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Effective noise of the Lorenz attractor

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The Lorenz equations are cast in the form of a single stochastic differential equation in which a "deterministic" part representing a bistable dynamical system is forced by a "noise" process. The properties of this effective noise are analyzed numerically. An analytically derived fluctuation-dissipation-like relationship linking the variance of the noise to the system's parameters provides a satisfactory fitting of the numerical results. The connection between the onset of chaotic dynamics and the breakdown of the separation between the characteristic time scales of the variables of the original system is discussed.

I. INTRODUCTION

Chaotic attractors constitute the natural models of physical systems whose evolution has a limited degree of predictability, even though it can be traced back to a well-defined set of rules. A probabilistic description is clearly the most adequate approach to such dynamical systems. In this respect the question can be raised, whether a system giving rise to a chaotic attractor can be cast in the form of a set of stochastic evolution laws, in which a "deterministic" part and an "effective noise" part can be identified. The present paper describes a procedure leading to this decomposition. The Lorenz model is used throughout the analysis, but it will become clear that the ideas are quite general and should be applicable to a wide class of systems.

In Sec. II we transform the Lorenz model to a *canonical form* in which the linearized part around the trivial solution $X_0=Y_0=Z_0=0$ becomes diagonal. This form is the precursor of the reduction to the center manifold which holds near the bifurcation point of nontrivial steady-state solutions. Here we follow, however, a different procedure which amounts to expressing the two variables which are fast near bifurcation, as functionals of the slow variable. The resulting equation is cast in Sec. III, in the form of a "stochastic" differential equation. Some properties of the deterministic and "noise" parts are identified analytically. Extensive numerical computations, carried out in Sec. IV, allow us to sharpen these properties further and suggest an interesting picture, whereby the chaotic dynamics on the Lorenz attractor can be viewed as a noise-induced transition. The main conclusions are drawn in Sec. V.

II. CANONICAL FORM OF THE LORENZ EQUATIONS

The Lorenz equations read[1,2]

$$\begin{aligned}\frac{dX}{dt}&=\sigma(Y-X)\,,\\ \frac{dY}{dt}&=rX-Y-XZ\,,\\ \frac{dZ}{dt}&=XY-bZ\,.\end{aligned}\qquad(1)$$

We have used standard notation. σ,b,r are real positive parameters. We hereafter fix $b=\frac{8}{3}$, $\sigma=10$, and use r as bifurcation parameter. It is well-known that at $r=1$ the trivial steady-state solution $X_0=Y_0=Z_0=0$ undergoes a bifurcation to nontrivial solutions. In phase space this is reflected by the fact that for $r\geq 1$ this state behaves like a saddle possessing a one-dimensional unstable manifold and a two-dimensional stable manifold. For $r_p\simeq 24.74$ the nontrivial steady states become unstable via a subcritical Hopf bifurcation. Furthermore, numerical simulations show that for $r_T \gtrsim 24.06$ a chaotic attractor emerges.

In order to incorporate explicitly in the description the unstable manifold of the reference state $X_0=Y_0=Z_0=0$, which is known to play an important role in the structure of the attractor, we transform Eqs. (1) into a canonical form in which the linearized part becomes diagonal. This amounts to solving the eigenvalue problem

$$\begin{pmatrix}-\sigma & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}=\omega\begin{pmatrix}u\\ v\\ w\end{pmatrix}.\qquad(2)$$

It is easy to see that the eigenvalues ω are given by

$$\omega_1=\tfrac{1}{2}\{-(\sigma+1)+[(\sigma+1)^2+4\sigma(r-1)]^{1/2}\}\ ,$$
$$\omega_2=\tfrac{1}{2}\{-(\sigma+1)-[(\sigma+1)^2+4\sigma(r-1)]^{1/2}\}\ , \qquad (3)$$
$$\omega_3=-b\ .$$

The transformation matrix, T^{-1}, of (1) into the canonical form is constructed in terms of the corresponding eigenvectors. A straightforward calculation gives

$$T^{-1}=\frac{r}{\omega_1-\omega_2}\begin{bmatrix} 1 & -\frac{\omega_2+1}{r} & 0 \\ -1 & \frac{\omega_1+1}{r} & 0 \\ 0 & 0 & \frac{\omega_1-\omega_2}{r}\end{bmatrix}\ . \qquad (4)$$

