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A FREE BOUNDARY VALUE PROBLEM ARISING IN CLIMATE DYNAMICS

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Abstract—A heat and mass transfer problem of geophysical interest involving coexisting phases is studied. The dynamical system considered is the atmosphere–hydrosphere–cryosphere, wherein the spatial degrees of freedom along the vertical and longitudinal directions have been lumped. The reduced one-dimensional system is modelled by a simple, yearly averaged, energy balance model taking into account the coupling between the two phases present: the ice sheets and the ocean. This is done self-consistently by introducing a Stefan type of boundary condition at the interface. The resulting balance equation is linearized and solved analytically using mode truncation and Galerkin's method. The analysis is centered on the stability of the present-day climatic regime with respect to small excursions of the ice boundary. Special emphasis is put on the thermodynamic aspects, as well as on the characteristic time scales of evolution.

NOMENCLATURE

A,	affinity of the phase transformation;
$a(x, x_s)$,	absorption function;
c,	heat capacity;
D',	eddy diffusivity;
E,	internal energy;
g,	acceleration of gravity;
h,	height of ice sheet;
$\mathbf{J}$,	energy flux;
L, r_{TP},	heat of melting ice;
l,	width of ice sheet; length along a meridian;
Q,	solar constant divided by 4;
R,	radius of the earth;
$S(x)$,	normalized distribution of solar radiation;
S,	entropy;
T,	surface temperature;
V,	section of ice sheet along the meridional direction;
x,	sine of the latitude;
x_s,	sine of the latitude of the ice boundary.

Greek symbols

α,	albedo;
φ,	latitude;
λ,	parameter given by $\lambda = (4/3)\,\tau/\rho g$;
ρ,	mass density of ice;
τ,	yield stress of ice;
ϕ_s,	latitude of the equatorward tip of the ice sheet in radians;
ϕ_M,	latitude of the poleward tip of the ice sheet in radians.

1. INTRODUCTION

IT IS WELL known that most situations involving energy transfer between two coexisting phases separated by an interface, give rise to free boundary value problems [1]. Typical problems of this kind refer to rather simple geometries with a high degree of symmetry: Solidification of a semi-infinite body, of a plate, a sphere, a cylinder, and so forth.

Large scale geophysical phenomena provide beautiful examples of heat and mass transfer in a somewhat less traditional context. The present paper is devoted to one such problem, namely, the interaction between an ice cap and the earth–atmosphere system. Interactions of this sort are known to play an important role in climate dynamics, especially in connection with the onset of glaciation cycles.

The mathematical modelling of the climate system received considerable attention recently [2]. One of the most powerful tools has been the systematic use of simple energy balance models where an average over the longitudinal and vertical coordinates is taken, and the only energy exchanges considered explicitly are along the meridional direction. Such models are reasonably tractable, and predict a variety of bifurcation phenomena associated with transitions between present day and less favorable climatic conditions [3]. They all involve a discontinuous element which marks the beginning of an ice sheet. Aside from this discontinuity, the dynamical aspects of the interaction between the ice sheet and the ice free part of the earth are discarded. It is only when the explicit ice sheet dynamics, and hence the coupling between energy balance and mass balance of the glaciers is considered, that one takes such interactions into account [4, 5]. On the other hand, the explicit form of the coupling requires a number of additional parameterizations of such quantities as the ablation rate of the ice sheet, the snow fall etc. Although plausible, such parameterizations certainly go beyond the basic assumptions underlying the balance equations.

The main hypothesis of the present paper is that the interaction between an ice sheet and the ice free part of the earth–atmosphere system can be viewed as a free boundary value problem. In Section 2 a 1-dim. energy balance model is introduced, and some problems related to the time scales of the evolution predicted by this model are raised. In Section 3 we construct the augmented model in which the position of the ice edge is related, self-consistently, to the energy balance equation through a Stefan type of boundary condition. In Sections 4 and 5 a linearized analysis around the present-day conditions using, respectively, mode truncation and Galerkin's method is outlined. Section 6 is devoted to the thermodynamic aspects, particularly the entropy and excess entropy balance equations as well as to the presentation of the main conclusions.

