Tellus (1984), *36A*, 1–10

Self-oscillations and predictability in climate dynamics

By C. NICOLIS, *Institut d'Aéronomie Spatiale de Belgique, Avenue Circulaire, 3, B-1180 Bruxelles, Belgium*

(Manuscript received October 13, 1982; in final form June 2, 1983)

ABSTRACT

A class of non-linear climatic models giving rise to sustained oscillations is considered. The evolution equations are cast into a "normal form" which allows one to distinguish between two different types of quantities: a radial variable, which obeys a closed equation and has strong stability properties; and a phase variable which is extremely labile. It is shown that as a result of these peculiar stability properties, the disturbances or random fluctuations acting on the system tend to deregulate the oscillatory behavior. It is concluded that such a phenomenon is a basic reason for progressive loss of predictability.

1. Introduction

In recent years it has been pointed out by several authors that simple climate models are capable of producing autonomous sustained oscillations. Consider first the range of 10^3 years which, as pointed out by Berger (1981), is an upper bound of the intrinsic time scale of variation of sea ice extent. Saltzman and coworkers (1978, 1980, 1981, 1982a) analyzed the interactions between the latter and the ocean surface temperature. They suggested the existence of a negative feedback loop arising from the positive ("insulating") effect of sea ice on temperature, and from the negative effect of temperature on sea ice. They concluded that such a system should give rise to autonomous sustained oscillations, and proposed a mathematical model for such a behavior. For one set of indicative parameter values, the periodicity indeed turns out to be of the order of 10^3 years.

Consider next the range of 10^4–10^5 years, which brings us to the time scale of quaternary glaciations. Saltzman et al. (1982b) worked out an extended version of the above-mentioned model which, for another set of parameter values, gives rise to self-oscillations of periods comparable to the glaciation time scales. A long period autonomous oscillation, involving ice sheet dynamics has also been studied by Källén et al. (1979) and Bhattacharya et al. (1982).

Let us now examine the climatic record to see if it carries some trace of such periodicities. As Saltzman et al. (1981) point out, an undulatory variation of sea ice extent presumably occurred at least once within the little ice age. Aside from this however, there seems to exist no compelling evidence of clearcut cyclic climatic change on the scale of 10^3 years. The situation is quite different for the scale of 10^4–10^5 years. In particular, it is well-known (see e.g. Berger, 1981) that quaternary glaciations present a cyclic character with a dominant periodicity of 10^5 years. This difference is reflected in a striking manner on the structure of variance spectra of paleoclimatic time series: The values around the frequencies corresponding to the 10^3 year range are indeed smaller by orders of magnitude than the values corresponding to variability in the range 10^4–10^5 years.

One is naturally tempted to correlate these facts with the apparent lack of systematic external forcings acting on the climatic system in the range of 10^3 years, and with the presence of a systematic astronomical forcing arising from the earth's orbital variations in the range of 10^5 years. In other words: might it not be that the absence of a "synchronizing agent" such as an external forcing, compromises the existence of a relatively well-defined cyclic variability despite the potential existence of autonomous oscillations?

The purpose of the present paper is to suggest that the answer to the above question is in the affirmative, for a reason that is deeply rooted in the dynamics of any non-linear non-conservative oscillator operating in the absence of a systematic forcing. In Section 2, the equations for the evolution of such an oscillator are cast into a "normal form" which allows us to undertake a qualitative study largely independent of the details of the particular model. In Section 3 it is observed that from this transformation a basic difference emerges between the *order parameter*, that is to say the variable that evolves at the slowest time scale, and the *phase* variable of the oscillator. Specifically while the order parameter shows a pronounced stability, the phase variable is extremely labile. As a result, there is an inherent tendency to desynchronization, chaotic behavior and loss of predictability. In Section 4 the phenomenological description is enlarged to take the fluctuations of the climatic variables into account. We show that desynchronization is reflected by the existence of a stationary state for the probability distribution as a result of which, statistically speaking, oscillation disappears altogether. Final comments are presented in Section 5. Throughout the work, the general ideas are illustrated on Saltzman's model oscillator.

2. Normal form and order parameter of a non-linear oscillator

Let $\{X_i\}$, $i = 1, \ldots, N$ be a set of climatic variables, $\{X_i^*\}$ a reference regime corresponding to a steady state. The evolution of the excess variables

$$x_i = X_i - X_i^* \tag{1}$$

can be written in the generic form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{N} a_{ij} x_j + g_i(\{x_j\}), \qquad i = 1, \ldots N, \tag{2}$$

in which the coefficients of the linearized part a_{ij} depend on the reference state and g_i contain the effects of non-linearity. Suppose that system (2) predicts the existence of sustained oscillations. In a typical situation this will be reflected by the fact that among the N eigenvalues of the matrix $\{a_{ij}\}$, 2 will be complex conjugate with positive real parts, and the real parts of the remaining $N-2$ will be negative. It is known from the qualitative theory of differential equations (see e.g. Arnold, 1980) that under these conditions the number of variables contributing effectively to the dynamics can be greatly reduced. Specifically, close to the bifurcation point and in the limit of long times we can cast eqs. (2) in the form

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = b_{11}\eta + b_{12}\theta + h_1(\eta,\theta),$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = b_{21}\eta + b_{22}\theta + h_2(\eta,\theta), \tag{3}$$

in which the eigenvalues of the 2×2 matrix $\{b_{ij}\}$ are the two privileged eigenvalues of $\{a_{ij}\}$ referred to above. The "master variables" η and θ are appropriate linear combinations of the x_j, while all other variables of the orignal problem are expressed entirely in terms of η and θ.

