attractor ($d = 0$) represents a stationary regime; a linear attractor ($d = 1$), a time-periodic regime; and a fractal attractor (d noninteger and larger than 2), a *chaotic* regime, characterized by a marked unpredictability similar to that usually attributed to atmospheric dynamics and climate.

As an example of the phase-space description we present, in Fig. 2, a projection of the trajectories generated by the time series of Fig. 1 in a two-dimensional space. The graph illustrates the complexity of the underlying motion. The principal questions one would like to answer are thus the following: Is it possible to obtain a lower bound on the number n of variables which capture the essential features of the long-term evolution of the climatic system? Is it possible to define a climatic attractor that represents this evolution and to estimate its qualitative properties, such as its dimensionality? Is it possible to assess the similarities and differences between dynamics and random noise?

The answer to all these questions is yes and has been developed in ref. 4. Specifically, we have been able to show that climatic evolution of the last million years can be

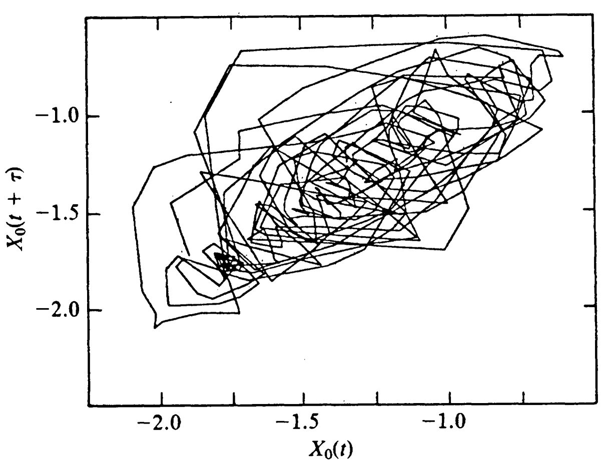

FIG. 2. Two-dimensional phase portrait generated by the time series of Fig. 1 ($\tau = 4\Delta t$; $\Delta t = 2000$ yr).

described in terms of an attractor of a fractal dimensionality $d \simeq 3.1$, embedded in a four-dimensional phase space. This substantiates further the difference between climate dynamics and random noise, which is already apparent from the structure of the power spectrum associated with Fig. 1. Moreover, that the dimensionality of the attractor is fractal provides us with a natural way to understand the intrinsic variability and unpredictability of the climatic system, since both of these features rank among the main characteristics of chaotic dynamics. A remarkable fact is that this dynamics is reducible to a limited set of key variables, although at this stage their specific nature cannot be identified.

In the subsequent sections, we develop two alternative descriptions of the climatic attractor, both of which will confirm the qualitative conclusions of our earlier work and will provide further insight to the salient features of long-term climatic evolution.

Probability Distribution on the Climatic Attractor

Despite the deterministic origin of the dynamics that may give rise to a chaotic attractor, the unpredictability of the underlying motion implies that individual trajectories in phase space lose their significance. A probabilistic description becomes therefore a natural tool for characterizing the behavior of the system. In this section we outline the structure of the probability distribution descriptive of the climatic attractor. We focus on the static properties of this function, such as the relative statistical weights of different parts of the attractor, postponing the study of the time-dependent aspects until the next section.

The general procedure is as follows. We introduce a coverage of the attractor by small hypercubes of size ε; we count the number of data points and of their shifts (see Eq. 1) in each of these hypercubes and divide the result by the total number of points. We obtain in this way a nonnegative quantity $P_\varepsilon(i)$, which is normalized to unity and which may be viewed as the probability for finding the system in the part of state space $X_0, \ldots, X_{n-1}$ lying within the hypercube $i = (i_0, \ldots, i_{n-1})$. Knowing this quantity, we then may compute the information theoretical entropy of the attractor (5):

$$S_I = -\sum_i P_\varepsilon(i) \ln P_\varepsilon(i). \qquad [2a]$$

In the limit $\varepsilon \rightarrow 0$, both P and S_I become singular. However, the quantity

$$D_I = -\frac{\sum_i P_\varepsilon(i) \ln P_\varepsilon(i)}{\ln(1/\varepsilon)} \qquad [2b]$$

can be shown to remain well-defined. In the literature of dynamical systems it is known as the *information theoretical dimensionality* (5). Actually, the fractal dimensionality d computed by the method described in the preceding section turns out to be less than or equal to D_I (3).

We have computed the probability distribution as well as S_I and D_I for the climatic attractor. The coverage was chosen in such a way that there are no empty hypercubes surrounded by hypercubes containing at least one data point. In a four-dimensional phase space this was achieved with a size of $\varepsilon \approx 0.2$, for which D_I turns out to be about 3.4, in good agreement with the results of the preceding section, considering the numerous uncertainties present in this problem.