Operating on both sides of (1) by T^{-1} we thus arrive at the form

$$\frac{d\xi}{dt}=\omega_1\xi+\frac{\omega_2+1}{r(\omega_1-\omega_2)}[(\omega_1+1)\xi+(\omega_2+1)\eta]\zeta\ ,$$
$$\frac{d\eta}{dt}=\omega_2\eta-\frac{\omega_1+1}{r(\omega_1-\omega_2)}[(\omega_1+1)\xi+(\omega_2+1)\eta]\zeta\ , \qquad (5)$$
$$\frac{d\zeta}{dt}=\omega_3\zeta+\frac{1}{r}[(\omega_1+1)\xi+(\omega_2+1)\eta](\xi+\eta)\ ,$$

in which ξ,η,ζ are related to the original variables X,Y,Z through

$$\begin{bmatrix}\xi\\ \eta\\ \zeta\end{bmatrix}=T^{-1}\begin{bmatrix}X\\ Y\\ Z\end{bmatrix}=\begin{bmatrix}\frac{r}{\omega_1-\omega_2}\left[X-\frac{\omega_2+1}{r}Y\right]\\ \frac{r}{\omega_1-\omega_2}\left[-X+\frac{\omega_2+1}{r}Y\right]\\ Z\end{bmatrix}\ . \qquad (6)$$

Let us first outline the properties of Eqs. (5) in the vicinity of the first bifurcation point $r=1$. The characteristic roots ω_1 and ω_2 behave in this range as [cf. Eq. (3)] $\omega_1\simeq(\sigma/\sigma+1)(r-1)$, $\omega_2=-(\sigma+1)+O(r-1)$. In other words the variable ξ varies on a much slower scale compared to η and ζ. By the Tikhonov theorem[3] we are thus allowed to set

$$\frac{d\eta}{dt}\simeq 0,\quad \frac{d\zeta}{dt}\simeq 0$$

and express η,ζ (hereafter denoted by η_0,ζ_0 in this approximation) as functions of ξ. This gives, to the dominant order in $r-1$

$$\zeta_0=\frac{1}{b}(\xi-\sigma\eta_0)(\xi+\eta_0)\ , \qquad (7a)$$

where η_0 satisfies the cubic equation

$$\eta_0^3+\left[1-\frac{2}{\sigma}\right]\xi\eta_0^2+\left[\left[\frac{1}{\sigma^2}-\frac{2}{\sigma}\right]\xi^2+b\frac{(\sigma+1)^2}{\sigma^2}\right]\eta_0+\frac{1}{\sigma^2}\xi^3=0\ . \qquad (7b)$$

For small ξ, a property expected to be valid near $r=1$, Eq. (7b) admits solutions such that $\eta_0\sim O(\xi^3)$, given by

$$\eta_0=-\frac{1}{b(\sigma+1)}\xi^3\ . \qquad (7c)$$

Substituting into the first equation (5) we then obtain, keeping again only the dominant terms in $r-1$,

$$\frac{d\xi}{dt}=\frac{\sigma}{\sigma+1}(r-1)\xi+\sigma(\sigma+1)\eta_0(\xi)\ ,$$

or, using (7c),

$$\frac{d\xi}{dt}=\frac{\sigma}{\sigma+1}(r-1)\xi-\frac{\sigma}{b(\sigma+1)}\xi^3\ . \qquad (8)$$

This equation turns out to be identical to the *normal form*[4] of the bifurcation equations of the Lorenz system near $r=1$. ξ is therefore the natural *order parameter* in this range. Alternatively, we may say that Eq. (8) represents the projection of the original equations into the *center manifold*[4] associated to the reference state $X_0=Y_0=Z_0=0$. Notice that, despite the linearity of the transformation T [Eq. (6)], nonlinear effects are kept through the adiabatic elimination procedure [Eqs. (7a) and (7b)]. In other words everything happens as if the transformation, when applied to ξ, is nonlinear. This procedure leads to completely consistent results near $r=1$.

In this paper we are interested in the behavior of the Lorenz system in the chaotic region for which the reduction leading to Eq. (8) is clearly inadequate, as $r\gg 1$ in this range. Nevertheless, we may still identify a part η_0 of η and a part ζ_0 of ζ corresponding to $d\eta/dt=0$, $d\zeta/dt=0$ as if Tikhonov's theorem were valid, and supplement the description by adding the deviations $\delta\eta,\delta\zeta$ around these parts. This is carried out explicitly in Sec. III.

