

where f is generally nonlinear in x and depends, in addition, on a number of external parameters λ . A typical example of Eq. (1) is the mean annual energy budget of the earth-atmosphere system. The variable x then represents the mean annual surface temperature, whereas λ may stand for such quantities as the solar constant, the albedo of land or ice, etc.

In most problems of interest the rate function f can be split into two parts corresponding to qualitatively different mechanisms: a highly nonlinear term $X(x, \lambda)$ describing the specific feedbacks inherent in the system's dynamics, and a quasi-linear, non-specific decay term. The latter usually ensures the existence of stationary solutions to Eq. (1). We thus write

$$\frac{dx}{dt} = X(x, \lambda) - kx \quad (2)$$

where k is some appropriate (positive) decay constant. For the illustration of Eq. (1) given above X would stand, for instance, for the effect of the surface-albedo feedback and k for the infrared cooling coefficient.

Consider now the following quite typical situation. Assume that for $0 < x < x_0$, X is very small but that for $x > x_0$ it suddenly becomes appreciable and saturates shortly thereafter to some threshold value X_{\max} . To capture the essence of this response, we regard X as a discontinuous, *Boolean function* of the discontinuous *Boolean variable* x . In other words, we consider that in the interval $0 < x < x_0$ both x and X are zero whereas for $x > x_0$, x and X are equal to one (cf. Fig. 1). We will say that x is 'off' and 'on' respectively when x is below and beyond the threshold value x_0 . Similarly for X . It may of course happen

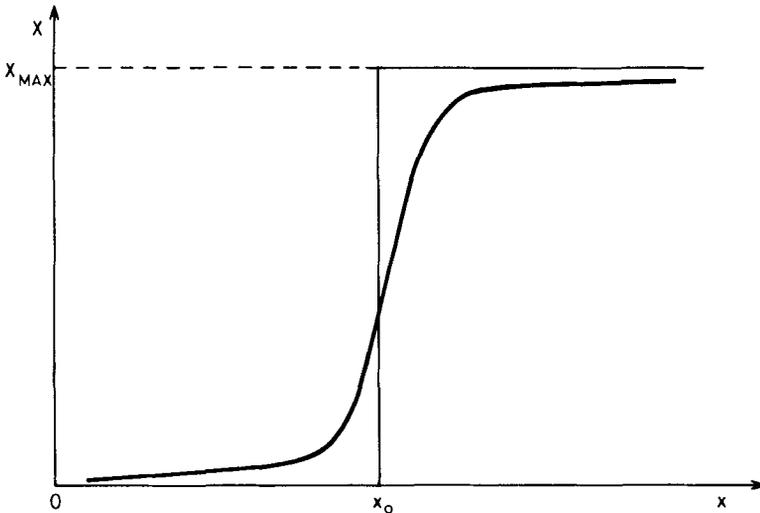


Figure 1. Boolean idealization of a sigmoidal dependence by a step function.

that intermediate levels between 0 and the maximum value have to be considered. An example is analysed later on in the present paper. Note that, because of the existence of the second term in Eq. (2), the temporal evolution of a function and of its corresponding variable will differ by a phase-shift, usually called *delay*, expressing the fact that when a function X becomes 'on' the corresponding variable x will build up only after some characteristic delay t_x depending on the value of k in Eq. (2). Similarly, when X is 'off' x will not disappear until another characteristic delay t_x has elapsed (Fig. 2).