A particular example of (2)–(3), which will be used frequently in the sequal for illustrative purposes is the sea ice ocean surface temperature model developed by Saltzman et al. (1981):

$$\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}\bar{t}} = -\phi_2\bar{\eta} + \phi_1\bar{\theta},$$

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\bar{t}} = -\psi_1\bar{\eta} + \psi_2\bar{\theta} - \psi_3\bar{\eta}^2\bar{\theta}, \tag{4}$$

in which all parameters are positive, $\bar{\eta}$ is the deviation of the sine of the latitude of the sea ice extent from the steady state, and $\bar{\theta}$ is the excess mean ocean surface temperature. Actually, instead of (4) we will prefer to work with a dimensionless form resulting from the following scaling transformation:

$$\bar{\eta} = \left(\frac{\phi_2}{\psi_3}\right)^{1/2}\eta, \qquad \bar{\theta} = \frac{\phi_2^{3/2}}{\psi_3^{1/2}\phi_1}\theta, \qquad \bar{t} = \frac{1}{\phi_2}t. \tag{5}$$

The result is:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\eta + \theta,$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\psi_1\phi_1}{\phi_2^2}\eta + \frac{\psi_2}{\phi_2}\theta - \eta^2\theta. \tag{6}$$

We see that the dynamical behavior depends on two dimensionless parameters only, playing a rôle analogous to that of the Rayleigh or Prandtl numbers in fluid dynamics. In what follows we shall fix ψ_1, ϕ_1 and ϕ_2 to the values given by Saltzman et al. (1981), and use ψ_2 as bifurcation parameter.

We now return to the general case, eq. (3). Let

$$\omega_1 = \omega_2^* = \beta + i\omega_0 \tag{7}$$

be the eigenvalues of the linearized operator $\{b_{ij}\}$. At $\beta = 0$ the system undergoes a bifurcation beyond which a *limit cycle* is expected to emerge. In general the full problem of exactly solving eq. (3) in this range is intractable. Let us therefore limit ourselves to a qualitative analysis. To this end it is convenient to cast the dynamics in a form in which the linear part becomes separable, corresponding to a full diagonalization of $\{b_{ij}\}$. This is achieved by a linear transformation T, which in the present case is a 2×2 matrix whose columns are the two right eigenvectors of $\{b_{ij}\}$. Operating on both sides of eq. (3) with the inverse matrix T^{-1} and introducing the new (complex) variables

$$\begin{pmatrix} z \\ z^* \end{pmatrix} = T^{-1} \begin{pmatrix} \eta \\ \theta \end{pmatrix}, \tag{8}$$

we obtain

$$\frac{dz}{dt} = (\beta + i\omega_0)\, z + H_2(z^2, z^* z, z^*; \beta) + H_3(z^2 z^*, zz^*, z^3, z^*; \beta) + \dots, \tag{9}$$

and a similar equation for z^*, obtained by taking the complex conjugate of both sides of eq. (9). The functions H_2, H_3 etc., contain the quadratic, cubic etc., non-linearities of the problem. They are obtained by operating with T^{-1} on h_1 and h_2 (eq. (3)) and switching subsequently to the variables z, z^* by the transformation law given in eq. (8).

As it stands, the system of eq. (9) is as complicated as the original system, eq. (3). However, the form displayed in eq. (9) is especially suitable for approximations. To see this we perform a final transformation, switching to "radial" and "angular" variables through

$$z = r\, e^{i\phi}. \tag{10}$$

Eqs. (9) then give rise to the following two equations for r and ϕ:

$$\frac{dr}{dt} = \beta r + \operatorname{Re} H_2(r^2 e^{i\phi}, r^2 e^{-i\phi}, r^2 e^{-2i\phi}) + \operatorname{Re} H_3(r^3, r^3 e^{2i\phi}, r^3 e^{-2i\phi}, r^3 e^{-4i\phi}) + \dots, \tag{11a}$$

$$\frac{d\phi}{dt} = \omega_0 + \frac{1}{r} \operatorname{Im} H_2(r^2 e^{i\phi}, r^2 e^{-i\phi}, r^2 e^{-2i\phi}) + \frac{1}{r} \operatorname{Im} H_3(r^3, r^3 e^{2i\phi}, r^3 e^{-2i\phi}, r^3 e^{-4i\phi}). \tag{11b}$$

The simplification to be made is now intuitively clear. Indeed, close to the bifurcation point the right-hand sides of eqs. (11) display a slow motion part, corresponding to the terms βr, r^3, and a fast motion corresponding to oscillating terms of the type $r^2 e^{\pm in\phi}$, $r^3 e^{\pm in\phi}$, with $n = 1, 2$ etc. It may thus be expected that for the long-time behavior, only the slow, "secular" part will be important. Actually, the full justification of this conjecture requires the use of a chapter of bifurcation theory known as the theory of normal forms (Arnold, 1980). As it turns out, discarding the angular dependence is a completely consistent procedure provided that the system operates relatively close to the bifurcation point, in the sense that

$$\frac{|\beta|}{\omega_0} \ll 1. \tag{12}$$

Throughout our qualitative analysis, we shall assume that in all cases of interest this inequality is satisfied. This will allow us to proceed a considerable way with an analytic approach. Subsequently we will relax this condition and we will verify numerically that the qualitative results hold even far from bifurcation.