III. FROM THE CANONICAL FORM TO THE STOCHASTIC DIFFERENTIAL EQUATION DESCRIPTION

We set

$$\eta=\eta_0(\xi)+\delta\eta(t)\ ,$$
$$\zeta=\zeta_0(\xi)+\delta\zeta(t)\ , \qquad (9)$$

where η_0,ζ_0 satisfy $d\eta_0/dt=0$, $d\zeta_0/dt=0$. From Eqs. (5) we see that ξ is connected to η_0,ζ_0 through the following generalization of Eqs. (7a) and (7b):

$$\zeta_0=\frac{1}{br}[(\omega_1+1)\xi+(\omega_2+1)\eta_0](\xi+\eta_0)\ , \qquad (10a)$$

$$\eta_0=\frac{\omega_1+1}{r\omega_2(\omega_1-\omega_2)}[(\omega_1+1)\xi+(\omega_2+1)\eta_0]\zeta_0\ . \qquad (10b)$$

From these relations we easily see that η_0 satisfies the cubic equation

$$\eta_0^3+\left[1+\frac{2(\omega_1+1)}{(\omega_2+1)}\right]\xi\eta_0^2+\left[\left[\frac{(\omega_1+1)^2}{(\omega_2+1)^2}+2\frac{(\omega_1+1)}{(\omega_2+1)}\right]\xi^2-\frac{r^2\omega_2 b(\omega_1-\omega_2)}{(\omega_1+1)(\omega_2+1)^2}\right]\eta_0+\frac{(\omega_1+1)^2}{(\omega_2+1)^2}\xi^3=0\;. \tag{10c}$$

The manifolds defined by Eqs. (10a) and (10b) enjoy some interesting properties. First, they contain the exact steady-state solution of the original system, which, in general, is not the case for the center manifold. Second, the velocity field on their intersection $\eta_0=\eta_0(\xi)$ is parallel to the direction of ξ while on traversing the manifolds it changes orientation with respect to the η and/or ζ directions. Equations (10a) and (10b) provide, therefore, the generalization of the notion of the null clines familiar from two-dimensional dynamical systems. Figures 1(a) and 1(b) depict the dependence of the η and ζ projections of the intersection curve upon ξ for $r=28$. For reference the steady-state points have been placed, and the projection of the chaotic attractor on the (η,ξ) and (ζ,ξ) planes drawn. The picture suggests that these curves represent some kind of "mean" motion. This idea will be implemented further in the sequel to this paper.

We now turn to $\delta\eta(t),\delta\zeta(t)$. Substituting Eqs. (9) into the last two of Eqs. (5) we obtain

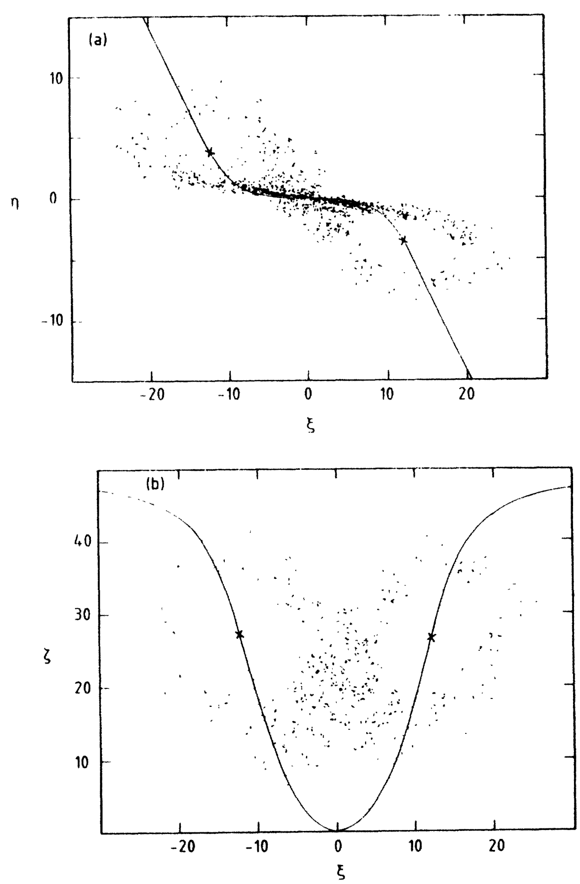

FIG. 1. The solid lines depict the dependence of η [Fig. 1(a)] and ζ [Fig. 1(b)] on ξ as given by Eqs. (10a)–(10c) for $r=28$. The dots stand for the projection of the Lorenz attractor, respectively, in the (η,ξ) and (ζ,ξ) planes. The crosses indicate the location of the nontrivial steady states.

