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Fluctuation-Dissipation Theorem and Intrinsic Stochasticity of Climate.

C. NICOLIS

Institut d'Aéronomie Spatiale de Belgique
3 Avenue Circulaire, 1180 Bruxelles, Belgium

J. P. BOON and G. NICOLIS

Faculté des Sciences de l'Université Libre de Bruxelles
Campus Plaine (C.P. 231), 1050 Bruxelles, Belgium

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Summary. — In climate dynamics the effect of internally generated fluctuations is usually described by augmenting the balance equations through the addition of *random forces*. In this paper the properties of these forces are investigated. A *fluctuation-dissipation theorem* relating their covariance matrix to the phenomenological coefficients such as eddy diffusivity is proposed. The theorem is subsequently used to identify the statistical properties of the climatic variables themselves, and thus to characterize climatic variability from the standpoint of the statistical theory of irreversible processes. Applications to a simple thermal convection problem and to a zonally averaged energy-balance model are presented; the possibility of experimental verification is discussed.

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1. – Introduction.

Stochastic analysis, incorporating statistical fluctuations in the description of climatic phenomena, is important for understanding climate variability. Ever since the classical study of Brownian motion by EINSTEIN (1905) and LANGEVIN (1908), it us customary to carry out such an analysis by adding *random forces* to the phenomenological balance equations. One thus obtains a set of expressions of the form

$$\partial_t x_\alpha = f_\alpha(\{x_\alpha\}) - \operatorname{div} J_\alpha + F_\alpha(\boldsymbol{r}, t)\,. \tag{1}$$

Here $\{x_\alpha\}$ represent climatic variables [1] like for instance temperature and convection velocity fields, f_α are the source or sink terms, J_α the flux of x_α and F_α the random force associated to x_α.

The problem of climatic variability can be formulated as finding the second- and higher-order statistics of the climatic variables. According to eq. (1), this should be possible if one could determine the statistical properties of the random forces. However, this is not an easy task in general. True, if climate dynamics were a near-equilibrium phenomenon, it would reduce to a straightforward problem since, then, the rate equations can be linearized and an important result of statistical mechanics, the *fluctuation-dissipation theorem*, provides us with explicit expressions for the covariance matrix and all further properties of F_α.

Let us outline the main steps of the procedure leading to the fluctuation-dissipation theorem in near-equilibrium systems.

i) First, one writes the entropy balance associated with eqs. (1). From irreversible thermodynamics, it is known that one has the general form [2]

$$\mathrm{d}S/\mathrm{d}t = \text{entropy flux} + \text{entropy production}\,,$$

where the entropy production, P, determines the rate of dissipation per unit time and assumes the following remarkable form:

$$P = \sum_\alpha J^{\mathrm{d}}_\alpha X_\alpha\,. \tag{2}$$

Here J^{d}_α are the fluxes associated with the various irreversible processes taking place in the system, such as heat and mass transfer, viscous dissipation, ...; and X_α represent the thermodynamic forces, or constraints, giving rise to these fluxes: temperature or concentration gradients, stress tensor, ...

ii) When the dissipative processes present in the system are identified, through eq. (2), one is able to incorporate the effect of the fluctuations. Specifically, one first observes that conservation of x_α in the absence of sources

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(1) K. HASSELMANN: *Tellus*, **28**, 473 (1976).
(2) I. PRIGOGINE: *Introduction to Thermodynamics of Irreversible Processes* (John Wiley, New York, N. Y., 1967).

or sinks requires that (see eq. (1))

$$F_\alpha(\boldsymbol{r}, t) = -\operatorname{div} j^{\mathrm{d}}_\alpha(\boldsymbol{r}, t) . \tag{3}$$

So j^{d}_α represent the *fluctuating fluxes*, which are to be added to the above-mentioned deterministic ones. Notice that eq. (3) imposes on the random forces F_α the severe restriction that only irreversible processes dissipating energy can give rise to a random force. For instance, if the flux J_α corresponds to energy or mass transfer via laminar flow, then $J^{\mathrm{d}}_\alpha = 0$ and $F_\alpha = 0$. Further examples of the important role of dissipation will be seen in the subsequent sections.

iii) Using eqs. (3), one can now write eqs. (1) in the form

$$\partial_t x_\alpha = f_\alpha(\{x_\alpha\}) - \operatorname{div} J^{\mathrm{nd}}_\alpha - \operatorname{div}(J^{\mathrm{d}}_\alpha + j^{\mathrm{d}}_\alpha) , \tag{4}$$

where J^{nd}_α are the nondissipative fluxes. One then requires that in the limit of infinitely long time, the probability distribution of the variables $\{x_\alpha\}$ reduces to the equilibrium distribution, as required by statistical mechanics [3,4].

The major conclusion emerging from such an analysis is that, near equilibrium, the fluctuating fluxes define a Gaussian white noise in time, and are fully uncorrelated in space

$$\begin{cases} \langle j_\alpha(\boldsymbol{r}, t)\rangle = 0 , \\ \langle j_\alpha(\boldsymbol{r}, t)\, j_\beta(\boldsymbol{r}', t')\rangle = 2Q_{\alpha\beta}\, \delta(\boldsymbol{r}-\boldsymbol{r}')\, \delta(t-t') . \end{cases} \tag{5}$$

This reflects the fact that fluctuations originate as localized small-scale events. On the other hand, as a rule the variables $\{x_\alpha\}$ themselves display correlations both in space and time as can be seen from the solution of the stochastic differential equations (1) or (4).

The study of climate dynamics incorporates an ensemble of highly nonlinear processes involving a large number of coupled variables which evolve under conditions far from thermodynamic equilibrium. Even at laboratory scale, there processes are poorly understood. Therefore, one would expect at first sight that the theory of fluctuations outlined above should break down. This is certainly true as far as the statistical properties of the climatic variables $\{x_\alpha\}$ are concerned. Indeed, one expects that under far from equilibrium conditions $\{x_\alpha\}$ should exhibit correlations with much longer range, in space and time than at equilibrium. On the other hand the stochastic forcing, to which the fluctuations of $\{x_\alpha\}$ constitute the response, is still likely to originate in the

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(3) L. Onsager: *Phys. Rev.*, **37**, 405 (1931); **38**, 2265 (1931).

(4) L. D. Landau and E. M. Lifshitz: *Fluid Mechanics* (Pergamon Press, Oxford, 1959), Chapt. 17.

form of *localized, small-scale events* which could not « sense » the overall constraints driving the dynamics away from equilibrium. For instance, the space-time scales of emergence of a surface temperature anomaly are short compared to the sometimes global scale of the climatic response induced by the temperature anomaly. It is, therefore, plausible that most of the properties of the random fluxes mentioned above—particularly the lack of correlations in time and space, see eq. (5)—would still hold.

The search of such a *generalized fluctuation-dissipation theorem* is the main purpose of the present work. This program should not be confused with a recently suggested fluctuation-dissipation theorem which involves directly the properties of the climatic variables rather than those of the random forces ([5,6]).