Summarizing, by retaining only the slow part of the motion we obtain:

$$\frac{dr}{dt} = \beta r - u r^3 + \dots, \tag{13a}$$

$$\frac{d\phi}{dt} = \omega_0 - v r^2 + \dots, \tag{13b}$$

where u and v are parameters arising from the real and imaginary parts of H_2 and H_3

In bifurcation theory (Arnold, 1980), eqs. (13) are known as the *normal form* of the initial dynamical system (eqs. (3)), whereas the radial variable r is referred to as the *order parameter*.

Let us illustrate this result for Saltzman's model. A straightforward but tedious algebra yields

$$\beta + i\omega_0 = \frac{1}{2}\left\{\frac{\psi_2}{\phi_2} - 1 + \left[\left(\frac{\psi_2}{\phi_2} - 1\right)^2 - 4\left(\frac{\psi_1\phi_1}{\phi_2^2} - \frac{\psi_2}{\phi_2}\right)\right]^{1/2}\right\}, \tag{14a}$$

and a transformation matrix T equal to

$$\mathrm{T} = \begin{pmatrix} 1 & 1 \\ \beta + i\omega_0 + 1 & \beta - i\omega_0 + 1 \end{pmatrix}, \tag{14b}$$

or, more explicitly

$$\eta = 2r\cos\phi,$$

$$\theta = 2r\{(\beta + 1)\cos\phi - \omega_0 \sin\phi\}. \tag{14c}$$

The evolution equations in the (r, ϕ) representation are

$$\frac{dr}{dt} = \beta r - \tfrac{1}{2} r^3, \tag{15a}$$

$$\frac{d\phi}{dt} = \omega_0 + \frac{3}{2\omega_0}(1 + \beta) r^2. \tag{15b}$$

The first of these equations allows us to evaluate analytically the radius of the limit cycle. Setting $dr/dt = 0$, which in the (r, ϕ) representation places us on the limit cycle, we obtain:

$$r_s = (2\beta)^{1/2}. \tag{16}$$

For the numerical values given in Saltzman et al. (1981) for the behavior in the 10^3 year range, we obtain $\beta = 1.5$, $r_s = 1.73$. Going back to the original variables one can see that this implies an amplitude of oscillation of $\bar{\eta}$ equal to $\bar{\eta}_{max} \sim 0.035$ in good agreement with the numerical simulations of the original paper. We therefore see that the normal form analysis gives the right trend even beyond the immediate vicinity of the bifurcation point.

The second relation (15) includes two types of contribution to the angular velocity $\Omega = d\phi/dt$. One arising through the dependence of the linearized frequency ω_0 on the bifurcation parameter β (see eq. (14a)), and another arising from the effect of non-linearities, $(3/2\omega_0)(1 + \beta)\, r^2$. Within the range of validity of the normal form, eqs. (15), only the first non-trivial correction to the value of ω_0 at the bifurcation point $\beta = 0$, $\omega_0(0)$, should be retained. We arrive in this way at the following expression for the angular velocity

$$\Omega = \frac{d\phi}{dt} = \omega_0(0) + \beta\left(\frac{d\omega_0}{d\beta}\right)_0 + \frac{3}{2\omega_0(0)} r^2. \tag{17a}$$

On the limit cycle itself, $r = r_s$, and the dominant correction to the angular velocity is:

$$\Omega = \left(\frac{d\phi}{dt}\right)_s = \omega_0(0) + \frac{2}{\omega_0(0)}\beta. \tag{17b}$$

We see that as the system moves further from bifurcation, the period is shortened with respect to the value $2\pi/\omega_0$ predicted by the linear theory. This agrees qualitatively with the numerical simulations reported in Saltzman et al. (1981). Indeed these authors report a periodicity of 1260 years far from bifurcation, whereas the linearized period $2\pi/\omega_0$, corresponding to $\beta = 0$ can be shown to be equal to 1740 years. Fig. 1 describes the dependence of both amplitude and period in terms of β. We obtain the right qualitative trend but, as expected from the perturbative nature of the normal form analysis, quantitative agreement fails beyond the range of small values of β.

Finally, one can compute analytically the phase lag α between the η and θ variables. From the first eq. (14c) one sees that $\alpha = \mathrm{arc\ tg}\,(\omega_0/(\beta + 1))$. Near bifurcation this yields $\alpha \sim 320$ years in excellent agreement with numerical results in this range.

