Chapter 4

ON A NEW FLUCTUATION-DISSIPATION THEOREM IN CLIMATE DYNAMICS

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4.1 ABSTRACT

In recent years, a fluctuation-dissipation theorem linking the time correlation of fluctuations of climatic variables to the matrix monitoring the linear response to an external forcing, has been extensively used in climate dynamics. In the present paper an extended fluctuation-dissipation theorem is suggested, which refers to the properties of the random forces rather than those of the climatic variables. The implications of this theorem are analyzed in the framework of an energy-balance model. In particular, some characteristic properties of the noise associated with climatic variability are identified.

4.2 INTRODUCTION

Climate variability,a central theme in the present volume, is an undisputable fact. Its effect both on the short term behavior and on the large scale transitions of the climatic system is widely recognized (Lorenz, 1976). On the other hand, its quantitative characterization is an extremely complex statistical problem. Indeed, climate dynamics is an ensemble of non-linear processes involving a large number of coupled variables, which evolve under conditions far away from thermodynamic equilibrium. Even at a laboratory scale, such processes are poorly understood. For instance the statistical foundations of hydrodynamic instabilities (see e.g. Swinney and Gollub, 1981) or of chemical instabilities (Nicolis and Prigogine, 1977) are still in their infancy.

An important step toward the quantitative formulation of climate variability was accomplished by Leith (1975, 1978), whose ideas were further discussed by Bell (1980) and North et al. (1981) among others. We briefly summarize the main points.

Let $\{x\}$ be a set of climatic variables. On the one side, their evolution is subjected to continuous random deviations from some macroscopic average, owing to the intrinsic stochasticity of climate. And on the other side, these same variables are capable of responding to any change of the external environment (like e.g. the solar output, the earth's orbital characteristics, the CO_2 level etc...) in the form of a deterministic signal. But the local dynamics of $\{x_\alpha\}$ has presumably no way to "know" whether the perturbation to which it responded was due to the system's intrinsic stochasticity or to the external forcing. Hence, the two responses must somehow be related. We call this relation the <u>fluctuation-dissipation theorem.</u> Leith wrote it in the form :

$$\langle \delta x_\alpha(0) \quad \delta x_\beta(\tau) \rangle = \sum_\gamma \langle \delta x_\alpha(0) \ \delta x_\gamma(0) \rangle \ . \ g_{\gamma\beta}(\tau) \tag{1}$$

The left hand side of Eq.(1) features the <u>time-correlation matrix</u> of the fluctuations $\{\delta x_\alpha\}$ in the absence of external forcing. In the right hand side we have the static quadratic average of these same fluctuations multiplied by the matrix $g(\tau)$, which monitors the linear response of $\{x_\alpha\}$ to a small change of the external environment. Specifically, the time integral of $g_{\alpha\beta}(\tau)$ may be viewed as some sort of transport coefficient,

$$D_{\alpha\beta} = \int_0^\infty g_{\alpha\beta}(\tau) \ d\tau \tag{2a}$$

relating the response $\delta\langle x_\alpha\rangle$ to the external forcing, δF_β :

$$\delta\langle x_\alpha\rangle = \sum_\beta D_{\alpha\beta} \ \delta F_\beta \tag{2b}$$

The brackets in Eqs (1) and (2b) denote the averaging over a statistical ensemble descriptive of the system. Presumably, both the time-correlation functions and the static quadratic averages of $\{\delta x_\alpha\}$ are accessible from meteorological data. Hence, from Eq.(1) one can evaluate the response function $g(\tau)$ and subsequently determine thanks to Eqs (2), the system's sensitivity to a variety of external signals. This, according to Leith, is the main usefulness of fluctuation-dissipation like relations in climate dynamics.

In view of the enormous complexity of the earth-atmosphere system, it is only unavoidable that many of the assumptions needed to arrive at the basic relation, Eq. (1), are not fully justified. Let us examine the most crucial of them in some detail.

First, in the original derivation (Leith, 1975, 1978) the Liouville equation for a conservative flow is the starting point. Now, the dynamics of the climatic variables is a dissipative, non conservative dynamics. As well known the passage from the microscopic degrees of freedom evolving according to the Liouville equation to the macrovariables obeying to such conservation laws as the Navier-Stokes equations is a complex process. Additional assumptions are therefore needed. Specifically, it is postulated that the probability ensemble of the macrovariables is given by a Gaussian distribution and that the process is ergodic. This is certainly true when the system evolves in the vicinity of a single globally stable steady state. If on the contrary the earth-atmosphere system is capable of performing transitions between different types of climatic states (Lorenz, 1976, North et al., 1981) this assumptions should break down : The probability ensemble is in this case multi-humped (Nicolis and Nicolis, 1981) and the process becomes "metrically almost intransitive" (Lorenz, 1976).