We consider in the sequel two types of problems of climatic interest, for which we make use of the statistical theory of irreversible processes to study the constraints imposed on the fluctuations. In sect. **2** we analyse thermal convection, an important part of the circulation of atmospheres and oceans, on the basis of the nonlinear Boussinesq equations ([7]) supplemented with appropriate random forces. We introduce additional simplifications, first suggested by SALTZMAN ([8]) and LORENZ ([9]), which amount to keeping only three Fourier modes of the original problem. The inclusion of fluctuations leads to an augmented set of equations reminiscent of systems studied recently by other authors ([10,11]). These authors adopted the *ad hoc* assumption that the noise strengths are identical in all three equations. Here we use a more basic and more general approach to derive explicit expressions for the statistical properties of the random forces using a generalized fluctuation-dissipation theorem. Consequences of the results on the statistics of the velocity and temperature fields are analysed in sect. **3**; these results exhibit experimentally verifiable features. In sect. **4** we carry out a similar analysis on a different problem, namely an energy balance model incorporating the surface albedo feedback. General conclusions are drawn in sect. **5**.

2. – Internal fluctuations in the Lorenz-Saltzman model for Rayleigh-Benard convection.

A full-scale analysis of climatic variability involves, per force, a very large number of variables coupled through balance equations of the type (1) or (4).

([5]) C. E. LEITH: *J. Atmos. Sci.*, **32**, 2022 (1975).
([6]) C. E. LEITH: *Nature (London,)* **276**, 352 (1978).
([7]) S. CHANDRASEKHAR: *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, London, 1961).
([8]) B. SALTZMAN: *J. Atmos. Sci.*, **19**, 329 (1962).
([9]) E. N. LORENZ: *J. Atmos. Sci.*, **20**, 130 (1963).
([10]) A. SUTERA: *J. Atmos. Sci.*, **37**, 245 (1980).
([11]) A. ZIPPELIUS and M. LÜCKE: *J. State Phys.*, **24**, 345 (1981).

The study of such a large set of equations in the presence of stochastic perturbations can only be performed numerically: this is an enormous task, and the chances that it could lead to a clear-cut identification of the major effects of the fluctuations are very slim. For this reason we focus in this section on a specific phenomenon taking part in climate dynamics, namely thermal convection. In addition to its presence in the circulation patterns of the atmosphere and of the oceans, convection is also important because it constitutes the prototype of systems capable of having unpredictable behaviour. It captures, therefore, albeit on a more modest scale, one of the essential characteristics of atmospheric dynamics and climate. In fact, the first model showing how unpredictability can arise from deterministic dynamics was constructed by LORENZ (9) in an attempt to simplify the analysis of thermal convection.

Consider a horizontal fluid layer heated from below and subject to the gravitational field. The mass density ϱ is assumed to be constant except in the term of the momentum balance equation expressing the coupling between the vertical velocity field and the density. We also neglect energy dissipation arising from dissipative stresses on the system. These are the basic assumptions contained in the Boussinesq equations for convection (7), *i.e.*

$$\varrho_0 C_p(\partial_t T + (\boldsymbol{v}\cdot\boldsymbol{\nabla})T) = \lambda\nabla^2 T + F_T\,, \tag{6a}$$

$$\varrho_0(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}) = -\boldsymbol{\nabla}p - \varrho g\boldsymbol{I}_z + \eta\nabla^2\boldsymbol{v} + \boldsymbol{F}_v \tag{6b}$$

with

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\,, \tag{6c}$$

$$\varrho \simeq \varrho(T) = \varrho_0(1-\alpha(T-T_0))\,. \tag{6d}$$

Here T, $\boldsymbol{v}$, ϱ and p are the temperature, velocity, mass density and pressure fields, respectively; ϱ_0 and T_0 denote reference values at the lower boundary of the layer, C_p is the specific heat at constant pressure and α the coefficient of thermal expansion, λ and η are the heat conductivity and shear viscosity coefficients, respectively; g is the acceleration of gravity, $\boldsymbol{I}_z$ the upward-directed unit vector and F_T and $\boldsymbol{F}_v$ the random forces describing the internally generated sources of fluctuations (see eq. (11)).

We first discuss the structure of the random forces. As pointed out in the introduction, see eq. (3), conservation of energy and momentum requires F_T and $\boldsymbol{F}_v$ to be the divergence of a vector and of a tensor field, respectively, $\boldsymbol{j}$ and $\underset{\approx}{s}$:

$$\begin{cases} F_T(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{j}(\boldsymbol{r},t)\,,\\ \boldsymbol{F}_v(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\underset{\approx}{s}(\boldsymbol{r},t)\,. \end{cases} \tag{7}$$

Moreover, near thermodynamic equilibrium, the properties of $\boldsymbol{j}$ and $\underset{\approx}{s}$ are essentially given by eqs. (5). Statistical mechanics allows one to identify the structure of the matrix in eq. (5), and one has [4]

$$(8)\qquad \begin{cases} \langle \boldsymbol{j}(\boldsymbol{r}, t)\rangle = 0\,, \\ \langle \boldsymbol{j}_k(\boldsymbol{r}, t)\, j_l(\boldsymbol{r}', t')\rangle = 2k_{\mathrm{B}}\overline{T}^2\lambda\delta_{kl}\delta(\boldsymbol{r}-\boldsymbol{r}')\,\delta(t-t') \end{cases}$$

and

$$(9)\qquad \begin{cases} \langle \underset{\approx}{s}(\boldsymbol{r}, t)\rangle = 0\,, \\ \langle s_{ik}(\boldsymbol{r}, t)\, s_{lm}(\boldsymbol{r}', t')\rangle = 2k_{\mathrm{B}}\overline{T}\eta\Gamma_{iklm}\,, \\ \Gamma_{iklm} = (\delta_{il}\delta_{km} + \delta_{im}\delta_{kl} - \tfrac{2}{3}\delta_{ik}\delta_{lm})\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,\delta(t-t')\,. \end{cases}$$

Here $\overline{T}$ is the temperature of the reference state around which fluctuations are generated, k_{B} is Boltzmann's constant, δ is the Kronecker symbol, and the brackets denote an equilibrium ensemble average. The presence of delta-functions in both space and time reflects the fact that fluctuations are generated by small scale, localized events without memory. Note, however, that the random forces, F_T and $\boldsymbol{F}_v$ themselves are *not* delta-correlated in space, while they exhibit Gaussian white noise in time. As will be seen below, despite the locality of the random fluxes, the state variables themselves (T and $\boldsymbol{v}$) will exhibit highly nonlocal correlations in both space and time.

In the original hydrodynamic fluctuation theory of Landau and Lifshitz, the reference state is thermodynamic equilibrium. Here we are interested in behaviour far from equilibrium and, particularly, in the fact that eqs. (6) present a sequence of bifurcations from regular to periodic convection and ultimately to chaotic convection (for a survey see, *e.g.*, ref. [12]). As explained in the introduction, even under these circumstances, we still expect fluctuation sources to arise from localized, small-scale events. So (see also ref. [13]), we assume that eqs. (8) and (9) can be extended to the nonlinear domain of irreversible processes, with a nonequilibrium reference state, taken here as the laminar convection state (*i.e.* the state just beyond the first bifurcation point).