3. Stability of the order parameter versus lability of the phase. Climatic "isochrons"

Inspection of the normal forms derived in Section 2 (eqs. (13) and (15)) reveals a fundamental difference between the evolution of the order parameter r and of the phase ϕ. The former obeys a closed equation that can be solved analytically. For Saltzman's model, starting with $r = r_0$ at $t = t_0$, one easily finds

$$r^2(t) = 2\beta \frac{1}{1 + D\, e^{-2\beta(t - t_0)}}, \tag{18}$$

with

$$D = \frac{2\beta - r_0^2}{2}.$$

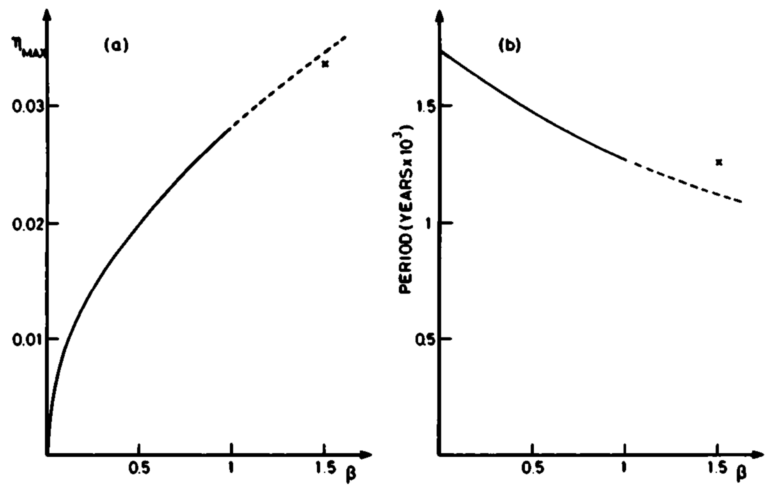

Fig. 1. Analytically computed dependence of the amplitude of oscillation of the sea ice extent (part (a)) and of the period of the limit cycle (part (b)) of the Saltzman model as a function of the bifurcation parameter β. Dotted lines are used for that part of the curves for which the validity of the perturbative calculation cannot be guaranteed. Crosses correspond to the values obtained from numerical solution of Saltzman's equations far from bifurcation.

As long as $\beta > 0$ and $r_0 \neq 0$, this predicts an exponential relaxation to the limit cycle. In other words the variable r enjoys asymptotic stability, in the sense that any perturbation that may act accidentally on r will be damped by the system.

The situation is very different for ϕ. Being an angular variable, the latter increases continuously in the interval $(0, 2\pi)$ from some initial value ϕ_0. If ϕ_0 is perturbed, this monotonic change will start all over again from the new value, and there will not be any tendency to reestablish the initial phase ϕ_0.

In order to better realize the consequences of this property, let us perform the following thought experiment. Suppose that the system runs on its limit cycle, $r = r_s$. At some moment, corresponding to a value ϕ_0 of the phase, we displace the system to a new state (cf. Fig. 2). Owing to the stability of the order parameter, the perturbed state will spiral to the limit cycle. Clearly however, when the limit cycle will be reached again, the phase will generally be different from the one that would characterize an unperturbed system following its limit cycle

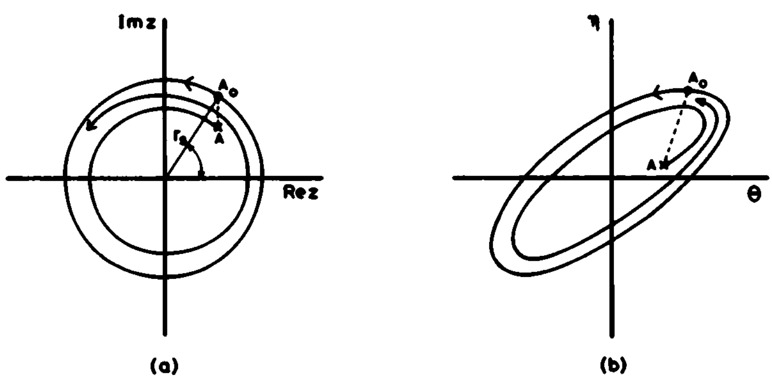

Fig. 2. Schematic representations of the evolution following the action of a perturbation leading from state A_0 on the limit cycle to state A. Parts (a) and (b) describe the situation, in the space of the variables of the normal form and in the space of the variables, η, θ, respectively.

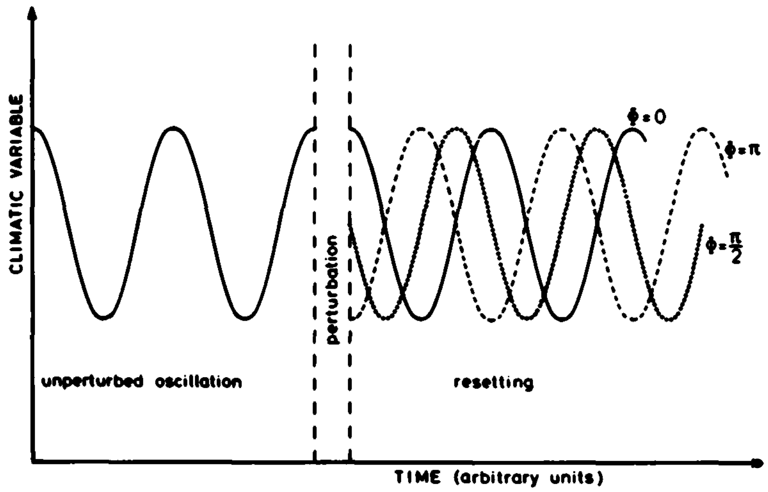

Fig. 3. Resetting of a perturbed oscillation on its limit cycle. Depending on the type of the perturbation, the reset phase may differ widely from that of the unperturbed situation.

during the same time interval. Inasmuch as the state at which the system can be thrown by a perturbation is unpredictable, it therefore follows that the reset phase of the oscillator will also be unpredictable (see Fig. 3). In other words, *a non-linear oscillator is bound to behave sooner or later in an erratic way* under the action of the perturbations. This already provides a qualitative answer to the question raised in Section 1.