Second, relation (1) involves two unknown sets of quantities : the correlation functions and the static quadratic averages of the fluctuations. As a result, it cannot give to us information on the source of intrinsic randomness of the climatic system. Moreover, a difficulty arises when one is interested in long term climatic changes, like those on which one focuses in the framework of energy-balance models. The measurability of the above mentioned two sets of average quantities becomes then questionable, and as a result the usefulness of a fluctuation-dissipation relation of the form given in Eq.(1) is reduced.

The purpose of the present paper is to present some results contributing to a partial resolution of the above mentioned difficulties. The starting point is to realize that climatic variability is the result of fluctuations. The latter are incorporated into the traditional description based on the phenomenological balance equations through the addition of random forces, as suggested by Hasselmann (1976) :

$$\frac{dx_\alpha}{dt} = f_\alpha \ (\{x_\beta\}) - \text{div}\ J_\alpha + F_\alpha(\underset{\sim}{r},t) \tag{3a}$$

Here f_α are the source or sink terms, J_α the flux of x_α, and F_α the random force associated to x_α. The problem of obtaining information on the statistics of climatic variables amounts therefore to finding the statistical properties of these random forces.

One, widely adopted, assumption is that the random forces define a Gaussian white noise in time :

$$<F_{\alpha}(\underset{\sim}{r},t)\, F_{\beta}(\underset{\sim}{r}'t')> \; = Q_{\alpha\beta}(\underset{\sim}{r},\underset{\sim}{r}')\,\delta(t-t') \tag{3b}$$

In order to characterize the statistics completely, it remains however to obtain information on the covariance matrix $\{Q_{\alpha\beta}(\underset{\sim}{r},\underset{\sim}{r}')\}$. This in turn can only be achieved if the physical mechanisms at the origin of fluctuations are identified. So far this aspect has been overlooked in the literature : either one assigns an arbitrary structure to $\{Q_{\alpha\beta}\}$ (see chapters by Benzi, de Elvira and Saltzman and Sutera, in this Volume) or, at best, one incorporates in the description some constraints arising from symmetry arguments (North and Cahalan, 1981). As a result, in systems involving several variables one is confronted with a large number of unknown parameters. This obscures considerably the understanding of the mechanisms which are at the origin of climatic variability.

Our work provides a first step toward a more fundamental characterization of the noise associated with climatic variability. The basic idea is the following : Fluctuations originate, typically, in the form of localized, small scale events and as a result they cannot perceive the nonequilibrium constraints conferring a nonlinear character to the overall dynamics. It is therefore highly plausible that such properties as microscopic reversibility and fluctuation-dissipation like relations still hold for the random forces descriptive of these fluctuations, even if they may be compromized for the state variables themselves.

The search of such a fluctuation-dissipation theorem of the "second kind" will be the central point of our work. As a byproduct, this procedure will enable us to determine the characteristics of the fluctuations of the state variables and thus obtain information on the type of noise characterizing climate variability.

The general formulation of these ideas, which are patterned after the Landau-Lifshitz theory of fluctuations in fluid dynamics, is presented in Section 4.3. In Section 4.4 we apply the theory to a zonally averaged energy balance climate model and derive expressions for the correlation matrix of the effective random forces. Section 4.5 is devoted to the detailed analysis of the implications of this result, using the two-mode truncation of the model of Section 4.4. In particular, we show that the globally averaged surface temperature constitutes a non-Markovian process with red noise spectrum. Some comments and suggestions are presented in Section 4.6.

4.3 A FLUCTUATION-DISSIPATION THEOREM FOR ENERGY-BALANCE MODELS : GENERAL FORMULATION

In order to avoid unnecessary complications and present the various ideas as clearly as possible we focus, in the major part of this paper, on energy balance climate models. We do not comment on the range of validity of the description afforded by such models, referring to North et al. (1981) for a recent survey. We want to emphasize however that most of the basic ideas expressed below are model-independent.

The general form of the evolution equation in an energy balance model is :

$$\frac{\partial e}{\partial t} \equiv c\,\frac{\partial T}{\partial t} = f(T,\underset{\sim}{r}) - \mathrm{div}\,\underset{\sim}{J} \qquad (4)$$

Here e (energy per unit surface) is the energy content of a column having a unit cross section and a height of the order of the depth of the mixed layer. c is the heat capacity of this column, T the surface temperature, f the radiation budget (difference between incident solar flux and outgoing infrared radiation) :

$$f(T,\underset{\sim}{r}) = F_S - F_{IR} \qquad (5)$$

Finally $\underset{\sim}{r}$ is the space coordinate, and $\underset{\sim}{J}$ the energy flux.