Introducing dimensionless quantities into eqs. (6*a*), (6*b*) and (7), one obtains

$$(10a)\qquad \partial_{t^*}\theta^* = Rw^* - (\boldsymbol{v}^*\cdot\nabla^*)\,\theta^* + \nabla^{*2}\theta^* - \boldsymbol{\nabla}^*\cdot\boldsymbol{j}^*\,,$$

$$(10b)\qquad \partial_{t^*}\boldsymbol{v}^* = P(\boldsymbol{\nabla}^*p^* + \nabla^{*2}\boldsymbol{v}^* + \theta^*\boldsymbol{I}_z) - (\boldsymbol{v}^*\cdot\boldsymbol{\nabla}^*)\,\boldsymbol{v}^* - \boldsymbol{\nabla}^*\cdot\underset{\approx}{s}^*\,.$$

Here θ^* is the dimensionless measure of the temperature deviation from the

(12) H. L. SWINNEY and J. P. GOLLUB (Editors): *Hydrodynamic Instabilities and Transition to Turbulence* (Springer-Verlag, Berlin, 1981).
(13) C. NICOLIS: *On a new fluctuation-dissipation theorem in climate dynamics*, in *New Perspectives in Climate Modelling*, edited by A. BERGER and C. NICOLIS (Reidel, Dordrecht, 1983).

linear profile, $T_0 - \beta z$. The asterisks indicate that both dependent variables $(\theta^*, w^*, \boldsymbol{v}^*)$ and independent variables $(\boldsymbol{r}^*, t^*)$ are dimensionless; w^* is the vertical component of $\boldsymbol{v}^*$; R and P are the Rayleigh and Prandtl numbers, respectively,

$$R = \alpha\beta g d^4(\nu\varkappa)^{-1}, \qquad P = \nu\varkappa^{-1}, \tag{11}$$

where ν is the kinematic viscosity ($\nu = \eta\varrho_0^{-1}$), $\varkappa$ the thermal diffusivity ($\varkappa = \lambda(\varrho_0 C_p)^{-1}$), d the height of the layer and β the average temperature gradient across the fluid layer. $\boldsymbol{j}^*$ and $\underset{\approx}{s}^*$ are the scaled noise fields given by

$$\begin{cases} \boldsymbol{j}^* = \boldsymbol{j}\alpha g d^4(\nu\varkappa^2\varrho_0 C_p)^{-1}, \\ \underset{\approx}{s}^* = \underset{\approx}{s}\,\alpha^2(\varrho_0\varkappa^2)^{-1}. \end{cases} \tag{12}$$

Equations (10a), (10b) are difficult to handle because of the presence of the highly nonlinear terms $(\boldsymbol{v}^*\cdot\boldsymbol{\nabla}^*)\theta^*$ and $(\boldsymbol{v}^*\cdot\boldsymbol{\nabla}^*)\boldsymbol{v}^*$, respectively; so we proceed by performing the following operations:

i) We neglect the nonlinear convection terms $((\boldsymbol{v}^*\cdot\boldsymbol{\nabla}^*)\boldsymbol{v}^*)$ in eq. (10b), which amounts to assuming that the Prandtl number P is large (note that certain bifurcation schemes may be excluded from the analysis by this assumption).

ii) We apply twice the curl operator on eq. (10) and project the result on the z-axis. The pressure term thus vanishes and one is left with an equation which involves only the vertical component the velocity w^* and the excess temperature.

iii) We focus on a particular type of convective state, characterized by a two-dimensional roll pattern. As a result all variables now depend only on x^* and z^*.

iv) We account for the large extension of the system along the horizontal direction by imposing periodic boundary conditions, and we consider that the layer is stress-free at the two vertical boundaries, so that [7]

$$\begin{cases} \theta^*(0) = \theta^*(1) = 0, \\ w^*(0) = w^*(1) = 0, \\ \partial^2 w^*/\partial z^{*2} = 0 \quad @ \quad z^* = 0, \quad z^* = 1. \end{cases} \tag{13}$$

Consequently θ^* and w^* can be Fourier expanded as

$$\begin{cases} \theta^*(x^*, z^*, t^*) = \sum\limits_{mn} \theta^*_{mn}(t^*) \exp[imk^*x^*] \sin(n\pi z^*), \\ w^*(x^*, z^*, t^*) = \sum\limits_{mn} w^*_{mn}(t^*) \exp[imk^*x^*] \sin(n\pi z^*). \end{cases} \tag{14}$$

k^* represents the characteristic wave number of the convection cells (k^* will be considered as a known quantity, determined by the geometrical conditions).

Keeping only the first nontrivial modes in the analysis which amount here to w_{11}^*, θ_{02}^* and θ_{11}^*, one obtains after some rather heavy algebra (see the appendix)

$$\left\{\begin{aligned} \dot{X} &= -\sigma X + \sigma Y + \phi_X\,, \\ \dot{Y} &= rX - Y - XZ + \phi_Y\,, \\ \dot{Z} &= -bZ + XY + \phi_Z\,. \end{aligned}\right. \tag{15}$$

Here X, Y, Z correspond to w_{11}^*, θ_{11}^* and θ_{02}^*, respectively, by the scaling indicated in eq. (A.3); σ is the Prandtl number, r the ratio of the Rayleigh number to its critical value and ϕ_X, ϕ_Y, ϕ_Z are effective random forces defining a white noise, with variance given in eqs. (A.6)-(A.8)

$$\langle\phi_i(t)\phi_j(t')\rangle = q_i^2\delta_{ij}^{\mathrm{Kr}}\delta(t-t')\,, \qquad i, j = X, Y, Z. \tag{16}$$

Equations (15) are the Lorenz-Saltzman equations including now internal-fluctuation terms. It is known that, although the Lorenz-Saltzman model does not describe quantitatively the behavior of experimental convection at very large R, it contains the essential qualitative features. Most important from our standpoint is that one is in a position to fully express the constraints imposed on the effective random forces by the statistical theory of irreversible processes. The procedure leading to eqs. (16) is outlined in the appendix.

It should be stressed that the strengths q_X^2, q_Y^2, q_Z^2 are *unequal*. This will not affect qualitatively the linear-response properties described in the next section, but is likely to introduce new features in the nonlinear domain. Indeed, in terms of stochastic theory the system does not satisfy the *potential conditions*, because of the difference in the strength of the random forces. As a result, the role of fluctuations will not necessarily be restricted to inducing aperiodic flip-flop between various simultaneously stable attractors (stable stationary, periodic or chaotic states) predicted by the deterministic equations. In fact, fluctuations may now modify the conditions of appearance of these attractors, by shifting the threshold values of the various parameters like the Rayleigh number, as well as their location and structure in phase space.

3. – Linear response theory.