We can go further and define the locus of perturbed states which, when the limit cycle will be reached again will be characterized by a given value of the phase, ϕ. We call this locus a *climatic isochron*, and the common phase Φ *latent phase*. These notions were first introduced by Winfree (1980) in connection with biological oscillations. For Saltzman's model, an analytic expression for Φ can be derived, by integrating eq. (15b) in time and using the fact that r satisfies eq. (15a). The interested reader is referred to Winfree (1980) for the details of this derivation. For our model the result is:

$$\Phi = \phi - \phi_0 + \frac{3}{\omega_0}(1 + \beta) \ln \frac{r}{r_s}, \qquad (19)$$

where ϕ_0 is the initial phase on the limit cycle (i.e. $\phi = \phi_0$, $\Phi = 0$, $r = r_s$) and (r, ϕ) are the values of the radial and phase variables descriptive of the perturbed state. The r-dependent part of the r.h.s. expresses the deviation from the isochronous motion, and is due to the fact that the equation for the phase of the oscillator (see (15b)) contains a contribution from the radial part of the motion. Fig. 4 gives plots in the (r, ϕ) plane corresponding to various values of Φ. As in Section 2, it is understood that only dominant terms in β are to be

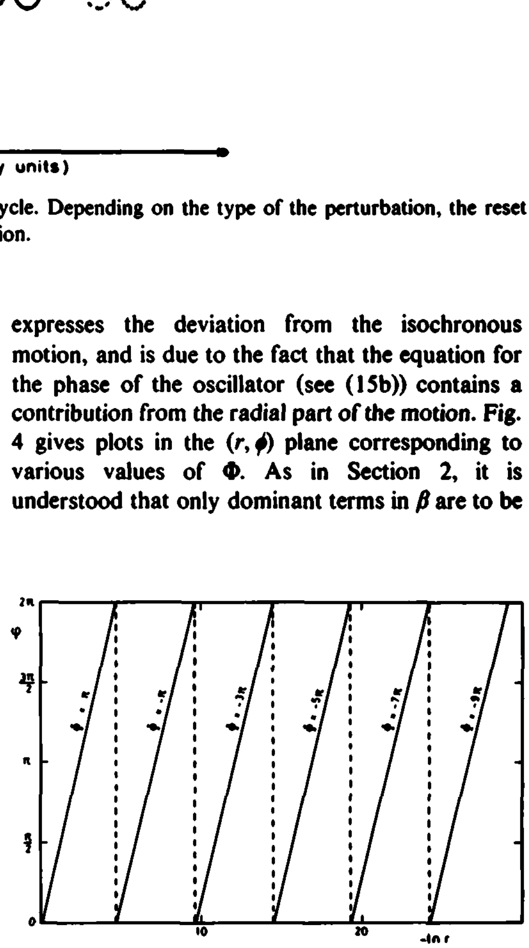

Fig. 4. Plot of climatic isochrons for the Saltzman model as given by eq. (19) for an initial phase $\varphi_0 = 0$ and for a value of bifurcation parameter $\beta = 0.005$. Motion along these isochrons leads to a resetting on the limit cycle in phase opposition, $\Phi = \pi$, with the unperturbed signal. The isochrons $\Phi = -\pi$, -3π etc. describe evolutions in which the system spirals once, twice etc. before resetting on the limit cycle.

retained in order to ensure consistency with the range of validity of the normal form.

As an example, suppose that an initial state on the limit cycle corresponding to the phase $\phi_0 = -1.114$ is perturbed. We want to characterize those perturbed states that will reset on the limit cycle in phase opposition, $\Phi = \pi$ with respect to the unperturbed situation. Choosing, for instance, $r = 0.5$, we find from eq. (19) that $\phi = -0.05$ near bifurcation ($\beta = 0.005$). From eqs. (14c) and (15) we can then express the perturbed state in terms of the original variables

$$\bar{\eta} \sim 0.01, \qquad \bar{\theta} \sim 0.14. \tag{20}$$

A numerical experiment, entirely confirms this analytic prediction: starting with the initial condition (20), we indeed find that the system reaches the limit cycle with a phase shift practically equal to π. It is furthermore found that the analytical result gives the right trend far from the bifurcation point, for the parameter values given by Saltzman et al. (1981).

From a practical point of view, we believe that the notion of climatic isochron will prove to be important. Such plots as in Fig. 4 give us information on the most "dangerous" perturbations that are likely to deregulate the oscillator completely, by inducing a resetting at a value nearly in phase opposition with respect to the unperturbed situation. They also show that there exist "benign" perturbations for which no sensible variation of phase is observed. Interestingly, both small and large amplitude perturbations may equally well belong to either of the above two classes.