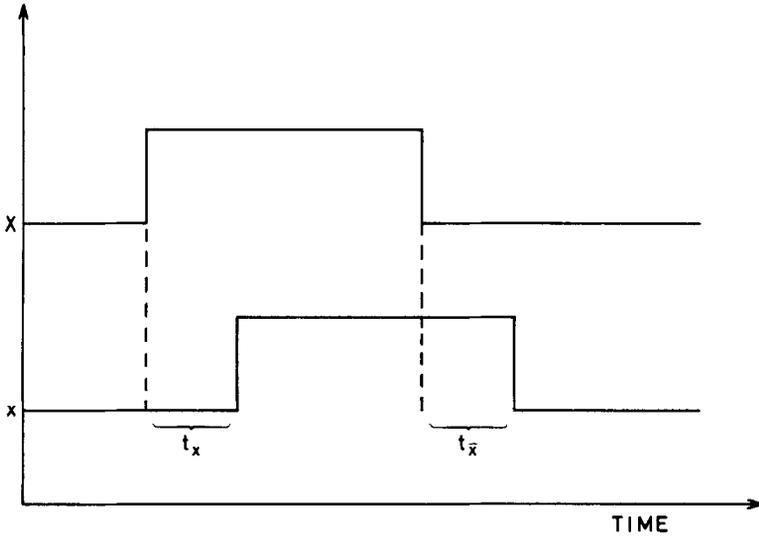


Figure 2. Schematic representation of the temporal evolution of a Boolean function X and its associated variable x .

More generally, let x, y, z, \dots be Boolean variables taking the values 0 or 1 according as the quantity they represent has negligible or appreciable values. The Boolean functions X, Y, Z, \dots will take the value 1 when a certain set of ‘commanding variables’ is building up or remains ‘on’ and the value 0 otherwise. We again emphasize that the functions X, Y, Z, \dots are taken to include only the feedbacks exerted in the system. It is assumed that there always exists a quasi-linear, non specific mechanism of the decay of x, y, z, \dots . This latter mechanism is not accounted for in the logical equations describing the system, but is incorporated in the delays.

Having adopted this representation, we immediately see that a Boolean state in which all variables take the same values as the corresponding functions is a stable steady state. Indeed, when such a condition is satisfied neither the variables nor the functions will need to change further in time. As regards the succession of the different states in time, during the approach to the stable steady state or to some other attractor, it will be dictated by appropriate *logical equations*, which link X, Y, Z, \dots to x, y, z, \dots and are the substitutes of the differential equations of energy, mass, or momentum balance written down in the traditional, continuous formalism. In writing such logical equations it will be useful to recall the following conventions:

$$\begin{aligned} \bar{x} & \quad \text{not } x \\ x \cdot y & \quad x \text{ and } y \\ x + y & \quad x \text{ or } y \text{ (inclusive)} \end{aligned}$$

We do not discuss the general theory any further, but rather proceed with applications to concrete situations of interest in climate modelling. We shall limit ourselves here to globally averaged, *zero dimensional* (0-d) models.

In Section 2 the logical formalism is applied to simple climatic models, and the results are compared with the predictions of the continuous formalism. Section 3 is devoted to more complex situations for which no continuous description has been set up so far. A brief discussion is presented in Section 4.

2. LOGICAL ANALYSIS OF SIMPLE MODELS

(a) *Surface albedo feedback*

Let θ be the surface temperature. Within the framework of a 0-d energy balance model only the surface albedo feedback is retained, which is positive. It expresses the fact that when θ increases, its rate of increase T is further enhanced because of the additional absorption of solar energy arising from the retreat of ice. Hence we write the logical scheme and the corresponding logical equation.

$$+ \begin{array}{c} \curvearrowright \\ \theta \\ T = \theta. \end{array}, \quad \text{or} \quad (3)$$

To analyse the predictions of this equation we arrange the states of the system in the form of a *matrix* or *truth table*, as follows

θ	T
0	①
1	①

For each level of the variable θ , the corresponding rate of change T is deduced from the logical equation (3). The circled elements in the matrix correspond to stable states. We see that we may obtain up to two such stable states. Of these, the high temperature state can be regarded as representing a favourable (present-day like) climate, whereas the low temperature one is descriptive of a cool, unfavourable climate. This agrees with the prediction of the continuous model analysed by Crafoord and Källén (1978) or of any other 0-d model where the albedo-temperature relationship saturates at low and high temperature values.