$$\frac{d\delta\eta}{dt}=\left[\omega_2-\frac{(\omega_1+1)(\omega_2+1)}{r(\omega_1-\omega_2)}\zeta_0(\xi)\right]\delta\eta$$
$$-\frac{(\omega_1+1)}{r(\omega_1-\omega_2)}[(\omega_1+1)\xi+(\omega_2+1)\eta_0(\xi)]\delta\zeta$$
$$-\frac{(\omega_1+1)(\omega_2+1)}{r(\omega_1-\omega_2)}\delta\eta\delta\zeta\;, \tag{11a}$$

$$\frac{d\delta\zeta}{dt}=\frac{1}{r}[(\omega_1+\omega_2+2)\xi+2(\omega_2+1)]\delta\eta$$
$$-b\delta\zeta+\frac{1}{r}(\omega_2+1)\delta\eta^2\;. \tag{11b}$$

Finally, introducing (9) into the first Eq. (5) we get

$$\frac{d\xi}{dt}=\omega_1\xi+\frac{\omega_2(\omega_2+1)}{\omega_1+1}\eta_0(\xi)+F(\xi,t)\;. \tag{12a}$$

Here $\eta_0(\xi)$ is provided by Eq. (10c), $F(\xi,t)$ is given by

$$F(\xi,t)=\frac{\omega_2+1}{r(\omega_1-\omega_2)}[(\omega_1+1)\xi+(\omega_2+1)\eta_0(\xi)]\delta\zeta(t)$$
$$+\frac{(\omega_2+1)^2}{r(\omega_1-\omega_2)}\zeta_0(\xi)\delta\eta(t)$$
$$+\frac{(\omega_2+1)^2}{r(\omega_1-\omega_2)}\delta\eta(t)\delta\zeta(t)\;, \tag{12b}$$

and it is understood that $\delta\eta,\delta\zeta$ are to be drawn from Eqs. (11a) and (11b).

We are now in the position to formulate the central idea of the present work. We suggest that Eq. (12a) can be viewed as a *stochastic differential equation,* in which a deterministic evolution described by the first two terms is continuously perturbed by the noise $F(\xi,t)$. The role of the latter is to correct the Tikhonov approximation, which is no longer adequate for $r\gg 1$, by incorporating the effect of the time change of the two variables $\delta\eta,\delta\zeta$. This becomes the source of a chaotic time evolution of the variable ξ, which can be thought of as a response to the stochastic forcing function $F(\xi,t)$.

Let us give some preliminary arguments in support of this idea. As expected from the remarks made after Eqs. (10), the zeros of the deterministic part,

$$\phi(\xi_s)\equiv\omega_1\xi_s+\frac{\omega_2(\omega_2+1)}{\omega_1+1}\eta_0(\xi_s)=0$$

are the exact steady-state solutions of the original system. Significantly, linear stability analysis shows that they remain asymptotically stable in the whole range of values of r, from $r=1$ to well within the chaotic region. A useful visualization of this property, to which we also come back later, is provided by Fig. 2, where the *kinetic potential*

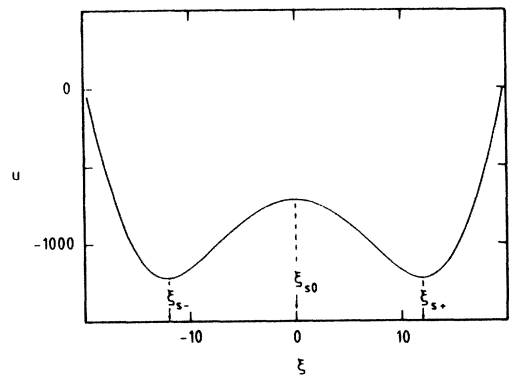

FIG. 2. Kinetic potential corresponding to the "deterministic" part of Eq. (12a) for $r=28$. ξ_{s0}: trivial steady state. $\xi_{s\pm}$: nontrivial steady states.

$$U(\xi)=-\int^{\xi}d\xi\,\phi(\xi) \tag{13}$$

is plotted against ξ for $r=28$. We observe two minima located on the nontrivial steady states and one maximum on the trivial steady-state indicating, respectively, the asymptotic stability and the instability of these states. The presence of the noise F perturbs this state of affairs by introducing a continuous hopping between the two potential wells, which will be perceived as chaos. In a sense therefore, the chaotic dynamics on the Lorenz attractor can be viewed as a *noise-induced transition*.[5] The capability of the noise to provoke such a transition is to be sought in the fact that F contains memory effects. Indeed, by expressing formally $\delta\eta,\delta\zeta$ as functions of ξ and t from (11a) and (11b) one gets an integral form displaying a memory kernel.