4. Stochastic analysis

In this Section we pursue the consequences of phase lability. The main point that we want to make is that in any complex physical system there is a universal mechanism of perturbations generated spontaneously by the dynamics, namely the *fluctuations*. Because of the complexity of the processes at their origin, fluctuations are basically random events. This implies that the state variables (η, θ) or (r, ϕ) themselves become random processes. Our purpose here is to examine whether this may affect the robustness of the oscillatory regime on the limit cycle. Intuitively, we expect from Section 3 that the radial variable r will remain robust, but the phase variable ϕ will become completely deregulated. It is shown in this Section that this is indeed the case. Moreover, because of the deregulation of ϕ, the oscillatory behavior will disappear altogether after a sufficiently long lapse of time.

Let us incorporate the effect of fluctuations by adding random forces F_η, F_θ to the deterministic rate of equations (Hasselmann, 1976). As usual we assume the latter to define a multi-Gaussian white noise:

$$\langle F_\eta(t) F_\eta(t') \rangle = q_\eta^2 \, \delta(t - t'),$$
$$\langle F_\theta(t) F_\theta(t') \rangle = q_\theta^2 \, \delta(t - t'),$$
$$\langle F_\eta(t) F_\theta(t') \rangle = q_{\eta\theta} \, \delta(t - t').$$

This allows us to write a Fokker–Planck equation for the probability distribution $P(\eta, \theta, t)$ of the climatic variables. In general this equation is intractable. However, the situation is greatly simplified if one limits the analysis to the range in which the normal form (eqs. (13) or (15)) is valid.

To see this we first express the Fokker–Planck equation in the polar coordinates (r, ϕ) defined in eq. (10). For Saltzman's model we obtain, after a long calculation (see also Baras et al., 1982):

$$\begin{aligned}\frac{\partial P(r, \phi, t)}{\partial t} = &- \frac{\partial}{\partial r}\left\{\beta r - \frac{1}{2} r^3 + \frac{1}{2r} Q_{\phi\phi}\right\} P \\ &- \frac{\partial}{\partial \phi}\left\{\omega_0 + \frac{1}{2\omega_0} 3(1 + \beta) r^2 - \frac{1}{r^2} Q_{r\phi}\right\} P \\ &+ \frac{1}{2}\left\{\frac{\partial^2}{\partial r^2} Q_{rr} + 2 \frac{\partial^2}{\partial r \partial \phi} \frac{Q_{r\phi}}{r} + \frac{\partial^2}{\partial \phi^2} \frac{Q_{\phi\phi}}{r^2}\right\} P,\end{aligned} \tag{21}$$

where

$$\begin{aligned} Q_{\phi\phi} &= q_R^2 \sin^2 \phi - 2 q_{RC} \sin \phi \cos \phi + q_C^2 \cos^2 \phi, \\ Q_{r\phi} &= - q_R^2 \sin \phi \cos \phi + q_{RC} (\cos^2 \phi - \sin^2 \phi) \\ &\quad + q_C^2 \sin \phi \cos \phi, \\ Q_{rr} &= q_R^2 \cos^2 \phi + 2 q_{RC} \sin \phi \cos \phi + q_C^2 \sin^2 \phi, \end{aligned} \tag{22}$$

and q_R^2, q_C^2 and q_{RC} are suitable linear combinations of q_η^2, q_θ^2, $q_{\eta\theta}$.

To go further it is necessary to introduce a perturbation parameter in the problem. It is reasonable to choose it to be related to the weakness of the noise terms. Mathematically, we express this through the following scaling:

$$Q_{\phi\phi} = \varepsilon \tilde{Q}_{\phi\phi},$$
$$Q_{r\phi} = \varepsilon \tilde{Q}_{r\phi},$$
$$Q_{rr} = \varepsilon \tilde{Q}_{rr}. \tag{23}$$

We next scale both the bifurcation parameter β and the deviation of the radius r from its value on the limit cycle, $r = r_s$, by suitable powers of ε. We do this in order to be in accordance with the conditions of validity of the normal form (eqs. (12) and (15)). However, we should keep in mind that according to the numerical results reported in Section 3, the qualitative predictions should still describe the general trend beyond the vicinity of the bifurcation point. Note that no scaling can be applied to the angular variable ϕ, as the latter increases in the interval $(0, 2\pi)$ and does not enjoy any stability property. Summarizing we write, using also eq. (16) and following Baras et al. (1982):

$$\beta = \tilde{\beta}\varepsilon^{2c},$$
$$r = r_s + \rho\varepsilon^b = (2\tilde{\beta})^{1/2}\varepsilon^c + \rho\varepsilon^b$$
$$\phi = \phi. \tag{24}$$

The (non-negative) exponents b and c are chosen in such a way that both the drift and diffusion terms contribute to the evolution of P in eq. (21). Indeed, should the diffusion terms be negligible, the probability P would be a delta function around the deterministic motion, and as a result the effect of fluctuations would be wiped out. If on the other hand the drift terms were negligible, P would exhibit a purely random motion similar to that of a Brownian particle in a fluid, and would tend to zero everywhere as $t \to \infty$. The best way to estimate the magnitude of these two terms is to introduce the conditional probability $P(\phi/\rho, t)$ through

$$P(\rho, \phi, t) = P(\phi/\rho, t)\, P(\rho, t), \tag{25a}$$

where

$$P(\rho, t) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, P(\rho, \phi, t). \tag{25b}$$