It should be mentioned that in the continuous analysis of this type of model a third, unstable steady state is obtained which lies between the two stable ones. In the discontinuous analysis unstable states can only manifest themselves whenever they can be approached through some exceptional path, like e.g. the separatrix associated with a saddle point. In our one-variable system such a path is excluded. As a result the Boolean analysis cannot account for the intermediate unstable state, which is physically insignificant anyway.

(b) *Effect of ice sheets*

The positive feedback, Eq. (3), may in fact be attributed to the interaction between energy and ice sheet balance equations. Let l be the mean annual latitudinal extent of ice sheets. The simplest such interactions are that θ and l exert a negative feedback on each other. This is represented by:

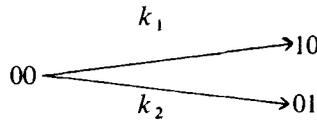
$$\begin{array}{c} \curvearrowleft \\ \theta \\ \curvearrowright \\ l \end{array}, \quad \text{or} \quad (4a)$$

$$\begin{array}{c} T = l \\ L = \theta \end{array} \quad \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \quad (4b)$$

The system has now $2^2 = 4$ states, and the analogue of the matrix of the preceding example becomes:

θ	l	T	L
0	0	1	1
0	1	0	1
1	1	0	0
1	0	1	0

We find two stable steady states corresponding, roughly, to an 'ice covered' and an 'ice free' earth. In actual fact, depending on the value of the threshold separating the two regimes, 'ice covered' may mean a climate with a large ice extent, and 'ice free' a warm climate with small or no ice extent. This result is in agreement with the conclusion of our previous example, thus illustrating a general theorem by Thomas (1978) that a logical loop involving an even number of negative elements behaves qualitatively as a positive one. Note that an initial condition may lead to either of these two stable states, depending on the relative values of the delays. For instance suppose that we start with a low temperature 'ice free earth' (0, 0). According to the first row of the logical matrix, both θ and l level will tend to change. As each of these changes has most probably different delay values a simultaneous switching to (1, 1) is improbable. Thus, the system will tend to relax to one of the two stable states according to



The $00 \rightarrow 10$ pathway will be preferred when the decay time of θ in Eq. (2) is faster, whereas $00 \rightarrow 01$ will be preferred in the case in which the decay time of l is faster.

It should of course be emphasized that the interaction between the two variables is not direct, but takes place through such processes as ablation, precipitation etc. We come to these points presently.

(c) *Free oscillations*

A set of two coupled differential equations for the energy and land ice sheet balance at the planetary scale has been recently suggested and analysed in detail by Källén *et al.* (1978 and 1979). One of the most important points in their model concerns the temperature dependence of the ratio of accumulation over ablation rates. They suggest that in some temperature range there must be a positive feedback exerted by the temperature on the land ice extent, since accumulation implies precipitation and snow fall which is very small for very low temperatures. In this way they find that the coupled atmosphere-hydrosphere-cryosphere system can undergo self-oscillations with periods comparable to those of the glaciation times.

Let us outline the logical formulation of this problem. According to the energy balance equation set up by the authors, there exists a large temperature range in which temperature θ , exerts a positive feedback on itself via the sea ice extent, in a way analogous to the one examined in case (a). In the same range however, the land ice extent l , has the opposite effect on θ . The logical scheme for the energy balance equation alone is therefore the following

$$T = \theta + l, \quad \text{or} \quad \theta \xrightarrow{+} \theta \xleftarrow{-} l \quad (5)$$

On the other hand, similarly to case B above, temperature exerts a negative feedback on land ice l through ablation; this becomes effective beyond a certain threshold value, say θ_1 . In addition, when globally averaged temperatures attain a higher level, say θ_2 ($\theta_2 > \theta_1$), significant evaporation can take place and consequently precipitation and snow fall rates are high. Thus, the overall effect is a positive feedback of θ on l through accumulation. Still, an excessively large size of the ice sheet will slow down its further increase because of the parabolic profile of the sheets (Weertman 1964).