2. THE MODEL

We begin with the yearly averaged energy balance equation of a column of unit surface extending from the top of the atmosphere until a certain ocean depth (mixed layer) within which most of the transport processes are assumed to take place. Quite generally, one can write

$$\frac{\partial E}{\partial t} = \text{Sources} - \text{Sinks} - \operatorname{div} \mathbf{J} \tag{1}$$

where E is the internal energy and $\mathbf{J}$ the energy flux. In principle, equation (1) is coupled to the momentum and mass transfer equations. However, because of the wide separation of the characteristic times associated with the vertical and longitudinal directions on the one side, and the meridional direction on the other side, it has been suggested [6, 7] that equation (1) can be averaged over the first two ones. In the resulting 1-dim. model (see Fig. 1), North [8] was able to obtain a reasonably satisfactory representation of the present-day meridional temperature distribution by modelling $\mathbf{J}$ as a (turbulent) diffusive heat transfer:

$$J_q = -D'(\nabla T)_x$$

or

$$J_q = -\frac{D'}{R}(1 - x^2)^{1/2}\frac{\partial T}{\partial x}. \tag{2}$$

Here D' is the eddy diffusivity, R the radius of the earth, T the surface temperature and x the sine of the latitude, $x = \sin\varphi$.

Within the same approximation the remaining terms in (1) are treated as follows: $\mathrm{d}E$ is replaced by $\mathrm{d}T$ through the thermodynamic relation

$$\mathrm{d}E = c\,\mathrm{d}T \tag{3a}$$

where c is a heat capacity (or thermal inertia coefficient). The source term is written as

$$\text{Source} = QS(x)\,a(x, x_s) \tag{3b}$$

where Q is the solar constant divided by 4, giving the value of the incoming solar radiative flux averaged over a year and over the surface of the earth (the factor 1/4 results from the earth's sphericity). $S(x)$ is the normalized distribution of solar radiation determined by astronomical calculations, and $a(x, x_s)$ is the absorption function, written as $1 - \alpha(x, x_s)$, α being the albedo. In climate modelling it is common to represent $a(x, x_s)$ as a function which changes in a step-like fashion in the vicinity of the ice edge, x_s, due to the marked difference between the reflectivities of ice and ocean or land. In particular, it is customary to consider symmetric hemispheres and write:

$$a(x, x_s) = \begin{cases} \beta_0 & x > x_s \\ \alpha_0 + \alpha_2 P_2 & x < x_s \end{cases} \tag{3c}$$

where β_0 is the absorption coefficient over ice or snow when 50% covered with clouds, and α_0, α_2 are the absorption coefficients over ice free areas obtained after analyzing the albedo distribution by Legendre series. Finally, the sink term expresses the effect of the infrared cooling and is approximated by

$$\text{Sink} \equiv I(x) = A + B\,T(x) \tag{3d}$$

provided that the range of variation of T around a reference value is not very high.

On substituting equation (2) as well as equations (3a)–(3d) into the energy balance equation (1), and setting $D = D'/R^2$ we obtain

$$c\frac{\partial T}{\partial t} = Q\,S(x)\,a(x, x_s) - (A + BT(x)) + \frac{\partial}{\partial x}\left[(1 - x^2)D\frac{\partial T(x)}{\partial x}\right]. \tag{4}$$

In order to have a closed form equation we still have to relate the position of the ice edge, x_s, to the temperature T. Following Budyko [6] we require that:

$$\left.\begin{array}{ll} T(x) > -10°\mathrm{C} & \text{no ice present,} \\ T(x) < -10°\mathrm{C} & \text{permanent ice present.} \end{array}\right\} \tag{5}$$

Equations (4) and (5) constitute a well posed problem if, in addition, appropriate boundary conditions are given at $x = 0$ and $x = \pm 1$. In the symmetric hemisphere case here considered the appropriate conditions are zero energy flux at the pole and across the equator. Finally, in all studies performed so far a physically motivated condition has been added, namely that the temperature and its gradient, giving the heat flux, must be continuous at the ice edge.