We would now like to express the mechanism by which fluctuations appear in the balance equation (4). As discussed extensively in the theory of irreversible processes (see e.g. Prigogine, 1967; Landau and Lifshitz, 1959) fluctuations originate from the dynamics of the dissipative degrees of freedom. In order to identify the latter we have to turn to the entropy balance equation associated to Eq.(4) and construct the expression of the rate of dissipation per unit time - the entropy production. For an energy balance model of the kind considered in the present Section this was accomplished in Nicolis and Nicolis (1980). The result is

$$\text{entropy production} \equiv \frac{d_i S}{dt} = \int d\underset{\sim}{r}\,\underset{\sim}{J}\cdot\underset{\sim}{\nabla}T^{-1} \qquad (6)$$

In other words the radiative terms $f(T,\underset{\sim}{r})$ do not contribute to the dissipative behavior of the system in this approximation. We are therefore forced to conclude that the only way fluctuations can influence the energy balance equation is through the heat flux $\underset{\sim}{J}$. In the framework of the deterministic description it is customary to express $\underset{\sim}{J}$ by a Fourier type law, the difference being that the proportionality coefficient is here the eddy diffusivity rather than the molecular heat conductivity. In the presence of fluctuations the situation will change, and we will have

$$\underset{\sim}{J}(\underset{\sim}{r},t) = - D'\underset{\sim}{\nabla}\ T + \underset{\sim}{j}(\underset{\sim}{r},t) \tag{7}$$

D' is the eddy diffusivity which for simplicity is taken to be both $\underset{\sim}{r}$ and t-independent and $\underset{\sim}{j}(\underset{\sim}{r},t)$ is the spontaneous heat flux arising from fluctuations. Substituting into Eq.(4) we obtain a stochastic differential equation of the form given in Eq.(3a) :

$$c\,\frac{\partial T}{\partial t} = f(T,\underset{\sim}{r}) + D'\ \nabla^2 T + F(\underset{\sim}{r},t) \tag{8a}$$

which displays the random force term

$$F(\underset{\sim}{r},t) = - \text{div}\ \underset{\sim}{j}(\underset{\sim}{r},t) \tag{8b}$$

We want now to establish the properties of the random field $\underset{\sim}{j}(\underset{\sim}{r},t)$ from which the properties of the effective random force F will follow automatically. Let us forget for a moment that we are dealing with the earth-atmosphere system and consider a simple heat conducting fluid operating in the vicinity of thermodynamic equilibrium. In this case a fluctuation-dissipation theorem linking $\underset{\sim}{j}(\underset{\sim}{r},t)$ to the phenomenological coefficient D' was derived by Landau and Lifshitz (1959) on intuitive grounds and further justified by Fox and Uhlenbeck (1970). It has the form :

$$\langle j_\ell(\underset{\sim}{r},t)\rangle = 0 \tag{9a}$$

$$\langle j_\ell(\underset{\sim}{r},t) j_m(\underset{\sim}{r},t')\rangle = 2\ \kappa_B\ D'\ \bar{T}^2 \delta_{\ell m}\ \ \delta(\underset{\sim}{r}-\underset{\sim}{r}')\,\delta(t-t') \tag{9b}$$

where κ_B is the Boltzmann's constant and $\bar{T}$ the equilibrium temperature. In other words different components of the random part of the heat flux are uncorrelated between themselves, the values of a given component at different space regions are also uncorrelated, and finally the whole process constitutes a Gaussian white noise in time.

As pointed out in the Introduction, fluctuations originate spontaneously in the form of localized, short scale events independently of the distance from the state of thermodynamic equilibrium. We therefore expect that the above properties will still hold under the highly nonequilibrium conditions characterizing the climatic system (see Nicolis and Prigogine, 1977; Keizer, 1978 for a general discussion of this point). We therefore write

$$< j_{\ell}(\underset{\sim}{r},t) > = 0 \tag{10a}$$

$$< j_{\ell}(\underset{\sim}{r},t) j_m(\underset{\sim}{r}',t') > = \sigma^2 \delta_{\ell m} \; \delta(\underset{\sim}{r}-\underset{\sim}{r}')\delta(t-t') \tag{10b}$$

In other words we require j_{ℓ} to have the same spatial and temporal properties as in equilibrium but allow for a different numerical factor, σ^2, which replaces the factor $2 \; \kappa_B \; D' \; \bar{T}^2$ in Eq.(9b) and expresses the strength of fluctuations far from equilibrium. In the sequel σ^2 will be treated as a parameter. Our analysis will focus solely on the consequences of the fact that j is delta-correlated both in space and time. Note that, because of Eq.(8b), the effective random force F appearing in the energy balance equation (8a), is not delta correlated in space. The consequences of this rather important property are examined in Section 4.5.