The fact that certain feedbacks change sign according to the state of the system suggests a decomposition of θ into at least three levels corresponding to the two thresholds θ_1 and θ_2 . The logical schemes and the corresponding logical equation for the rate of change L of the land ice extent are given by

$$\begin{array}{ccc}
 \theta_1 \xrightarrow{-} l & \text{and} & \theta_2 \xrightarrow{+} l \text{ (with a feedback loop)} \\
 L = \bar{\theta}_1 + \theta_2 \cdot l & &
 \end{array} \quad (6a)$$

The subdivision of θ into levels requires a slight modification of Eq. (5). It is replaced by the set of equations

$$T_1 = (\theta_1 + l) + \theta_2 \quad (6b)$$

$$T_2 = T_1 \cdot (\theta_2 + l) \quad (6c)$$

The feedbacks intrinsic to each level are $\theta_1 + l$ and $\theta_2 + l$, respectively. Eq. (6b) expresses the fact that θ_1 is automatically 'on' if θ_2 is 'on'; similarly in Eq. (6c), T_2 cannot be 'on' unless T_1 has already been activated. In other words, the system cannot get at level θ_2 without getting first through level θ_1 .

We are now in a position to write down the logical matrix for this problem. Of the $2^3 = 8$ states of the system only 6 are acceptable, because as we saw previously θ_2 cannot be 'on' when θ_1 is 'off'. We thus obtain the following representation of Eq. (6).

θ_1	θ_2	l	T_1	T_2	L
0	0	0	1	1	1
0	0	1	0	0	1
1	0	0	1	1	0
1	0	1	1	0	0
1	1	0	1	1	1
1	1	1	1	1	0

We find one stable state corresponding to a very low temperature earth with an important land ice extent. The new point however is that there exists no Boolean stable state at higher temperatures. Instead, the system, as we will see, can perform a cycle of the form

$$1 \ 1 \ 0 \rightleftharpoons 1 \ 1 \ 1$$

To show this, suppose that we start with the initial condition (1 1 0). According to the logical matrix (row before last), l is then bound to change. Once in state 1 1 1 however, the last row commands that l be changed again. This cycle suggests therefore the possibility of an oscillation in the vicinity of the θ_2 level with an important advance and retreat of the land ice extent. In actual fact, according to the threshold values given by Källén *et al.* (1978, 1979), this range of θ happens to be between the present day climate and the glacial one.

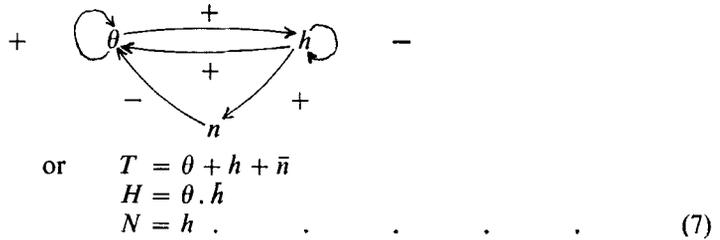
A still finer description of the system, permitting a closer agreement with the quantitative results of Källén *et al.* would be to include four θ levels, thus extending the range to even higher global temperature values. One finds an additional stable state in the high temperature range as well as the possibility of a competition between this state and the self-oscillation. These conclusions are in qualitative agreement with the results of the original paper.

3. LOGICAL ANALYSIS OF A COMPLEX NETWORK

Having illustrated the Boolean method on known simple examples, we proceed to set up plausible models for more complex situations.

Consider for instance, the still debatable coupling between global temperature θ , relative humidity h , and cloudiness n . In addition to the positive feedback of θ into itself discussed already in our previous examples, we expect a negative feedback of h into itself because of the slowing down of evaporation when relative humidity increases. A positive effect of h on θ expresses the greenhouse effect and a positive effect of h on n expresses the simple idea that the cloud build up is facilitated in a medium of high h . There remain some uncertainties in the effect of θ on h and of n on θ (θ is taken to act on n only indirectly through h). According to Sasamori (1975), $\partial h/\partial \theta > 0$. On the other hand, using a general circulation model Roads (1978) suggests that the effect should be just in the opposite direction. As regards the $n - \theta$ interaction, it actually depends on the type of cloud and on latitude (Paltridge, 1980 and references therein). Moreover, it is often suggested (Paltridge and Platt 1976) that n should have a globally negative effect on θ , in order to ensure stability.