Let (X_s, Y_s, Z_s) be a steady-state solution of eqs. (15) in the absence of fluctuations. When $r < 1$, there is only one such solution that is asymptotically stable [9,14]

$$X_s = Y_s = Z_s = 0 . \tag{17}$$

According to the definitions of w_{11}^*, θ_{11}^* and θ_{02}^*, the steady-state solution given by (17) characterizes a purely conductive state (no convection). For $r > 1$, (17) remains a mathematical solution, but the corresponding state has now lost its stability. Two new stable branches of solutions emerge in this range which bifurcate from (17) at the critical value $r = 1$ and are given by

$$X_s = Y_s = \pm(b\varepsilon)^{\frac{1}{2}} , \qquad Z_s = \varepsilon , \qquad \varepsilon = r - 1 . \tag{18}$$

The steady-state as defined by solutions (18) is the convection state which in turn loses its stability at a second bifurcation point $r = r_T$ [9]

$$r_T = \sigma(\sigma + b + 3)(\sigma - b - 1)^{-1} \tag{19}$$

with the condition that σ be sufficiently large, $\sigma > b - 1$. At this point, the bifurcation leads to undamped oscillatory solutions with frequency

$$\Omega_T = \pm(b(\sigma + r_T))^{\frac{1}{2}} . \tag{20}$$

These solutions bifurcate subcritically for $r \leqslant r_T$ and are, therefore, unstable [15,16]. Figure 1 summarizes the above-described bifurcation phenomena. The second bifurcation point marks the transition towards the onset of complex dynamics

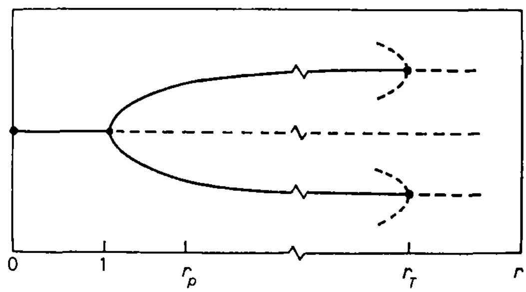

Fig. 1. – Bifurcation scheme of solutions; r denotes the ratio of the Rayleigh number to its critical value at the first bifurcation point.

([14]) C. Sparrow: *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors* (Springer-Verlag, New York, N. Y., 1983).

([15]) P. C. Martin: in *Proceedings of the International Conference on Statistical Mechanics* (North-Holland, Amsterdam, 1975).

([16]) J. Marsden and M. McCracken: *The Hopf Bifurcation and its Applications* (Springer-Verlag, Berlin, 1976).

in the form of chaotic behaviour; the corresponding set of solutions (for $r > r_T$) that appears at finite distance from the steady state defined by (18) are asymptotically stable. Note also that oscillatory behaviour appears already below r_T; indeed when r reaches a value r_p, with $1 < r_p < r_T$, damped oscillations arise whose damping rate decreases to zero, as r tends to r_T, where their frequency reaches the value Ω_T. Note however, that no bifurcation occurs at $r = r_p$.

We now investigate the behaviour of the system in the vicinity of the two bifurcation points. More specifically we consider the statistical properties of the temperature and velocity fields around $r = 1$ and $r = r_T$. We first linearize eqs. (15) around the steady-state solution (X_s, Y_s, Z_s). After time-Fourier transformation of the quantities X, Y, Z and ϕ_X, ϕ_Y, ϕ_Z, *i.e.* using

$$X(t) = \int_{-\infty}^{+\infty} \mathrm{d}\omega \exp[-i\omega t]\, \tilde{X}_\omega$$

and similarly for the other quantities, we obtain from eqs. (15) and (16)

$$\text{(21)} \qquad \begin{cases} (\sigma - i\omega)\tilde{X}_\omega - \sigma\tilde{Y}_\omega = \tilde{\phi}_{X\omega}\,, \\ (Z_s - r)\tilde{X}_\omega + (1 - i\omega)\,\tilde{Y}_\omega + X_s\tilde{Z}_\omega = \phi_{Y\omega}\,, \\ -Y_s\tilde{X}_\omega - X_s\tilde{Y}_\omega + (b - i\omega)\tilde{Z}_\omega = \phi_{Z\omega}\,. \end{cases}$$

Equations (21) constitute the set that we now study for variability around the trivial state (17) and the nontrivial state (18) successively.

For the sake of conciseness, we restrict the discussion, without loss of generality, to the behaviour of the thermal mode $\tilde{Y}_\omega$.

$r < 1$: *Variability around the conduction state*, $X_s = Y_s = Z_s = 0$. From eqs. (20), one finds that the amplitude is given by

$$\text{(22)} \qquad \tilde{Y}_\omega = [r\tilde{\phi}_{X\omega} + (\sigma - i\omega)\tilde{\phi}_{Y\omega}]/P_0(-i\omega)\,,$$

where P_0 is the characteristic polynomial

$$\text{(23)} \qquad P_0(z) = z^2 + (\sigma + 1)z - \sigma\varepsilon\,, \qquad z = -i\omega,$$

with $\varepsilon = r - 1 < 0$. The explicit expressions for the two modes, *i.e.* for z_+ and z_-, are easily computed from (23). More interesting is the behaviour in the vicinity of the bifurcation point. Close to $r = 1$, for small values of ε, one obtains

$$\text{(24)} \qquad z_- \simeq -\sigma\varepsilon(\sigma + 1)^{-1}, \qquad z_+ \simeq (\sigma + 1) + \sigma\varepsilon(\sigma + 1)^{-1}.$$

We observe that the mode corresponding to z_- exhibits a slower and slower decay rate as r approaches unity ($z_- \to 0$ as $\varepsilon \to 0$). This is the analogue of *critical slowing down* in equilibrium phase transitions. On the other hand, the mode corresponding to z_+ keeps a finite decay rate, which increases to the value $\sigma + 1$ at $r = 1$. In the literature of fluid dynamics z_- and z_+ are referred to as the thermal and the vorticity mode, since they are dominated, respectively, by heat conduction and viscous effects ([17]).

An interesting quantity from the experimental view point as a useful index of variability is the power spectrum (see, *e.g.*, ref. ([18])). Noting that

$$\langle \tilde{\phi}_{X\omega} \tilde{\phi}_{X\omega'} \rangle = \frac{1}{\pi} q_X^2 \delta(\omega + \omega') \tag{25}$$

and similarly for $\tilde{\phi}_{Y\omega}$ and $\tilde{\phi}_{Z\omega}$, we obtain from the ensemble averaged complex conjugates product of (22)

$$C_{YY}^{(0)}(\omega) = \langle \tilde{Y}_\omega \tilde{Y}_{\omega'} \rangle = \gamma^{(0)}(\omega)\,\delta(\omega + \omega') \tag{26}$$

with

$$\gamma^{(0)}(\omega) = \frac{1}{\pi} \frac{r^2 q_X^2 + (\omega^2 + \sigma^2) q_Y^2}{(\omega^2 + z_+^2)(\omega^2 + z_-^2)}, \tag{27}$$

where $z_\pm = \frac{1}{2}(\sigma + 1) \pm \frac{1}{2}[(\sigma + 1)^2 + 4\sigma\varepsilon]^{\frac{1}{2}}$ are the roots of the polynomial $P_0(z)$, eq. (23). The spectrum $\gamma^{(0)}(\omega)$ is a single-peaked curve centred around $\omega = 0$. The peak intensity reads

$$\gamma^{(0)}(\omega = 0) = \frac{1}{\pi} (r^2 q_X^2 + \sigma^2 q_Y^2)(\sigma^2 \varepsilon^2)^{-1}, \tag{28}$$

which depends on the Prandtl number (σ), the noise strength factors (q_X^2, q_Y^2) and the value of the Rayleigh number ($|\varepsilon| = (R_c - R)/R_c$). On the approach to the first bifurcation point ($R = R_c$) from below ($r < 1$), the peak intensity diverges as ε^{-2}. However, the peak intensity is not an easily accessible quantity from the experimental point of view. In this respect a preferable quantity is the integrated intensity

$$\gamma^{(0)} = \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \gamma^{(0)}(\omega), \tag{29}$$