Eqs.(10) are somewhat reminiscent of the formalism developed recently by North and coworkers (North et al., 1981; North and Cahalan, 1981). However, to our knowledge these authors have not incorporated in their description the physical constraint that the effective random force F is the divergence of a random vector field and that the latter is a white noise both in space and time. True, our expressions may be considered as a special case of a general covariance matrix $\{ Q_{\alpha\beta}(\underset{\sim}{r},\underset{\sim}{r}')\}$ of the random force field (cf. Eq.(3b)). However, in the theory of fluctuations it is important to become gradually as specific as possible in order to incorporate the maximum amount of information available from first principles.

Finally, let us emphasize that the procedure outlined above can be extended readily to any other kind of conservation equation like, for instance, the momentum balance equation :

$$\rho \frac{d\mathbf{v}}{dt} = - \nabla p - \operatorname{div} \boldsymbol{\sigma}$$

in which ρ is the mass density, $\mathbf{v}$ is the velocity field, p the hydrostatic pressure, $\boldsymbol{\sigma}$ the dissipative part of the stress tensor, and d/dt the hydrodynamic derivative. For an incompressible fluid the analogue of Eq.(7) is :

$$\sigma_{ij} = -\eta \left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right) + S_{ij}$$

η being the shear viscosity and S_{ij} the fluctuating stress tensor. Eqs.(9) are now to be replaced by (Landau and Lifshitz, 1959) :

$$\langle S_{ij}(\mathbf{r},t)\rangle = 0$$

$$\langle S_{ij}(\mathbf{r},t)S_{\ell m}(\mathbf{r}',t')\rangle = 2\,\kappa_B\,\bar{T}\,\eta\,(\delta_{i\ell}\,\delta_{jm} + \delta_{im}\,\delta_{j\ell} - \frac{2}{3}\,\delta_{ij}\,\delta_{\ell m})\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t')$$

fixing the statistics of the effective random force present in the momentum balance equation. The transition from this near-equilibrium result to the highly nonlinear regime of climate dynamics would naturally follow the same lines as above and would lead to an equation similar to (10b), in which the factor $2\kappa_B\bar{T}\eta$ is replaced by a single parameter σ^2 expressing the strength of fluctuations far from equilibrium. Some properties of the noise of the vector field $\mathbf{v}$ are reported in Nicolis et al. (1983) along with an application to the Lorenz model.

4.4 APPLICATION TO A ZONALLY-AVERAGED MODEL

We now simplify further the model introduced in the previous Section by taking into account only the energy transfer in the meridional direction. In this zonally-averaged, one-dimensional (1-d) model Eqs.(8) take the form (North, 1975)

$$c\,\frac{\partial T}{\partial t}(x,t) = f(T,x) + D\,\frac{\partial}{\partial x}(1-x^2)\,\frac{\partial T}{\partial x} - \frac{1}{R}\,\frac{\partial}{\partial x}(1-x^2)^{1/2}\,j(x,t)$$

$$\equiv f(T,x) + D\frac{\partial}{\partial x}\,(1-x^2)\frac{\partial T}{\partial x} + F(x,t) \qquad (11)$$

To arrive at Eq.(11) one has also to perform a change to spherical coordinates on the earth's surface. x is the sine of the latitude, R the earth's radius, and D an effective heat diffusion coefficient defined by $D=D'/R^2$.

In order to find the form of the generalized fluctuation-dissipation theorem, Eqs.(10), in these coordinates, we first observe that in the framework of our 1-d model, only the localization in the θ-direction needs to be expressed. The ϕ-dependence contained in the delta function $\delta(\underset{\sim}{r}-\underset{\sim}{r}')$ should therefore merely enter as part of the Jacobian of the transformation to spherical coordinates, namely $R^2\Delta\phi \cos\theta$. Thus

$$\delta(\underset{\sim}{r}-\underset{\sim}{r}') = \frac{1}{R^2\Delta\phi \cos\theta} \delta(\theta-\theta')$$

The size of $\Delta\phi$ has to be chosen in such a way that $R^2\Delta\phi$ reduces to the unit surface element for which the energy balance equation is usually written. Furthermore,

$$\delta(x-x')=\delta(\sin\theta - \sin\theta') = \frac{1}{\cos\theta} \delta(\theta-\theta')$$