We have examined the consequences of all these conjectures. Hereafter we only report the results in the case where $\partial n/\partial \theta < 0$, and $\partial h/\partial \theta > 0$. A plausible logical scheme incorporating the above interactions is as follows:



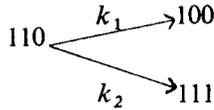
The corresponding matrix now comprises $2^3 = 8$ states and is:

θ	h	n	T	H	N
0	0	0	1	0	0
0	0	1	0	0	0
0	1	1	1	0	1
0	1	0	1	0	1
1	0	0	1	1	0
1	0	1	1	1	0
1	1	1	1	0	1
1	1	0	1	0	1

It gives no Boolean stable state but predicts, as we see presently, the possibility of a surprising variety of oscillatory behaviour.

As an exercise, suppose that the initial state is a climate characterized by a high temperature, high humidity level but no (or small) cloud extent (state 110). The logical matrix

(last row) commands that both humidity and cloudiness are bound to change. As the characteristic times (or delays) of these two processes are most probably different, a simultaneous change of h and n is to be ruled out. Instead, the system is given the following choice:



As in Section 2(b), the particular path to be followed will depend on the relative values of the delays, k_1 and k_2 . In other words, state 100 will be reached if the decay time of h is faster than that of n . In the opposite case state 111 will be reached.

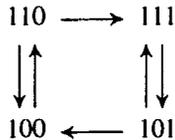
Suppose now that the first transition is realized and the system is at state 100. Going back to the logical matrix we see that h is again bound to change. Consequently, the following oscillatory behaviour is expected

$$110 \rightleftharpoons 100$$

If on the contrary $k_2 > k_1$ and state 111 is reached, one will have the scenario

$$110 \rightarrow 111 \rightleftharpoons 101$$

The following graph summarizes the various possibilities:



It should be clear by now that the main difficulty in the analysis of complex networks is the choice between different pathways. As pointed out above the answer to this question requires a careful examination of the delays and will be further discussed elsewhere in a more technical communication (Nicolis 1982), along with the analysis of other models. In any case one is tempted to correlate such cycles with the presence of periodicities in the climatic record (Lamb 1977).

4. DISCUSSION

In this paper we have tried to convey our conviction that a *qualitative* analysis of the Boolean type is likely to add new insights in the understanding of the complex problems of climate modelling. Such an analysis can in no way be a substitute for more quantitative studies. Rather, we envisage a *complementary* approach based on the use of both Boolean and more quantitative types of models. When a complex system is considered and some rough consensus on the types of feedbacks involved is reached, it would be appropriate to first look at it as a network of interacting binary elements, and to sort out the different potentialities through the analysis outlined in the present paper. If some of these behaviours agree qualitatively with observation, then one can reasonably expect that the feedbacks adopted in the logical treatment are the right ones. A more refined study based on continuous models and utilizing these feedbacks would then enable one to go further and reach quantitative agreement. If on the other hand the outcome of the Boolean analysis contradicts observation, one would have a compelling evidence that the consensus reached was not justified. A new set of hypotheses would be necessary, and the iterative process would continue until a qualitative agreement is reached.

Needless to say, the climatic problems analysed in the present paper have been relatively simple and straightforward. Several extensions motivated by our results can however be envisaged. Of particular interest would be the possibility of handling systems involving a large number of coupled variables. Such situations are intractable by continuous

models, but remain within the limits of feasibility of the Boolean approach. Nevertheless, as the number of states increases rapidly with the number of variables N (essentially like 2^N), numerical techniques will be an indispensable complement of the approach. A promising tool appears to be the use of logical machines, like those invented by Florine (1973), Van Ham (1977) and co-workers.

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