As long as the value of the parameter r is below the onset of chaos r_T, $F(\xi,t)$ is bound to decay to zero as $t\rightarrow\infty$. For most of these r's the decay will be accompanied by damped oscillations, a remnant of the Hopf bifurcation occurring at $r=r_P$. For $r>r_T$ however, we expect that $F(\xi,t)$ will exhibit a permanent aperiodic behavior, which will entrain the otherwise stable dynamics of ξ and bring it on the Lorenz attractor. In the next section we compare these ideas to the results of numerical simulations, which will also allow us to characterize more sharply the nature of the noise term.

IV. NUMERICAL EXPERIMENTS

The numerical simulation of the deterministic and stochastic parts of Eq. (12a) is carried out as follows. We first integrate numerically the full set of Eqs. (5). For each value of ξ and t, the quantities $\eta_0(\xi)$ and $\zeta_0(\xi)$ are determined through Eqs. (10a)–(10c). Subtracting η_0,ζ_0 from the numerically computed η and ζ [Eqs. (5)] one deduces the values of the excess quantities $\delta\eta,\delta\zeta$. Substitution of these values into (12b) gives finally the instantaneous value of the noise term, $F(\xi,t)$.

Figure 3(a) depicts the time dependence of F for $r=28$. Although the variation looks rather erratic, there exists

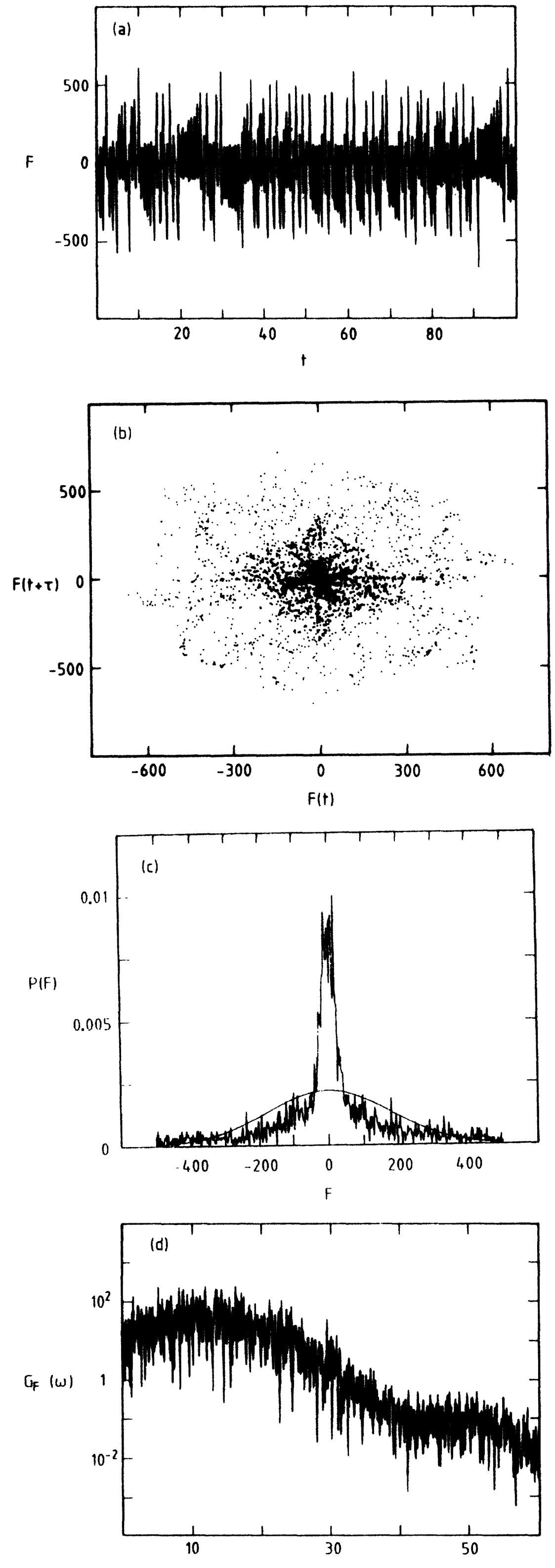

FIG. 3. (a) Time dependence of the effective noise term F, Eq. (12a). (b) Two-dimensional phase portrait constructed from the time series of Fig. 3(a) and the delayed time series corresponding to $\tau=1.5$. (c) Stationary probability distribution of F compared to a Gaussian (smooth line) having the same variance. (d) Power spectrum of F. All calculations have been performed for $r=28$.

some structure which persists when the integration is carried out for longer periods of time. The two-dimensional phase-space picture obtained from the time series using time delayed values [Fig. 3(b)] allows one to visualize the kind of structure involved. As we see the noise visits the state space in a more isotropic fashion as compared to Fig. 1(a). Moreover, the density of states is highly nonuniform with clear preference for states near the origin.