([17]) J. P. Boon: *Hydrodynamic instability: structure and chaos*, in *Scattering Techniques Applied to Supramolecular and Nonequilibrium Systems* (Plenum Press, New York, N. Y., 1981).
([18]) J. P. Boon and S. Yip: *Molecular Hydrodynamics*, (McGraw-Hill, New York, N. Y., 1980).

which is easily computed here from (27). One finds

$$\gamma^{(0)} = q_Y^2 + r\,\frac{rq_X^2 + \sigma q_Y^2}{\varepsilon\sigma(\sigma+1)}\,. \tag{30}$$

So the integrated spectrum has a regular part (the equilibrium value of $\gamma^{(0)}$ at $r = 0$) and a critical part that diverges like ε^{-1} as the first bifurcation point is approached. The critical behavior of the integrated spectrum is a consequence of the critical slowing-down of the thermal diffusivity mode z_-, see (24).

$r > 1$: *Variability around the convective state,* $X_s = Y_s = \pm\sqrt{b\varepsilon}$, $Z_s = \varepsilon$. Inserting the steady-state expressions (18) into the set of eqs. (21) and solving the latter for the amplitude of the thermal mode, one obtains

$$\tilde{Y}_\omega = \Big[\big(b(2-r) - i\omega\big)\tilde{\phi}_{X\omega} + (b - i\omega)(\sigma - i\omega)\tilde{\phi}_{Y\omega} - \sqrt{b\varepsilon}\,(\sigma - i\omega)\phi_{Z\omega}\Big]\Big/P_+(-i\omega) \tag{31}$$

with the characteristic polynomial

$$P_+(z) = z^3 + (\sigma + b + 1)z^2 + b(\sigma + r)z + 2b\varepsilon\,, \qquad z = -i\omega, \tag{32}$$

where $\varepsilon = r - 1 > 0$. There are now three modes that emerge as the roots of the characteristic equation $P_+(z) = 0$. When r does not exceed the value r_p, the three roots are real. Although the cubic can be solved formally the explicit expressions of the roots are not very enlightening as to their physical content. In this respect a computation of the roots to first order in ε is valuable in order to investigate the mode behaviour in the vicinity of $r = 1$ when the bifurcation point is approached from above. One finds

$$z_0 \simeq \frac{2\sigma\varepsilon}{\sigma+1}, \qquad z_1 \simeq (\sigma+1) + \frac{b(\sigma-1)\varepsilon}{(\sigma+1)(\sigma+1-b)}, \qquad z_2 \simeq b - \frac{(2\sigma-b)\varepsilon}{\sigma+1-b}, \tag{33}$$

which shows that the thermal mode (z_0) exhibits critical slowing-down ($z_0 \to 0$ when $\varepsilon \to 0$) according to the same power law as for the approach to $r = 1$ from below, eq. (24), but with an amplitude factor twice as large, due to the emergence of a third mode when $r > 1$.

The power spectrum for the thermal mode is computed along the same lines as in the previous case ($r < 1$) to yield

$$C_{YY}^{(+)} \equiv \langle \tilde{Y}_\omega \tilde{Y}_{\omega'} \rangle = \gamma^{(+)}(\omega)\,\delta(\omega + \omega')\,, \tag{34}$$

where

$$\gamma^{(+)}(\omega) = \frac{1}{\pi}\,\frac{\big(b^2(2-r)^2 + \omega^2\big)q_X^2 + (\sigma^2 + \omega^2)(b^2 + \omega^2)q_Y^2 + b\varepsilon(\sigma^2 + \omega^2)q_Z^2}{(\omega^2 + z_0^2)(\omega^2 + z_1^2)(\omega^2 + z_2^2)}, \tag{35}$$

with

$$(36)\qquad \begin{cases} z_0 + z_1 + z_2 = \sigma + b + 1 \,, \\ z_0 z_1 + z_1 z_2 + z_2 z_0 = b(\sigma + r) \,, \\ z_0 z_1 z_2 = 2\sigma b \varepsilon \,. \end{cases}$$

In the range $1 < r < r_p$, the spectrum $\gamma^{(+)}(\omega)$ is a single-peaked function of ω centred around $\omega = 0$, thus exhibiting the same qualitative structure as below $r = 1$. The peak intensity is given by

$$(37)\qquad \gamma^{(+)}(\omega = 0) = \frac{1}{\pi}\left(\frac{(r-2)^2 q_X^2 + \sigma^2 q_Y^2}{4\sigma^2 \varepsilon^2} + \frac{q_Z^2}{4b\varepsilon}\right).$$

On the approach to the bifurcation point $r = 1$ from above, the strongest divergence causes the peak intensity to blow up like ε^{-2}, in the same way as when the bifurcation point is approached from below ($r < 1$). Note also the difference in the amplitude factor (compare (37) with (28)).

Here again a most interesting quantity, from the theoretical view point as well as from the experimental view point, is the integrated intensity, which is obtained by frequency integration of (35) to yield

$$(38)\qquad \gamma^{(+)} = \frac{q_X^2 + [\sigma^2 + b(\sigma + b + r)]q_Y^2 + \varepsilon b q_Z^2}{(\sigma + b + 1) b(\sigma + r)}\left(1 + \frac{\sigma - b - 1}{2\sigma}\frac{\varepsilon}{\delta}\right) + \\ + \frac{b(r-2)^2 q_X^2 + b\sigma^2 q_Y^2 + \varepsilon q_Z^2}{2\sigma b(\sigma + r)}\left(\frac{1}{\varepsilon} + \frac{2\sigma}{\sigma - b - 1}\frac{1}{\delta}\right)$$

with $\delta = r_T - r$, where r_T, the value of r at the second bifurcation point, is given by (19).