We obtain therefore

$$\delta(\underset{\sim}{r}-\underset{\sim}{r}') = \frac{1}{R^2\Delta\phi \cos\theta} \cos\theta \ \delta(x-x')=\delta(x-x') \tag{12}$$

and hence (cf. Eq.(10b)):

$$\langle j(x,t)j(x',t')\rangle = \sigma^2 \delta(x-x')\delta(t-t') \tag{13}$$

From this relation as well as Eq.(8b) we may compute the autocorrelation function of the effective random force

$$\langle F(x,t)F(x',t')\rangle =$$

$$\langle\frac{1}{R}\frac{\partial}{\partial x}(1-x^2)^{1/2}j(x,t) \cdot \frac{1}{R}\frac{\partial}{\partial x'}(1-x'^2)^{1/2}j(x',t')\rangle$$

$$= (\frac{\sigma}{R})^2 \frac{\partial}{\partial x}(1-x^2)^{1/2} \frac{\partial}{\partial x'}(1-x'^2)^{1/2}\delta(x-x') \tag{14}$$

As usual (North, 1975) we expand both the temperature and the random force fields in series of Legendre polynomials :

$$T = \sum_n T_n(t)\, P_n(x) \tag{15a}$$

$$F = \sum_n F_n(t)\, P_n(x) \tag{15b}$$

Inserting relation (15b) into Eq.(14) multiplying by $P_m(x)\, P_\ell(x')$ and integrating over x and x' we obtain :

$$\frac{2}{2\ell+1}\,\frac{2}{2m+1} < F_m\, F_\ell > = (\frac{\sigma}{R})^2 \quad \delta(t-t') .$$

$$\int dx\, dx'\, P_m(x)\, \frac{\partial}{\partial x}\, (1-x^2)^{1/2}\, P_\ell(x')\, \frac{\partial}{\partial x'}\, (1-x'^2)^{1/2}\, \delta(x-x')$$

Integrating by parts and performing the delta function we finally obtain :

$$<F_m\, F_\ell> = q_m^2\, \delta_{\ell m} \quad \delta(t-t') \tag{16a}$$

with

$$q_m^2 = \frac{2m+1}{2} \quad m(m+1)\, (\frac{\sigma}{R})^2 \tag{16b}$$

Several important conclusions can be drawn from these expressions. First, the random forces associated to different Legendre modes are uncorrelated. Second, there is no random force acting directly on the equation for the globally averaged temperature T_0, as $q_0=0$ from Eq.(16b). And third the importance of the fluctuations depends on the order of the Legendre mode. For the first few modes q_m is expected to be small, because of the presence of the inverse square of the earth's radius in Eq.(16b). For higher m however q_m rapidly increases, varying roughly as m^3. Now as well known higher order Legendre modes represent localized disturbances. We reach therefore a very natural conclusion, namely that the importance of fluctuations depends on their space scale. This is reminiscent of the phenomenon of nucleation frequently encountered in phase transitions. The above points would of course have been missed completely if we had proceeded formally, keeping a general, unspecified structure for the covariance matrix of the random forces.

4.5 THE TWO-MODE TRUNCATION

In this Section we explore the consequences of Eqs.(16a) and (16b) on the two-mode truncation of Eq.(4), in order to obtain explicit information on the statistics of the temperature field. As well known (North et al., 1981) the first two Legendre modes provide a reasonably good description of the large scale features of the meridional temperature distribution and heat flux.

The starting equations are obtained by inserting (15b) into (4), multiplying by $P_m(x)$, integrating over x, and recalling that $F_0=0$. We thus have :

$$c \frac{dT_0}{dt} = Q\, H_0(x_s) - (A+B\, T_0) \tag{17a}$$

$$c \frac{dT_2}{dt} = Q\, H_2(x_s) - (B+6D)T_2+F_2(t) \tag{17b}$$

Here Q is the solar constant divided by 4, A and B are cooling coefficients arising in the parameterization of the infrared radiation flux :

$$F_{IR} = A+BT(x,t)$$

and H_0, H_2 are the first two Legendre moments of the coalbedo a(x) multiplied by the mean annual distribution, S(x), of radiation reaching the top of the atmosphere (North, 1975). Both H_0 and H_2 depend on the position of the ice edge x_s, which is related to T_0 and T_2 through the Budyko boundary condition (Budyko, 1969)

$$T_0(t) + T_2(t)\, P_2(x_s) = -10°C \tag{18}$$

Let T_0^*, T_2^*, x_s^* be a reference steady-state solution of Eqs.(17) representative, say, of the present-day climate. We first focus on the short term variability of this climate. To this end it is sufficient to analyse the linear response by introducing the small deviations

$$\begin{aligned} \theta_0 &= T_0 - T_0^* \\ \theta_2 &= T_2 - T_2^* \end{aligned} \tag{19}$$

$$\xi = x_s - x_s^*$$

Linearizing Eqs.(17), (18) with respect to these quantities we obtain :

$$\xi = - \frac{1}{T_2^* \, P_2'(x^*)} [\theta_0 + \theta_2 \, P_2(x_s^*)] \\ = \alpha\theta_0 + \beta\theta_2 \tag{20}$$

and

$$c \frac{d\theta_0}{dt} = (Q \, H_0' \, \alpha - B) \, \theta_0 + Q \, H_0' \, \beta \, \theta_2 \tag{21a}$$

$$c \frac{d\theta_2}{dt} = Q \, H_2' \, \alpha \, \theta_0 - (B + 6D - Q \, H_2' \, \beta)\theta_2 + F_2(t) \tag{21b}$$

where the prime denotes differentiation of H_0, H_2, P_2 with respect to their argument.