Fig. 3(c) depicts the stationary probability distribution of F. We see that this function exhibits a long tail and a very sharp peak around zero. It is thus very different from a Gaussian having the same variance [smooth line of Fig. 3(c)]. In addition [Fig. 3(d)], the power spectrum presents a maximum at some finite frequency close to the imaginary part of the eigenvalues of the linearized Eqs. (1) around the *nontrivial* steady states, and falls off rather slowly for larger frequencies. This kind of spectrum is qualitatively similar to that of a damped oscillator forced by white noise $W(t)$:

$$\frac{d^2F}{dt^2}+2\omega_0\zeta\frac{dF}{dt}+\omega_0^2F=W(t)\ . \qquad (14a)$$

Indeed, taking the Fourier transform of (14a) and denoting the variance of W by Δ^2 we obtain

$$P_F(\omega)=\frac{\Delta^2}{(\omega_0^2-\omega^2)^2+4\omega^2\omega_0^2\zeta^2}\ , \qquad (14b)$$

which presents a maximum at $\omega=\omega_0(1-2\zeta^2)^{1/2}$. Although no quantitative fit is claimed, a qualitative connection along the above lines is natural indeed, since the noise F incorporates information about the Hopf bifurcation occurring in the original equations (1) around the nontrivial steady states $X_{s\pm}=Y_{s\pm}=\pm b(r-1)$, $Z_s=r-1$. This observation should also be compared to Takeyama's analysis[6] in which the Lorenz equations are reduced to a single second-order equation for the variable X containing a part describing a damped oscillator and a part associated to memory effects.

Let us now turn to the variable ξ, Eq. (12a). Since the noise F is nonwhite, ξ is bound to be non-Markovian. Figure 4(a) depicts its time dependence. We see a very clear structure, to be contrasted from the rather erratic time dependence of F in Fig. 3(a). The stationary probability distribution of ξ is also much closer to the Gaussian [Fig. 4(b)]. Finally, its power spectrum decreases monotonously, being qualitative similar to a red noise intermediate between white noise and Ornstein-Uhlenbeck process.

Figure 4(a) also substantiates the idea expressed earlier in this paper, namely that in the representation afforded by Eq. (12a) the system performs rapid transitions between states of positive and negative ξ after spending in the vicinity of these states an amount of time which, typically, is longer than the transition time. This is reminiscent of the diffusion of a particle in the double potential well of Fig. 2, and one is tempted to inquire whether a Kramers-type theory[7] can account at least qualitatively for the observed behavior.[8] Now, for $r=28$ the variance of F turns out to be about 32 200. The variance of a white noise having the same correlation integral as F turns out to be about 30 000. The potential barrier associated to the