Fig. 3 depicts a cut of the probability surface along the variable X_0. We see that despite the existence of a few distinguished peaks, many other states away from the positions of these peaks are visited with appreciable probability.

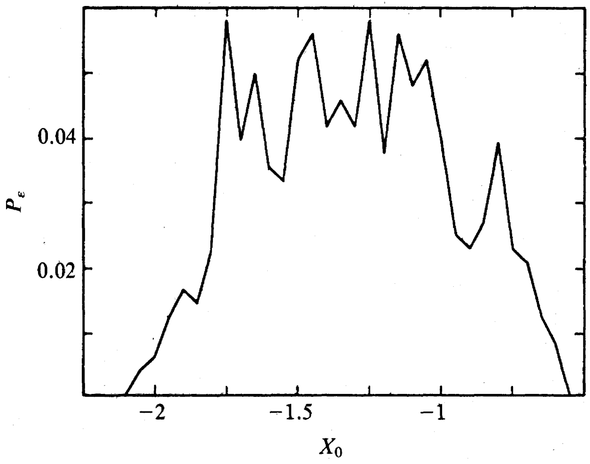

FIG. 3. One-dimensional cut of the probability surface associated with the climatic attractor.

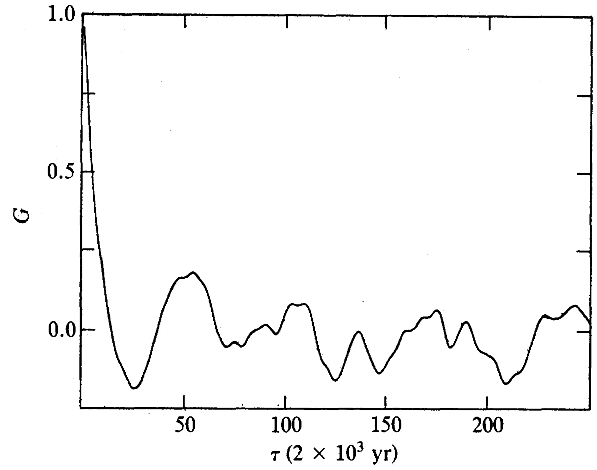

FIG. 4. Time-correlation function (G) of the variable X_0 on the climatic attractor.

This adds further credence to our previous conclusion that the dynamics is taking place on a single, chaotic attractor.

This result depends on our consideration of the full length of the record available. Had an interval of 400,000 years been considered instead, two more clear-cut peaks would have emerged. This reflects the existence of a low-frequency drift in the record, a situation common to many problems dealing with complex dynamics.

Time-Dependent Properties of the Climatic Attractor

In this section we explore further the properties of the climatic attractor and comment on the insight afforded by the developments of the previous sections in the understanding of climate dynamics.

We have already pointed out that the fractal nature of the attractor provides a natural explanation of unpredictability, one of the salient features of climate. In order to examine more closely how unpredictability is manifested from our time-series data, it is instructive to compute the time-correlation function of, say, the variable X_0:

$$G(\tau) = \frac{\frac{1}{N}\sum_{i=1}^{N} (X_0(t_i) - \overline{X}_0)(X_0(t_i + \tau) - \overline{X}_0)}{\frac{1}{N}\sum_{i=1}^{N} (X_0(t_i) - \overline{X}_0)^2}, \quad [3a]$$

where $$\overline{X}_0 = \frac{1}{N}\sum_{i=1}^{N} X_0(t_i). \quad [3b]$$

This quantity allows us to estimate the persistence of a relation built between the values of the variable X_0 in two nearby instances, over an increasingly long period of time. The sooner the relation dies out, the more pronounced the unpredictability of our system will be. Two extreme examples are a sustained periodic oscillation and a white noise for which $G(\tau)$ is respectively, an undamped periodic function (complete predictability) and a delta function (complete unpredictability).

Fig. 4 depicts the correlation function calculated from the time-series data of Fig. 1. We see that this function decays for small-to-intermediate times, and with increasing time it becomes negative and subsequently oscillates irregularly around zero. This leads us to two important conclusions. First, that the correlation function is not merely periodic or quasiperiodic proves that climate dynamics cannot be viewed as a passive response to the astronomical forcings. True, the periodicities of the latter are found in the peaks of the power spectrum of the data (6). This, however, is far from providing a full characterization of the nature of the underlying dynamics. Second, that the correlation function decays for short-to-intermediate times proves that we are dealing with a process displaying unpredictability. Barring white noise, the simplest example of such a process is a Markovian red noise generated, for instance, by an Ornstein–Uhlenbeck process (12). However, if this were the case here, the correlation function would never go to zero and would decay strictly exponentially. Neither of these two properties holds for Fig. 4. We conclude that we deal with a process possessing "memory."