The first point we make in connection with these equations is motivated by a result of one of the authors (Nicolis, 1980) concerning the passage from 1-d to 0-d climate models. Specifically, by eliminating θ_2 in terms of θ_0 from Eq.(21b), we obtain a closed equation for θ_0. Contrary to the deterministic analysis reported (Nicolis, 1980), this equation contains now an effective noise term, as θ_2 depends on the random force $F_2(t)$. To evaluate the characteristics of this noise we first solve Eq.(21b) with the initial condition that $\theta_2=0$ at $t=-\infty$. We find :

$$\theta_2(t) = e^{-\gamma t} \, \frac{1}{c} \int_{-\infty}^{t} dt' \, [Q \, H_2'\alpha \;\; \theta_0(t') + F_2(t')] e^{\gamma t'} \tag{22}$$

with

$$\gamma = \frac{1}{c} (B + 6D - QH_2' \, \beta) \tag{23}$$

We thus obtain a stochastic differential equation for θ_0 :

$$\frac{d\theta_0}{dt} = \frac{1}{c}(Q H_0'\alpha - B)\,\theta_0 + \frac{1}{c^2} Q^2 H_0' H_2' \alpha\beta \int_{-\infty}^{t} dt'\, e^{-\gamma(t-t')}\, \theta_0(t')$$

$$+ \frac{1}{c^2} Q H_0' \beta \int_{-\infty}^{t} dt'\, e^{-\gamma(t-t')}\, F_2(t') \qquad (24)$$

Let us denote by $\Phi_0(t)$ the last term of the right hand side. The correlation function of this effective noise term is :

$$\langle\Phi_0(t)\ \Phi_0(t')\rangle =$$

$$\left(\frac{QH_0'\beta}{c^2}\right)^2 \int_{-\infty}^{t} d\tau_1 \int_{-\infty}^{t'} d\tau_2\, e^{-\gamma(t-\tau_1)}\, e^{-\gamma(t'-\tau_2)}\ \langle F_2(\tau_1)\ F_2(\tau_2)\rangle$$

Utilizing Eq.(16a) for m=2 we obtain after some algebra :

$$\langle\Phi_0(t)\ \Phi_0(t')\rangle = \left(\frac{QH_0'\beta}{c^2}\right)^2 q_2^2\, \frac{1}{2\gamma}\, e^{-\gamma|t-t'|}$$

$$\equiv r_0^2\, \frac{1}{2\gamma}\, e^{-\gamma|t-t'|} \qquad (25)$$

This relation is characteristic of an Ornstein-Uhlenbeck process (Soong, 1973). In other words, the effective noise acting on the globally averaged surface temperature θ_0 is <u>not</u> a white noise, characteristic of a diffusion process, but a colored noise having an exponentially decaying time correlation function. Inasmuch the only Markovian process with continuous realization is a diffusion process, it therefore follows (Arnold, 1974) that θ_0 is a non-Markovian process. Therefore, if fluctuations are incorporated in a reduced description (like for instance in a zero-dimensional climate model) involving a limited number of variables (like for instance the mean average surface temperature), one should be particularly careful in the modelling of the corresponding random forces. On the other hand, the pair (θ_0, θ_2) is still Markovian and thus amenable to a description in terms of a bivariate Fokker-Planck equation.

In order to see the repercussions of this property on the characteristics of θ_0 and θ_2 themselves, we solve Eqs.(21) by the method of Fourier transforms. Defining

$$\theta_0(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \bar{\theta}_0(\omega) \qquad (26a)$$

$$\theta_2(t) = \int_{-\infty}^{\infty} d\omega \; e^{-i\omega t} \; \bar{\theta}_2(\omega) \tag{26b}$$

we obtain by a straightforward algebra :

$$\bar{\theta}_2(\omega) = \frac{1}{-i\omega+\gamma} \, Q \, H_2' \, \alpha \, \bar{\theta}_0(\omega) + \frac{1}{-i\omega+\gamma} \, \bar{F}_2(\omega) \tag{27}$$

and

$$\bar{\theta}_0(\omega) = - Q \, H_0' \beta \, \frac{\bar{F}_2(\omega)}{Q^2 H_0' H_2' \alpha\beta + \gamma(QH_0'\alpha - B) + \omega^2 + i\omega(\gamma + B - QH_0'\alpha)} \tag{28}$$