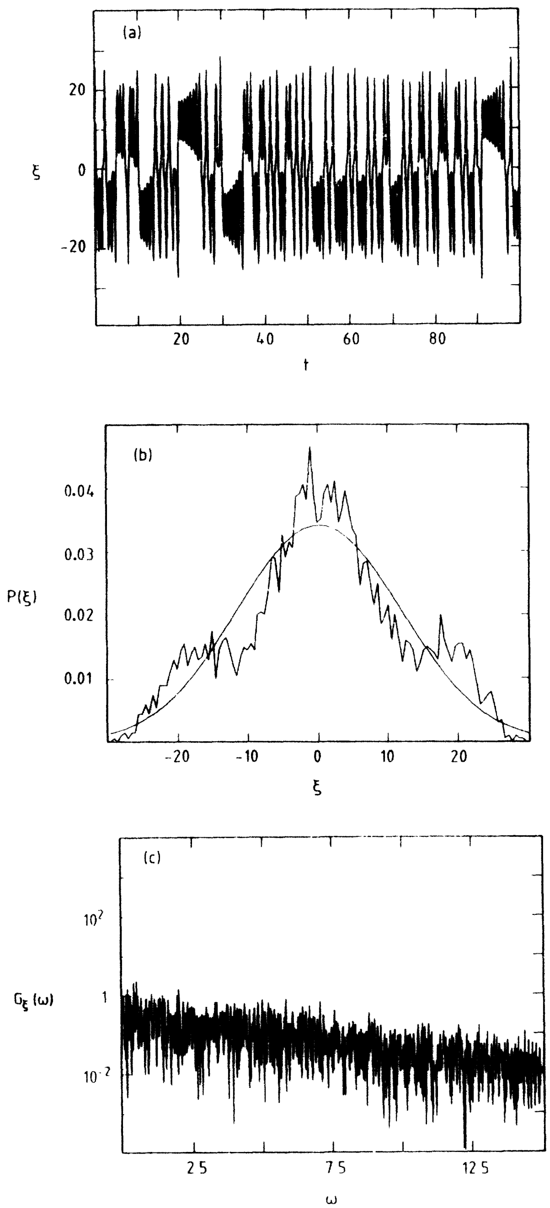

FIG. 4. (a), (b), and (c) as (a), (c), and (d) in Fig. 3 for the variable ξ.

transition, $\Delta U=U(\xi_{s0})-U(\xi_{s-})$ is about 498, whereas the second derivative of U at ξ_{s0} and ξ_{s-} is, respectively, -12 and 31. This gives a Kramers time of about 0.2, to be contrasted with the numerically computed residence time $\tau\simeq1.7$. This failure is to be attributed primarily to the large value of the variance compared to the height of the potential barrier, a property that automatically places us outside the range of validity of Kramers's theory. An alternative manifestation of it is the fact, clearly recognizable in Fig. 4(b), that the system visits with appreciable probability an extended interval of states and not only the immediate vicinity of the minima of the kinetic potential.

So far we reported the results of numerical simulations for fixed parameter values $r=28$. We now investigate the dependence of the variable ξ and of the effective noise F on r, starting from values r close to the threshold of chaotic motion, $r_T \simeq 24.06$ and going up to $r=30$. Figure 5 describes the main result. We see that the variance of ξ (dotted line in the figure) vanishes for $r<r_T$ (in agreement with the remarks made at the end of Sec. III), experiences a finite jump at $r=r_T$, and varies subsequently with r in an almost linear fashion. This result can be understood qualitatively as follows. Since the random character of ξ is connected to the jumps performed between the two deterministic solutions $\xi_{s\pm}$ across the value $\xi_{s0}=0$ under the effect of noise, one expects that

$$(\text{variance of } \xi)^{1/2} \simeq (\text{distance between } \xi_{s+} \text{ and } \xi_{s0})$$

or, using (6) and (3)

$$\langle(\delta\xi)^2\rangle^{1/2} \simeq \frac{r}{\omega_1-\omega_2}\left[X_{s+}-\frac{\omega_2+1}{r}Y_{s+}\right] = \left[\frac{r+\frac{1}{2}(\sigma-1)}{[(\sigma+1)^2+4\sigma(r-1)]^{1/2}}+\frac{1}{2}\right] \times [b(r-1)]^{1/2} . \quad (15)$$

Figure 5 (solid line) depicts this dependence for $r \geq 24.1$. We see that the agreement with the numerically computed variance is good, considering the simplicity of the arguments involved. We are therefore allowed to refer to relation (15) as an approximate *fluctuation-dissipation theorem*[9] linking the statistics of ξ to the phenomenological coefficients appearing in the equations of evolution.

Let us now turn to the dependence of the variance of F on r. In view of the structure of the Lorenz attractor and, in particular, its connection with the separatrices of the unstable state $X_0=Y_0=Z_0=0$, we expect that the strength of F should be related to the rate with which the system leaves the unstable state ξ_{s0}. According to Eq. (12a) this rate is given by the second derivative of the kinetic potential evaluated at $\xi=\xi_{s0}$. A reasonable estimate would therefore be to linearize (12a) around $\xi_{s0}=0$,

$$\frac{d\delta\xi}{dt} = U''(\xi_{s0})\delta\xi + \delta F , \quad (16)$$

or

$$\langle(\delta F)^2\rangle = [U''(\xi_{s0})]^2\langle\delta\xi^2\rangle + \left\langle\left[\frac{d\delta\xi}{dt}\right]^2\right\rangle . \quad (17a)$$

A lower bound of $\langle\delta F^2\rangle$ is therefore given by

$$\langle(\delta F)^2\rangle \geq \{U''[\xi_{s0}(r)]\}^2\langle\delta\xi^2\rangle . \quad (17b)$$

Figure 6 (solid line) depicts the right-hand side as a function of r. The result, viewed as a lower bound, is rather satisfactory. Estimating $\langle\delta\xi^2\rangle$ from Eq. (15) leads us to a relation linking the statistics of F to the phenomenological coefficients, which could therefore be viewed as an approximate fluctuation-dissipation theorem of the second kind.[10]

V. CONCLUDING REMARKS

The principal idea followed in this work is that it might be useful to cast the evolution of a multivariable system into a form displaying a limited number of privileged variables subjected to stochastic forcings. When applied to the Lorenz model this idea leads to the selection of a single variable, ξ, and of a single noise process, F. As F is neither Gaussian nor white, ξ has complex statistical properties and, in particular, it is a non-Markovian process.