The integrated intensity $\gamma^{(+)}$ exhibits interesting features: it contains regular and critical parts. There is a critical part $\propto \varepsilon^{-1}$ that diverges when r approaches the first bifurcation point and another critical part $\propto \delta^{-1}$ which blows up on the approach to the second bifurcation point ($r \to r_T$). The first divergence arises from the behaviour of the thermal mode which becomes critical when $r \to 1$ (see z_0 (32)); the critical behaviour in the vicinity of the second bifurcation point has its origin in the vanishing of the damping of the conjugate modes that emerge from $r = r_p$. At this point oscillatory behaviour sets in as two roots of the characteristic equation $P_+(z) = 0$ become complex conjugates. Correspondingly the power spectrum $\gamma^{(+)}(\omega)$ contains one central line and two side peaks when r exceeds the value r_p. When r increases, the central peak broadens and its line width reaches the value $\Delta\omega = \sigma + b + 1$ at $r = r_T$; simultaneously the satellite peaks narrow in width and are symmetrically shifted away from the central peak. At $r = r_T$, the central line appears as a broad background with low peak intensity

$$\gamma_T^{(+)}(\omega = 0) = \frac{1}{4\pi}\left[\left(\frac{r_T - 2}{r_T - 1}\right)^2 \frac{q_X^2}{\sigma^2} + \frac{q_Y^2}{(r_T - 1)^2} + \frac{q_Z^2}{b(r_T - 1)}\right] \tag{39a}$$

and the satellite peaks become δ-functions located at $\omega = \pm\Omega_T$, eq. (20), as indicated by the δ^{-2} behaviour of the spectrum intensity

$$\begin{cases} \gamma^{(+)}(\omega = \pm\Omega_T) = A/B\,, \\ A = [b(r-2)^2 + \sigma + r_T]\, q_X^2 + (\sigma + b + r_T)[\sigma^2 + b(\sigma + r_T)]\, q_Y^2 + \\ \qquad + \varepsilon[\sigma^2 + b(\sigma + r_T)]\, q_Z^2\,, \\ B = \pi b[4\sigma^2 + b(\sigma + r_T)]\, \delta^2\,. \end{cases} \tag{39b}$$

The changes in the power spectrum between the first and the second bifurcation are illustrated in fig. 2. Most interesting is the appearance of side peaks when r becomes larger than r_p. These shifted spectral lines are the man-

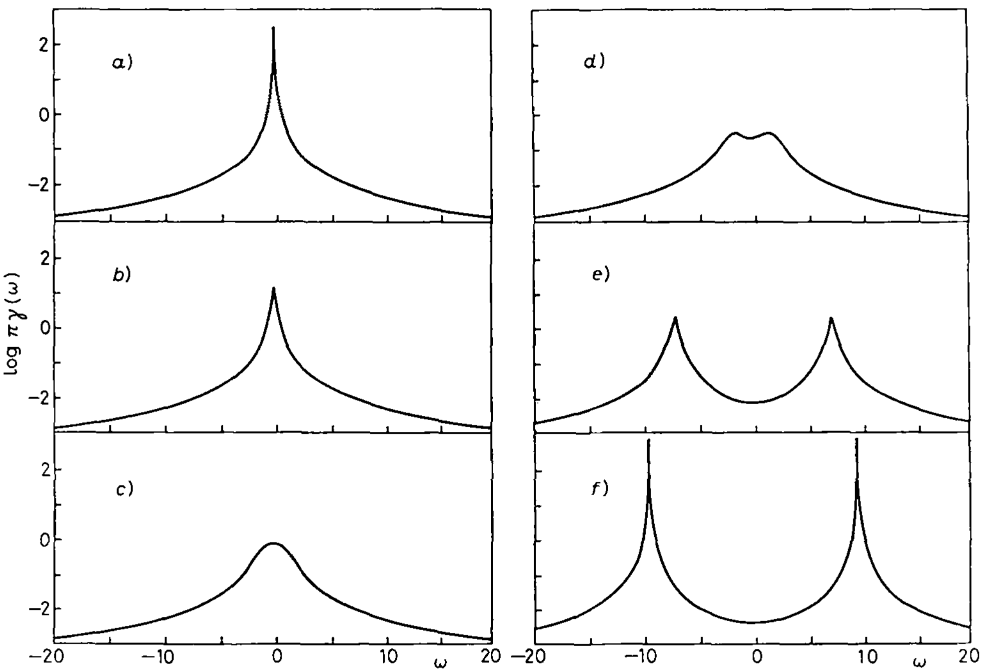

Fig. 2. – Power spectrum of thermal mode between first and second bifurcation points, $r = 1$ and $r = r_T$, respectively. The case illustrated here is for water at 20 °C; the values of the parameters are: $b = 8/3$, $\sigma = 27/4$, $q_X = 0.12$, $q_Y = 0.75$, $q_Z = 1$. With these values, periodic behaviour sets in at r_p with $1.5 < r_p < 2$, and chaotic behaviour appears at $r_T = 27.1824\ldots$ with $\Omega_T = 9.5124\ldots$. Note the logarithmic scale for $\gamma(\omega)$. *a*) $r = 1.02$, *b*) $r = 1.1$, *c*) $r = 1.5$, *d*) $r = 2$, *e*) $r = 15$, *f*) $r = 27$.

ifestation of the damped oscillations which arise from mode coupling and mark the onset of a regime with propagating waves in the system. Note that propagating modes also exist in the region $r < 0$, when $|r|$ exceeds a certain value $|r^*|$, and have been observed experimentally ([19]). So it would be interesting to probe experimentally the region $r > r_p$ where propagating modes with a different nature are present in the convective regime. Note that the accessible range of Rayleigh number where experiments could be performed to detect these propagating modes does not extend all the way to r_T because of the inverted bifurcation (see fig. 1) as a consequence of which the narrowing of the satellites will not be observable in the vicinity of the second bifurcation. Experimental measurements of the detailed structure of the power spectrum (or even of the integrated intensity) would also provide a valuable test of validity for the generalized fluctuation-dissipation theorem presented in sect. **2.**

Note that the power spectrum of the velocity fluctuations, C_{XX}, can be computed along the same lines as described above. Interesting information can be obtained from the peak intensity of the power spectrum C_{XX} $(\omega = 0)$. Indeed the diffusion coefficient D of suspended particles can be expressed by means of Faxén's theorem (see ref. ([20,21])) in terms of the fluid velocity field fluctuations, through their power spectrum at zero frequency. Fluid-dynamics measurements are conveniently performed by laser light spectroscopy to probe fluid motion via the light scattered by suspended (dust or seeded) particles. Thus enhancement of hydrodynamic fluctuations near bifurcation points can be probed by measuring the diffusion coefficient of such suspensions, *e.g.*, it was shown that D diverges like $\varepsilon^{-\mu}$ when $r \to 1$ for $r < 1$ ($\mu = \frac{2}{3}$, ref. ([22]); $\mu = 2$, ref. ([23])). This result can now be extended to investigate the vicinity of the first transition when the bifurcation point is approached from above. The present theory predicts that the diffusion coefficient behaves critically with the same power law ($\varepsilon^{-\mu}$) for $r > 1$ as well as for $r < 1$, but shows no divergence on the approach to the second bifurcation point as C_{XX} $(\omega = 0)$ is well behaved at $r = r_T$. (An expression similar to (39*a*) is obtained for the power spectrum (at $\omega = 0$) of the velocity field fluctuations).