Integrating eq. (21) over ϕ and taking the scaling (23) and (24) into account, one can see that $\partial P(\rho, t)/\partial t$ is of order ε. $P(\rho, t)$ is therefore a slow variable: this is the probalistic analogue of the fact stressed in Section 2, that the order parameter varies at the slowest of all time scales present in the problem. Using this property and dividing through eq. (21) with $P(\rho, t)$ we obtain a closed equation for $P(\phi/\rho, t)$ which to dominant order in ε reads:

$$\frac{\partial P(\phi/\rho, t)}{\partial t} = -\frac{\partial}{\partial\phi}\omega_0 P(\phi/\rho, t) \tag{26}$$

This equation admits a properly normalized *stationary* solution,

$$P(\phi/\rho, t) = 1/2\pi. \tag{27}$$

Introducing this into eq. (25a) and integrating eq. (21) over ϕ, we can obtain a closed equation for the slow variable $P(\rho, t)$. From this equation one can see that the drift and diffusion terms are of the same order of magnitude if and only if

$$b = c = \frac{1}{4}. \tag{28}$$

The equation for $P(\rho, t)$ then reads:

$$\frac{\partial P(\rho, t)}{\partial t} = -\varepsilon^{1/2}\frac{\partial}{\partial\rho}\left[-2\tilde{\beta}\rho - 3\left(\frac{\tilde{\beta}}{2}\right)^{1/2}\rho^2 - \frac{1}{2}\rho^3 + \frac{\tilde{Q}}{2((2\tilde{\beta})^{1/2} + \rho)}\right]P + \varepsilon^{1/2}\frac{\partial}{\partial\rho^2}P, \tag{29}$$

where

$$\tilde{Q} = \frac{1}{2}(q_R^2 + q_C^2). \tag{30}$$

This equation admits a steady-state solution. Switching back to the original variables and parameters r, β and Q through the inverse scaling (eqs. (23) and (24)), we can see that this steady-state is of the form

$$P_s(r) \sim r\exp\frac{2}{Q}\left(\frac{\beta r^2}{2} - \frac{r^4}{8}\right). \tag{31}$$

The terms in parenthesis in the exponential feature the integral of the right-hand side of the deterministic evolution equation (15a). This quantity is known as *kinetic potential*, and its importance in climate has been stressed by Ghil (1976), North et al. (1981) and Nicolis and Nicolis (1981). In the present case, denoting the potential by $U(r, \beta)$, we can write

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\partial U(r, \beta)}{\partial\beta},$$

with

$$P_s(r) \sim \exp -\frac{2}{Q} U(r,\beta) . \tag{32}$$

Fig. 5 represents function (31) or (32) in terms of the variables (η, θ). We obtain a crater-like distribution. The projection of the lip of the crater is identical to the deterministic limit cycle.

Let us summarize the situation. We have shown that starting from the initial variables η, θ one can switch to two combinations which have a completely different status. The angular variable ϕ which, according to the transformation laws given in Section 2, is related to η, θ through

$$\phi = \text{arc tg}\left[\frac{1}{\omega_0}\left(\beta + 1 - \frac{\theta}{\eta}\right)\right] \tag{33}$$

has a completely "flat" probability distribution (cf. eq. (27)). It may therefore be qualified as "chaotic", in the sense that the dispersion around its average will be of the same order as the average value itself. This is to be correlated with the remarks made in Section 3 about the labile behavior of the phase under the action of perturbations.

The situation is different as far as the statistics of the radial variable r is concerned. This variable, related to η, θ through

$$r^2 = \frac{1}{4}\left\{\eta^2 + \frac{1}{\omega_0^2}[(\beta + 1)\eta - \theta]^2\right\} \tag{34}$$

has a stationary probability distribution (eqs. (31) and (32)) such that the dispersion around the most probable value $r = r_s$ is small. Nevertheless, the mere fact that the probability distribution is stationary rather than time-periodic implies that a remnant of the chaotic behavior of ϕ subsists in the statistics of r. In a sense we arrive at a picture whereby if an average over a large number of samples is taken, the periodicity predicted by the deterministic analysis will be wiped out as a result of destructive phase interference. We believe that this property may be at the origin of the lack of pronounced systematic periodicities in the climatic record for time scales for which no systematic external forcing is acting, like for instance in the range of 10^3 years.

It should be pointed out that the above result is to be understood in an *asymptotic sense*. In a given system which represents a particular realization of the stochastic process described by the Fokker–Planck equation, a trace of the periodic behavior may subsist for a substantial amount of time and show up as a peak in a power spectrum computed over that interval. Eventually however, the oscillatory behavior is bound to disappear. How soon this will happen depends on the strength, Q of the random force. The computer simulations on the effect of noise reported in the paper by Saltzmann et al. (1981) corroborate this view.

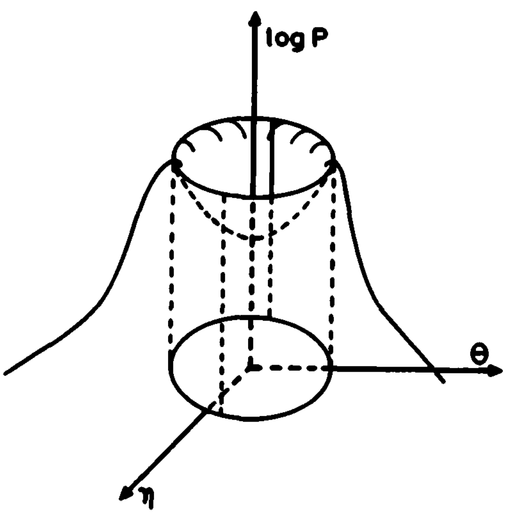

Fig. 5. Schematic representation of the steady state probability distribution eq. (31). Note that the projection of the lip of the crater on the phase plane of the variables reproduces the deterministic limit cycle.