Our results give some interesting insights on chaotic dynamics. Thus, we have seen that the emergence of chaos in the Lorenz model is associated to the breakdown of the time scale separation between different variables. This gives a precise meaning to the assertion frequency found in the literature that the elimination of fast variables generates noise in the subset of the slow variables. This belief, which is at the heart of the crucial separation between "diagnostic" and "prognostic" variables in meteorology, should be reconsidered carefully in the light of the results reported in the present paper.

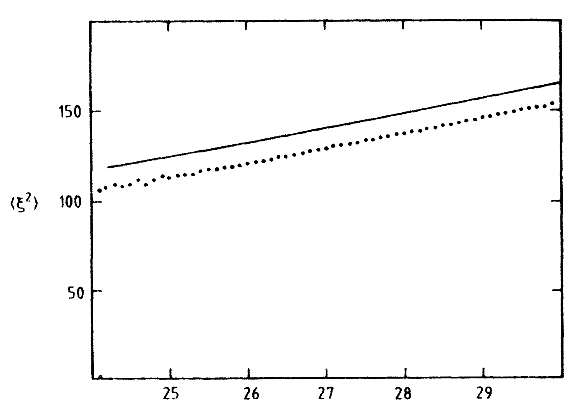

FIG. 5. Numerically computed variance of ξ (dots) and its analytical estimate from Eq. (15) (solid line).

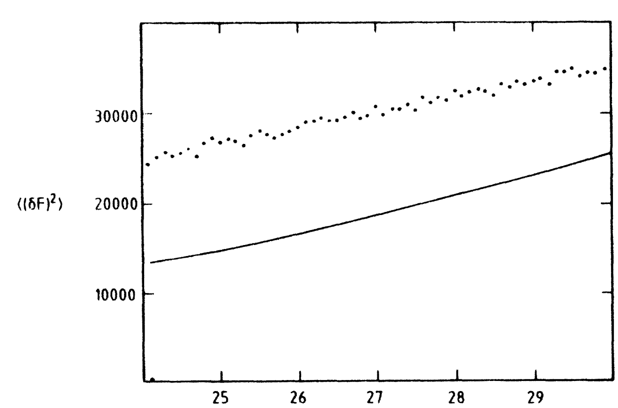

FIG. 6. Numerically computed variance of the effective noise F (dots) and the lower bound provided by Eq. (17b) (solid line).

An interesting extension of our work would be to define, in chaotic dynamics, universality classes corresponding to typical separation mechanisms between deterministic part and effective noise part. For instance, the Rössler model[11] is likely to belong to a class in which the deterministic part is associated to a limit cycle or to a homoclinic orbit,[12] contrary to the bistable system representation of the Lorenz model. When applied to the Rössler model, our procedure would lead to an order parameter related to the radius of the limit cycle. In principle, the equations replacing the auxiliary relations (10a) and (10b) need no longer yield the exact steady state solutions for this system. Such classification schemes should be useful in tackling the "inverse" problem, namely, given an experimental time series exhibiting an appreciable randomness determine the mechanisms that might be responsible for the observed behavior. For instance, many complex systems encountered in nature are probed experimentally through a single variable or through a limited number of key variables. The resulting time series (think, for instance, of meteorological or climatic data) is almost always noisy. Separating the deterministic and the "random" part of such a series is therefore a necessary task which, usually, is carried out by statistical regression methods.[13] In as much as many of the signals observed in nature result primarily from an intrinsic dynamics rather than from extrinsic noise, statistical methods cannot tell the whole story. We therefore suggest that complex time series describing natural phenomena should first be analyzed from the standpoint of the theory of dynamical systems to see whether the phenomenon of interest can be associated to a low-dimensional attractor.[14] If the answer to this question is in the affirmative, like, for instance, in recent work by the authors on the climatic system,[15] the analysis reported in the present paper should be applicable and lead to a simplified representation of the evolution, which might be useful for the purposes of modeling.

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