4. – Internal fluctuations in a zonally-averaged energy-balance model.

In this section we briefly analyse the effect of internal fluctuations for a simple energy-balance model incorporating the surface-albedo feedback and

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(20) R. Zwanzig: *J. Res. Nat. Bur. Stand., Sect. B*, **68**, 143 (1964).
(21) P. Mazur and D. Bedeaux: *Physica*, **76**, 235 (1974).
(22) H. N. K. Lekkerkerker and J. P. Boon: *Phys. Rev. Lett.*, **36**, 724 (1976).
(23) J. B. Lastovka and J. P. Boon: *Phys. Rev. A*, **14**, 1583 (1976).

the energy transfer in the meridional direction [24,25]. The only variable retained in the description is the surface temperature T. Choosing a spherical co-ordinate system on the Earth's surface and denoting by s, a and χ the sine of the latitude, the Earth's radius, and the effective heat diffusion coefficient, respectively, we write the energy balance equation in the form

$$\mathscr{C}\partial_t T(s,t) = f_T(T,s) + \chi\partial_s(1-s^2)\,\partial_s T + \mathscr{F}_T(s,t)\,. \tag{40}$$

Here $\mathscr{C}$ is the heat capacity of a column with a unit cross-section and a height of the order of the depth of the mixed layer; f_T is the radiation budget (difference between incident solar flux and outgoing infra-red radiation); and $\mathscr{F}_T$ the effective random force. According to eq. (3)

$$\mathscr{F}_T(s,t) = -\nabla\cdot\boldsymbol{J}(s,t) \tag{41a}$$

or, in the chosen spherical co-ordinate system,

$$\mathscr{F}_T(s,t) = -a^{-1}\partial_s(1-s^2)^{\frac{1}{2}}J(s,t)\,. \tag{41b}$$

The random flux correlation function is obtained from eq. (4) by switching to spherical co-ordinates to yield

$$\langle J(s,t)J(s',t')\rangle = q^2\delta(s-s')\delta(t-t')\,, \tag{42}$$

where q^2 is a quantity similar to the strength factors in eqs. (15) and (16). It follows that

$$\begin{cases} \langle\mathscr{F}_T(s,t)\mathscr{F}_T(s',t')\rangle = \varphi_T\delta(t-t')\,, \\ \varphi_T = q^2a^{-2}\partial_s(1-s^2)^{\frac{1}{2}}\partial_{s'}(1-s'^2)^{\frac{1}{2}}\delta(s-s')\,, \end{cases} \tag{43}$$

i.e. the random forces themselves ($\mathscr{F}_T$) are not delta-correlated in space. Notice that, despite the highly singular character of the random force, the thermal response will behave in a perfectly regular fashion.

We proceed by expanding both the temperature and random force fields in Legendre polynomial series

$$T = \sum_n T_n(t)P_n(s)\,,\qquad \mathscr{F}_T = \sum_n \mathscr{F}_n(t)P_n(s)\,. \tag{44}$$

Inserting (44) in eq. (43), multiplying by $P_m(s)P_l(s')$ and integrating over s

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([24]) G. R. NORTH: *J. Atmos. Sci.*, **32**, 2033 (1975).
([25]) G. R. NORTH, F. R. CAHALAN and J. A. COAKLEY: *Rev. Geophys. Space Phys.*, **19**, 91 (1981).

and s', we obtain, assuming symmetric hemispheres,

$$\langle \mathscr{F}_m(t)\mathscr{F}_l(t')\rangle = q_m^2 \delta_{ml}\delta(t-t') \tag{45a}$$

with

$$q_m^2 = \tfrac{1}{2}(2m+1)m(m+1)q^2/a^2 . \tag{45b}$$

Interesting conclusions can be drawn from these results. First, the random forces associated with different Legendre modes are uncorrelated. Second, there is no random force acting directly on the equation for the globally averaged temperature T_0, because it follows from eq. (45b) that $q_0 = 0$. Still, T_0 keeps a stochastic character through its coupling with the modes T_2, etc., which are affected directly by a random force. Third, the importance of the fluctuations increases rapidly with the order of the Legendre mode. Inasmuch as higher-order Legendre modes represent localized disturbances, we, therefore, see that the strength of fluctuations depends on their space scale. This is reminiscent of the phenomena of nucleation frequently encountered in phase transitions ([26]).

To obtain the statistical properties of the temperature field from those of the random force, one would have to solve the stochastic differential equations (41a) with (41b). This is a difficult task because of the nonlinearities involved in the radiative term f_T. One can, however, obtain explicit results by truncating the infinite set of coupled equations for $T_n(t)$ keeping only the first two modes T_0 and T_2, and by linearizing around a reference state corresponding to the present-day climate. Such an analysis has been reported elsewhere ([13]). We simply mention that the power spectra obtained from the time-Fourier transforms of T_0 and T_2 present a structure similar to that given by eq. (26) and illustrated in fig. 2. Such forms are characteristic of *red noise spectra* which are commonly observed in atmospheric phenomena ([27,28]), whereby most of the power is concentrated in the low-frequency range.

The presence of the function $\delta(t-t')$ in eq. (43) is primarily responsible for this quite general property of climatic spectra. Nevertheless, the spatial structure of the noise plays an important role in the following respect. In the absence of spatially inhomogeneous fluctuations the temporal Fourier transform of the correlation function of the mean surface temperature would reduce to a Lorentzian, exhibiting the characteristic $(\varkappa^2+\omega^2)^{-1}$ dependence. The presence of spatial modes modifies this structure and gives rise to correlation functions such as given in eqs. (27) and (35), which have an altogether different frequency dependence.

([26]) G. Nicolis and I. Prigogine: *Self-organization in Non-equilibrium Systems* (John Wiley, New York, N. Y., 1977).
([27]) D. L. Gilamn, F. J. Fuglister and J. M. Mitchell: *J. Atmos. Sci.*, **20**, 182 (1963).
([28]) K. Hasselmann: in *Climatic Variations and Variability: Facts and Theories*, edited by A. Berger (Reidel, Dordrecht, 1981).

5. – Concluding comments.

By applying a generalized fluctuation-dissipation theorem to the random forces appearing in the balance equations of the climatic system, we obtain extensive information on the characteristics of the fluctuations of the climatic variables. The formulation outlined in the present paper is quite general, and two specific applications have been considered, where we analyse the vicinity of a given climatic state. It would be interesting to extend this analysis so as to take into account the multiplicity of solutions of the fully nonlinear problems. The most important point to investigate in this framework would be the passage times between different climatic states. It would also be desirable to develop a suitable description of localized, short wave-length fluctuations.

Finally, we note that the role of stochastic perturbations in more complex models like spectral models involving several modes could now be considered in the light of the the qualitative trends suggested by the analysis given in the present paper.

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Appendix

The Lorenz equations in the presence of fluctuations.