5. Discussion

In this paper we explored some qualitative properties of autonomous oscillations in climate dynamics. Our analysis led us to raise some general questions concerning predictability. Specifically we have shown that, as a result of the poor stability properties of the phase variable, the systematic oscillation tends to disappear in the climatic record and to be gradually replaced by a random looking motion. Concomitantly, in a time series corresponding to a long sampling interval, detailed predictability would be compromised.

It is important to realize that the origin of this mechanism of loss of predictability is completely different from that associated with the appearance of aperiodic or chaotic solutions (strange attractors) of the deterministic balance equations. More work is necessary to differentiate between the consequences of "probabilistic" and "deterministic" chaos.

Throughout this work we have argued in terms of an oscillatory behavior of mean surface properties, encompassing the entire earth–atmosphere–cryosphere system. It is clear however that if spatial inhomogeneities are allowed, such a global oscillator can only result from the dynamics of localized oscillators, each of them representing the climate of a certain region of the globe. In this picture, the lability of the phase of each local oscillator will be reflected by the desynchronization of this oscillator with respect to its neighbors. Thus, if an average over a large space region is taken, the oscillatory behavior will be wiped out. The behavior of coupled Saltzman oscillators is analyzed in a forthcoming paper by the author (Nicolis 1983a).

The analysis carried out in this work is valid both for intermediate and for longer period oscillations, like those developed in connection with quaternary glaciations. In this latter case, however, an additional argument comes into play: because of the existence of a systematic *external* forcing of comparable period, related to the earth's orbital variations, the oscillator may be entrained and thus be less sensitive to external disturbances or fluctuations. As a result, a relatively well-defined periodicity may exist in the climatic record. We arrive in this way at a synthesis between internally generated and external mechanisms of climatic change. In Nicolis (1982, 1983b) the behavior of forced oscillators is analyzed, and the mechanisms of phase stabilization by resonant or harmonic coupling to the forcing are explored.

6. Acknowledgements

We acknowledge fruitful discussions with Dr. M. Malek-Mansour. This work is supported, in part, by the EEC under contract no. CLI-027-B(G).

REFERENCES

Arnold, V. 1980. *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*. Mir, Moscow.

Baras, F., Malek-Mansour, M. and Van Den Broeck, C. 1982. Asymptotic properties of coupled non-linear Langevin equations in the limit of weak noise (II). Transition to a limit cycle. *J. Stat. Phys. 28*, 577–587.

Berger, A. 1981. Spectrum of climatic variations and possible causes. In: *Climatic variations and variability: facts and theories* (ed. A. Berger). Reidel, 411–432.

Bhattacharya, K., Ghil, M. and Vulis, I. L. 1982. Internal variability of an energy balance model with delayed albedo effects. *J. Atmos. Sci. 38*, 1747–1773.

Ghil, M. 1976. Climate stability for a Sellers-type model. *J. Atmos. Sci. 33*, 3–20.

Hasselmann, K. 1976. Stochastic climate models, Part 1. Theory. *Tellus 28*, 473–485.

Källén, E., Crafoord, C. and Ghil, M. 1979. Free oscillations in a climate model with ice-sheet dynamics. *J. Atmos. Sci. 36*, 2292–2303.

Nicolis, C. 1982. Self-oscillations, external forcings and climate predictability. In: Proc. Symp. "Milankovitch and Climate", Palisades, N.Y.

Nicolis, C. 1983a. Coupled oscillator models in climate dynamics: synchronization, phase lags and non-periodic behavior. To be published.

Nicolis, C. 1983b. Self-oscillations and predictability in climate dynamics—Periodic forcing and phase locking. *Tellus*, submitted.

Nicolis, C. and Nicolis, G. (1981). Stochastic aspects of climatic transitions—additive fluctuations. *Tellus 33*, 225–237.

North, G. R., Cahalan, F. R. and Coakley, J. A. 1981. Energy balance climate models, *Rev. Geophys. Space Phys. 19*, 91–121.

Saltzman, B. 1978. A survey of statistical-dynamical models of the terrestrial climate. *Adv. Geophys. 20*, 183–304.

Saltzman, B. 1982a. Stochastically-driven climatic fluctuations in the sea-ice, ocean temperature, CO_2 feedback system. *Tellus 34*, 97–112.

Saltzman, B. and Moritz, R. E. 1980. A time-dependent climatic feedback system involving sea-ice extent, ocean temperature, and CO_2. *Tellus 32*, 93–118.

Saltzman, B., Sutera, A. and Evenson, A., 1981. Structural stochastic stability of a simple auto-oscillatory climate feedback system. *J. Atmos. Sci. 38*, 494–503.

Saltzman, B., Sutera, A. and Hansen, A. R. 1982b. A possible marine mechanism for internally generated long-period climate cycles. *J. Atmos. Sci. 39*, 2634–2637.

Winfree, A. T. 1980. *The geometry of biological time*. Springer, Berlin.