The behavior displayed in Fig. 4 is strikingly similar to the behavior of the correlation function of the Lorenz model, computed recently by Grossman and Sonneborn-Schmick (7). This latter model is the prototype of a system generating unpredictability and chaos by its intrinsic dynamics, independently of the intervention of any external forcing, yet its power spectrum is peaked around some preferred frequencies, just like the power spectrum of the climatic data. Clearly, a major reconsideration of the role of orbital variations and of the information drawn from spectral analysis in climatology becomes necessary, in the light of these results.

Let us now turn to the kinetics of the attractor embedded in phase space. Fig. 5 (*A–D*) describe the way the trajectory unfolds on the attractor in the course of time for the last 400,000 years. We see clearly that the system winds repeatedly around the warm (low X_0) and the cold (high X_0) climates, but this motion is executed in a rather erratic manner and is interrupted by major excursions away from these climates. The direction of winding is apparently always the same. It seems therefore that the climatic attractor belongs to the family of attractors exhibiting "spiral chaos" and "screw" chaos, two terms coined by Rössler (8) in the theory of nonlinear dynamical systems.

In the theory of dynamical systems, the usual way to characterize quantitatively the wandering behavior of chaotic orbits is through the Lyapounov exponents (9). These numbers contain the following information: throughout the attractor, trajectories initially close to each other are considered, and the rate at which they diverge or converge for later times is computed. The Lyapounov exponent, σ, is the average of these individual results over a large number of data.

It can be shown that there exist as many Lyapounov exponents as phase-space dimensions. One of them is necessarily equal to zero, expressing the fact that the relative distance of initially close states on a given trajectory varies slower than exponentially. Others are negative, expressing the exponential approach of initial states to the attractor.

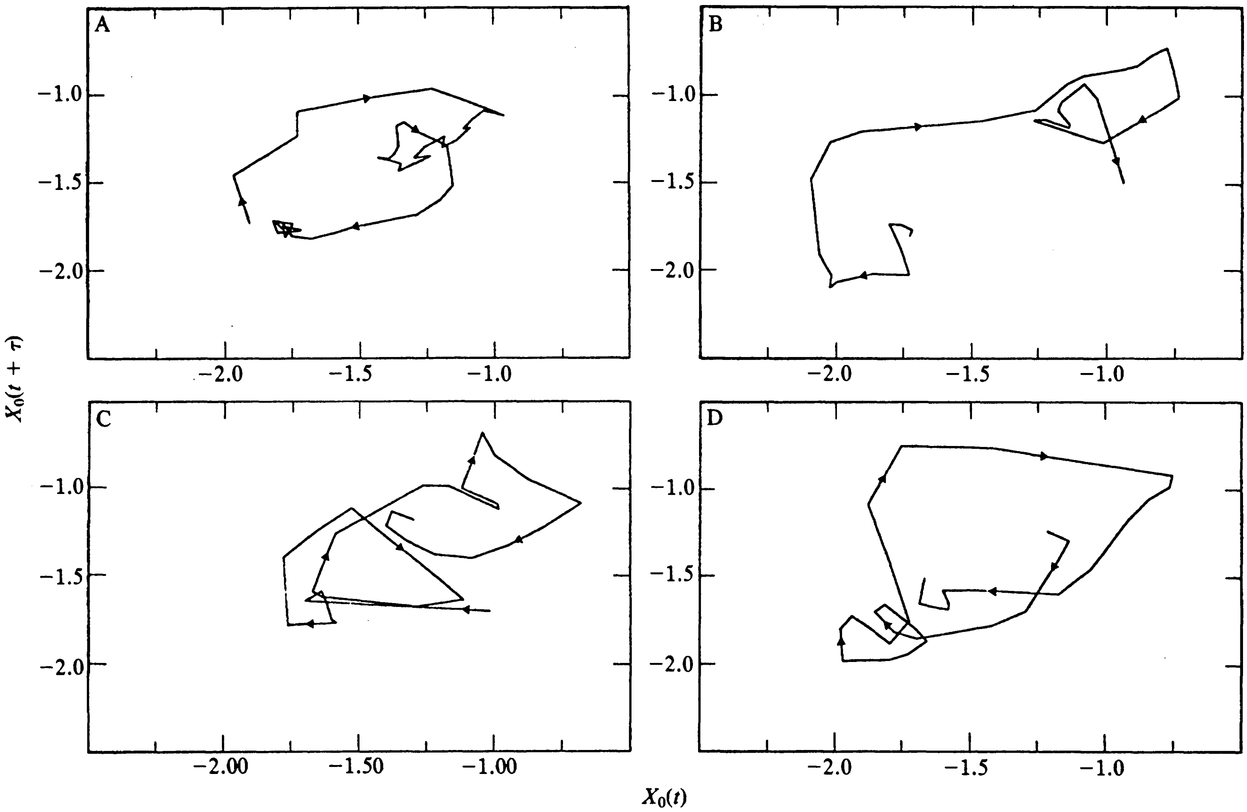

FIG. 5. Successive stages of the unfolding of phase-space trajectory (Fig. 2) on the climatic attractor. Each panel (*A–D*) corresponds to an interval of $50\Delta t$, where $\Delta t = 2000$ yr.