For simplicity we have normalized our time scale so that c=1. From relation (28) we can evaluate the autocorrelation function of $\bar{\theta}_0(\omega)$, noting that from Eq.(16a) one has

$$\langle \bar{F}_2(\omega) \; \bar{F}_2(\omega') \rangle = q_2^2 \, \frac{1}{2\pi} \, \delta(\omega+\omega') \tag{29}$$

By switching back to the time variables one finally obtains :

$$\langle \theta_0(t) \; \theta_0(t') \rangle = \frac{(QH_0'\beta)^2}{2\pi} \, q_2^2 \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega|t-t'|}}{(K+\omega^2)^2 + \omega^2 L^2} \tag{30}$$

where we set

$$K = Q^2 \, H_0' \, H_2' \, \alpha\beta + \gamma(Q \, H_0' \, \alpha - B)$$

$$L = \gamma + B - Q \, H_0' \alpha \tag{31}$$

The integral in Eq.(30) can be performed by the calculus of residues. The result is

$$\langle \theta_0(t) \; \theta_0(t') \rangle = \frac{QH_0'\beta}{|L| \, (L^2+4K)^{1/2}} \, q_2^2$$

$$\left[\frac{1}{2\omega_1} \, e^{-\omega_1 (t-t')} + \frac{1}{2\omega_2} \, e^{-\omega_2 (t-t')} \right] \tag{32}$$

Here ω_1, ω_2 are the singularities of the integrand of Eq.(30) :

$$\omega_1 = \frac{1}{2} [|L| + (L^2 + 4K)^{1/2}]$$

$$\omega_2 = \frac{1}{2} [|L| - (L^2 + 4K)^{1/2}]$$

If $L^2 + 4K > 0$, expression (32) describes the superposition of two decaying exponentials with different characteristic times. This is to be contrasted with the Ornstein-Uhlenback process, whose autocorrelation function is a single decaying exponential (cf. Eq.(25)). If the effective noise acting on the equation for θ_0 were a Gaussian white noise, the autocorrelation function of θ_0 would be of this latter type. The difference with the present situation, Eq.(32), is therefore due to the fact that the effective noise acting on θ_0 is not a white noise, which in turn implies that θ_0 is a non-Markovian process.

Fig. 4.1 illustrates the difference between the power spectra of the two types of stochastic process. Both spectra are characteristic of red noise processes which are commonly observed in atmospheric phenomena (Gilman et al., 1962; Hasselmann, 1981). However, in the case analyzed in the present paper (curve (a)) the spectrum is localized more sharply in the low frequency range as compared to the spectrum characterizing the Ornstein-Uhlenbeck process (curve (b)). This reflects the additional "filtering" achieved by the spatially inhomogeneous fluctuations. The evaluation of the power spectrum of θ_2, using Eq.(27), leads to similar conclusions. The situation would be altogether different for $L^2+4K<0$, as in this case the correlation function of θ_0 would exhibit damped oscillations. However, for the parameter values usually adopted in the deterministic analysis of Eqs.(17), L^2+4K turns out to be positive. This possibility has therefore to be ruled out.

A very interesting case arises when $K \to 0$. One can easily see from the deterministic version of Eqs.(21) and the definitions (31) that the limit corresponds to the occurrence of a climatic transition in the form of bifurcation of new branches of solutions of the initial nonlinear equations[(*)]. From Eq.(32) it then follows that the second exponential acquires a long time tail and a divergent amplitude. In other words, a climatic change appears to be "signalled" by an enhancement of the amplitude and lifetime of the correlation function of the fluctuations of the climatic variables. This is reminiscent of the phenomenon of critical divergencies familiar from phase transitions. Actually, this conclusion is characteristic of a mean-field theory of phase transitions (Ma, 1976), whereby only the effect of long wavelength spatial fluctuations is taken into account.