We first expand the stochastic fields $\boldsymbol{j}^*$, $\underset{\approx}{s}^*$ (see eqs. (10a), (10b)) in Fourier series, analogously to eqs. (14). However, since their effect on the balance equations appears through the divergence operator (for $\boldsymbol{j}^*$) and the $\nabla\times\nabla\times\nabla$, operator (for $\underset{\approx}{s}^*$), care should be taken in the series expansion to satisfy the parity and other symmetry requirements. For instance,

$$j_x^* = \sum_{mn} f_{mn}^x \exp[imk^*x^*] \sin(n\pi z^*) , \tag{A.1a}$$

but since j_z^* appears in the energy equation as $\partial j_z^*/\partial z^*$

$$j_z^* = \sum_{mn} f_{mn}^z \exp[imk^*x^*] \cos(n\pi z^*) . \tag{A.1b}$$

Similarly, one has

$$s^*_{xx} = \sum_{mn} f^{xx}_{mn} \exp[imk^*x^*] \cos(n\pi z^*) \,, \tag{A.2a}$$

$$s^*_{xz} = \sum_{mn} f^{xz}_{mn} \exp[imk^*x^*] \sin(n\pi z^*) \,, \tag{A.2b}$$

$$s^*_{zz} = \sum_{mn} f^{zz}_{mn} \exp[imk^*x^*] \cos(n\pi z^*) \,. \tag{A.2c}$$

Insertion of the above expansions and of eqs.(14) reduces the nonlinear partial differential equations (10*a*) and (10*b*) to an infinite set of coupled nonlinear stochastic ordinary differential equations for the mode amplitudes θ^*_{mn} and w^*_{mn}. Following SALTZMAN (8) and LORENZ (9), we truncate the set and restrict to the first three nontrivial modes, which amounts to keeping w^*_{11}, θ^*_{02} and θ^*_{11}. Introducing the reduced quantities,

$$\left\{\begin{aligned} X &= k^*(k^{*2} + \pi^2)^{-1} w^*_{11} \,, \\ Y &= -2^{\frac{3}{2}}\pi k^{*2}(k^{*2} + \pi^2)^{-3}\theta^*_{11} \,, \\ Z &= 2\pi k^{*2}(k^{*2} + \pi^2)^{-3}\theta^*_{02} \,, \\ r &= k^{*2}(k^{*2} + \pi^2)^{-3} R \,, \\ b &= 4\pi^2(k^{*2} + \pi^2)^{-1} \,, \\ \sigma &= P \,, \end{aligned}\right. \tag{A.3}$$

one obtains

$$\dot{X} = -\sigma X + \sigma Y + \phi_X \,, \tag{A.4a}$$

$$\dot{Y} = rX - Y - XZ + \phi_Y \,, \tag{A.4b}$$

$$\dot{Z} = -bZ + XY + \phi_Z \,, \tag{A.4c}$$

where ϕ_X, ϕ_Y, and ϕ_Z are given below, see (A.5).

These are the Lorenz-Saltzman equations including now internal fluctuation terms. It is known that the Lorenz-Saltzman equations do not describe correctly the behaviour of experimental convection at very large r ($r \lessdot r_T$); however, they contain the essential features for the purpose of the present analysis. Most important is that one is in a position to fully express the constraints imposed on the effective random forces by using statistical theory of irreversible processes. Indeed, the fact that the random force is the divergence of a random vector or tensor field is already built into eqs. (A.4). In addition using eqs. (8), (9), (A.1) and (A.2), one can determine the statistical properties of the three effective random forces in eqs. (A.4)

$$\left\{\begin{aligned} \phi_X &= 2\sqrt{2}\pi(k^{*2} + \pi^2)^{-3}[\pi k^{*2}(f^{zz}_{11} - f^{xx}_{11}) - ik^* f^{xz}_{11}] \,, \\ \phi_Y &= 2\sqrt{2}\pi k^{*2}(k^{*2} + \pi^2)^{-4}(ik^* f^x_{11} - \pi f^z_{11}) \,, \\ \phi_Z &= 4\pi^2 k^{*2}(k^{*2} + \pi^2)^{-4} f^z_{02} \,. \end{aligned}\right. \tag{A.5}$$

Because the computation is rather long and tedious, it is not displayed here; we merely outline the general procedure. By expanding eqs. (8) and (9) in Fourier series one obtains the two-time correlation functions of the Fourier components of the random forces f^i_{mn}, f^{ij}_{mn}. For f^i_{mn} one finds that all correlations of the form $\langle f^i_{mn}(t) f^j_{mn}(t')\rangle$; $i \neq j$ vanish; on the other hand for f^{ij}_{mn} one has a nonvanishing correlation $\langle f^{xz}_{11}(t) f^{zz}_{11}(t')\rangle$. It follows that the effective random forces ϕ_X, ϕ_Y, ϕ_Z are superpositions of three random processes f^{xz}_{11}, f^{zz}_{11}, f^{z}_{11}, two of which are correlated.

So we finally obtain the following nonvanishing contributions:

(A.6) $$\langle \phi_X(t)\, \phi_X(t')\rangle = q_X^2 \delta(t-t')$$

with

$$q_X^2 = 8\pi\eta k_B T(\varrho_0 \varkappa^2)^{-2} d^4 k^{*3} \frac{4\pi^2 k^{*2} + 2(k^{*4} + \pi^4)}{(k^{*2} + \pi^2)^6};$$

(A.7) $$\langle \phi_Y(t) \phi_Y(t')\rangle = q_Y^2 \delta(t-t')$$

with

$$q_Y^2 = 16\pi\lambda k_B T^2 (\alpha\varkappa^2 C_v)^{-2} (\alpha g d^4)^2 k^{*5} (k^{*2} + \pi^2)^{-7}$$

and

(A.8) $$\langle \phi_Z(t) \phi_Z(t')\rangle = q_Z^2 \delta(t-t')$$

with

$$q_Z^2 = 2\pi^2 (k^{*2} + \pi^2)^{-1} q_Y^2 .$$

These results show that, in general, the strengths of the random forces are *unequal*. An evaluation from numerical values describing laboratory conditions ($T = 293$ K, $d = 1$ cm, $\varrho = 1$ g·cm^{-3}, $\alpha = 10^{-4}$ K^{-1}, $\nu = 10^{-2}$ stokes, $\varkappa = 1.5 \cdot 10^{-3}$ cm^2 s^{-1}) yields $q_X/q_Z = 0.12$ and $q_Y/q_Z = 0.75$.

● RIASSUNTO (*)

Nella dinamica del clima si descrive solitamente l'effetto delle fluttuazioni generate internamente aumentando le equazioni del bilancio con l'aggiunta di forze casuali. In questo lavoro si studiano le proprietà di queste forze. Si propone un teorema di fluttuazione-dissipazione che correla la matrice di covarianza ai coefficienti fenomenologici come la diffusività turbolenta. Il teorema è usato successivamente per identificare le proprietà statistiche delle variabili climatiche stesse e per caratterizzare la variabilità climatica dal punto di vista della teoria statistica dei processi irreversibili. Si presentano applicazioni a un problema semplice di convezione termica e ad un modello di bilancio d'energia mediato a zone; si discute la possibilità di verifica sperimentale.

(*) *Traduzione a cura della Redazione.*

Резюме не получено.