Finally, if the dynamics is chaotic, there will be at least one positive Lyapounov exponent witnessing the exponential divergence of nearby initial conditions on the stable attractor. Clearly, such a quantity is the ideal tool for describing the limits of predictability in climatic change. Note that in a well-behaved dissipative system the sum of all Lyapounov exponents must be strictly negative.

Recent algorithms (10) allow the calculation of the largest positive Lyapounov exponent of a dynamical system from time-series data. We have applied this procedure to the data set of Fig. 1 and found indeed a positive number σ, between 2.5×10^{-5} and 4×10^{-5} yr^{-1}. Its inverse, σ^{-1} (25–40 kyr), gives the limits of predictability of the long-term behavior of the system. We see that it corresponds, roughly, to the region between the vanishing and first minimum of the correlation function depicted in Fig. 4. It is interesting that the above limits of predictability seem to correspond to the limits beyond which a spectral multivariate regressive model becomes unstable (11).

Another useful way to characterize the wandering motion on the climatic attractor is to compute residence and exit times from a certain region of state space. To this end, we consider a particular hypersurface in phase space, say the hypersurface on which X_0 is equal to its time average value $\overline{X}_0$, see Eq. **3b**. We register the times the trajectory crosses the hypersurface, say toward the region of high X_0, and the times at which it crosses the hypersurface in the opposite direction, as it exits from this region. Repeating this procedure over a long time, we can then determine the residence-time probability in the cold climate. Fig. 6 describes the result. We find a multihumped probability, which turns out to be very similar to the corresponding quantity computed for Rössler's model of screw chaos (8). This shows that the process described by the variable X_0 is not Markovian since, as well known from the theory of stochastic phenomena, the residence-time probability of a Markovian process decreases exponentially with its argument. This corroborates the conclusion drawn earlier in this section on the basis of the behavior of the correlation function. Note, however, that the full set of variables defining the climatic attractor constitutes a Markovian process.

Concluding Remarks

Current progress in the theory of dynamical systems allows us to identify certain important features of climate from geological data, independent of any modeling. We believe

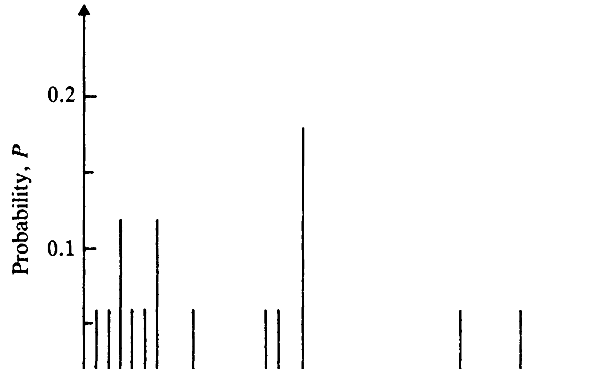

FIG. 6. Normalized residence-time probability distribution in the "cold" climate. Comparison with residence times in the "warm" climate shows that the overall times spent in the two regions are roughly equal.

that we have produced strong evidence that climate is to be viewed as a dynamical system which, by its own unstable dynamics, has the ability to capture and amplify small external effects. This view is different from the one in which climate dynamics is driven entirely by astronomical factors. To begin with, had climatic variability been a linear response to the Milankovitch periods of about 19, 23, 41, and 100 kyr, the attractor would have been quasiperiodic and its dimensionality greater than 4. Both properties are contradicted by our results. More important perhaps, it should be realized that the climatic record integrates in a complex way a variety of environmental inputs, of which the astronomical forcings are by no means the only example. Which regime will be realized in the "output space" as a result of these inputs depends on many factors such as nonlinearities, parameter values, or initial conditions, and there is absolutely no reason to believe that the record should keep track of every single input variable in an explicit manner. We believe therefore that a reassessment of the various mechanisms of climatic change becomes necessary in the light of our results. Finally, it would be important to extend the type of analysis reported in the present paper to problems pertaining to short- or intermediate-term climatic variability.

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