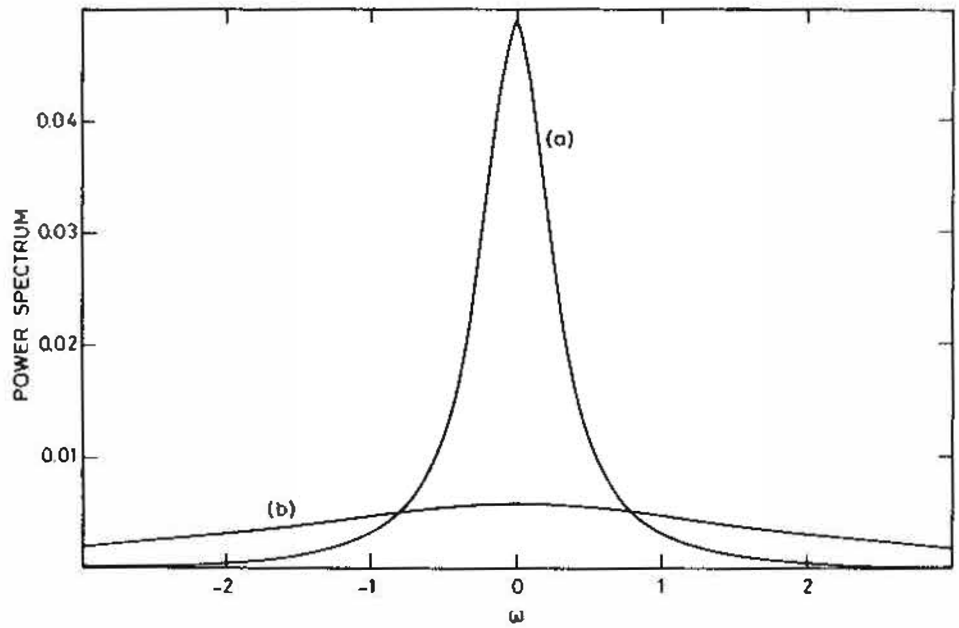

Fig. 4.1. Power spectrum associated with the autocorrelation function of the mean surface temperature. Eq.(32) (curve (a)) and an Orstein-Uhlenbeck process (curve (b)), for $q_2^2 = 1$ Kcal cm^{-2} yr^{-1}. The parameters of the model described in Eq.(21) are as follows : Q=252.76 Kcal cm^{-2} yr^{-1}, A=159.24 Kcal cm^{-2} yr^{-1}, B=1.17 Kcal cm^{-2} yr^{-1}, D=0.44 Kcal cm^{-2} yr^{-1}, $S(x)=1-0.477\ P_2(x)$, $a(x)=0.697-0.00779\ P_2(x)$ for $x < x_s$ and $a(x)=0.38$ for $x > x_s$, $T_0^*=14.9°C$, $T_2^*=-28.2°C$, $x_s^*=0.96$.

(*) Note that for the parameters values fitting the present climate the case K→0 cannot be realized for the energy balance model corresponding to Eqs.(17).

4.6 DISCUSSION

Our principal goal in this work was to incorporate into the description of fluctuations in climate dynamics certain physical constraints suggested by the statistical theory of irreversible processes. This was achieved by applying a fluctuation-dissipation type of relation to the random forces appearing in the deterministic balance equations. Clearly, the above idea holds for quite general situations and can thus be applied to complex sets of coupled evolution equations, as long as the local description of the processes involved remains a legitimate approximation. In the present paper however we preferred to keep the formalism as simple and explicit as possible, and for this reason we carried out the main part of the analysis for the particular - although quite representative - class of zonally averaged energy balance models. This provided us with extensive information on the characteristics of the fluctuations of the temperature field.

We have first shown that when a Legendre expansion of the noise and temperature field is performed there is no random force associated with the equation for the globally averaged surface temperature. As a byproduct, in the framework of a two-mode truncation, it follows that the noise source

$$F(x,t) = F_2(t)\ P_2(x)$$

is more important in high latitudes ($P_2 \to 1$) than in equatorial latitudes ($P_2 \to 1/2$). This might provide a plausible interpretation for the observations (see for instance Leith, 1978 and quite recently, North et al., 1982).

A second result of interest is that the effective noise acting on the equation for the globally averaged surface temperature θ_0 is an Ornstein-Uhlenbeck rather than a Gaussian white noise. As a result, θ_0 is a non-Markovian process. It also displays the characteristics of red noise and is picked around $\omega = 0$ more sharply than if the effective noise were white. These features should play an important role in the interpretation of power spectra associated with climatic variability.

Finally, we have seen that a transition in the climatic system in the form of a bifurcation of new branches of solutions is reflected by the emergence of persistent, high amplitude correlations. We believe that this property should provide a useful index of climatic change.

The specific application considered in Section 4.5 was limited to a linear response analysis around the present climate. On the other hand the fluctuation-dissipation theorem of the second kind utilized in the present work should remain valid in the nonlinear range since, as we repeatedly emphasized, it reflects the short range character of the fluctuation sources. It follows that the results of Sections 4.3 and 4.4 provide also the basis of an analysis in which the multiplicity of solutions of the fully nonlinear model, Eqs(17), is taken into account. This would allow us to investigate for instance, the passage times between the different climatic states (see Nicolis and Nicolis, 1981 for the estimation of these times in a zero-dimensional model). Moreover, it would be highly desirable to set up a suitable description of localized, short wavelength fluctuations. We will report on these problems in future work.

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