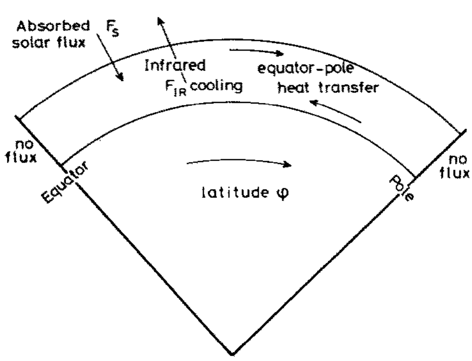

FIG. 1.

As mentioned in the introduction, the analysis of equations (4)–(5) gives rise to an amazing variety of bifurcation phenomena corresponding to climatic transitions. Rather than dwell on these results however, we prefer to insist on the limitations arising from some of the assumptions adopted in this formalism, which in our opinion are particularly stringent.

As is known, one of the ubiquitous features of the climatic system is the coexistence of processes characterized by widely separated time scales. Thus, our usual perception of climate is associated with variations of temperature, humidity etc. on a scale of a few years; the atmosphere–hydrosphere–cryosphere system brings about new features with characteristic scales of 10^3–10^4 yr, the onset of a glaciation time; finally, in a longer time scale the interaction with external variations (orbital parameters, solar output or geological environment) begin to play a non-negligible role [9].

Now, in the modelling based on equations (4) and (5) these various processes have been completely decoupled. More specifically, the assumption of continuous temperature gradient across the ice edge implies equality of the heat fluxes on both sides. As a result, the ice edge follows passively the temperature variations even if the latter occur with the characteristic relaxation time c/B of equation (4) which is typically of the order of a few years. This is clearly wrong, as the enormous inertia of the ice sheets should imply a scale of variation of the ice edge of at least several hundred years.

As mentioned in the introduction, the most satisfactory way to account correctly for the atmosphere–hydrosphere–cryosphere coupling would be to appeal both to the energy balance equation and to the mass balance of the ice sheets. However, in view of the complexity of this project, and the concomitant uncertainties involved in the parameterization of the various quantities involved in the theory, we adopt hereafter an alternative point of view. We show that independently of explicit ice sheet dynamics, there exists a completely self-consistent coupling mechanism between atmosphere, hydrosphere and cryosphere which is based solely on the energy balance equation, and which is sufficient to generate the long time scale missing in equations (4) and (5).

3. AN AUGMENTED ENERGY BALANCE MODEL

The starting point is to realise that the continuity of the flux across the ice edge may still be a satisfactory assumption for steady states, but should break down completely for time dependent ones. As is well known from free boundary value problems, the excess between "right" and "left" fluxes around a boundary separating two phases can serve for the advance of one of them at the expense of the other [1]. Let L denote the heat of melting of ice per unit mass, ρ its mass density, and V the section of the ice sheet along the meridional direction. Then, remembering that the origin of the coordinate system in the energy balance equation is at the equator,

$$J_q(x_s - \varepsilon) - J_q(x_s + \varepsilon) = L\rho \frac{dV}{dt} \tag{6}$$

where $\varepsilon\,(\varepsilon > 0)$ denotes a small distance from the ice boundary x_s.

We proceed to the evaluation of dV/dt. If present-day configurations of ice sheets are to be modelled, two different cases can be envisaged: (i) a full ice cap centered at the pole (southern hemisphere), and (ii) a circumpolar ring of ice delimited by the presence of sea (northern hemisphere).

According to Weertman [10], the ice sheet flows as a perfectly plastic substance and the flow is only in the meridional direction. It follows that the ice sheet profile remains always parabolic around its centre of symmetry. Choosing the latter as the origin of a *local* coordinate system:

$$h(u) = \lambda^{1/2}(l - |u|)^{1/2} \tag{7}$$

where h is the elevation above sea level, l is the width of the sheet and λ a parameter depending on the yield stress of ice.

Consider first the case of a full ice cap. The cross-section V is then

$$V = \lambda^{1/2}\int_0^l du(l - |u|)^{1/2} = \frac{2}{3}\lambda^{1/2}l^{3/2}. \tag{8}$$

Hence

$$\frac{dV}{dt} = (\lambda l)^{1/2}\frac{dl}{dt}. \tag{9a}$$

Now, dl/dt can be related straightforwardly to the motion of the ice boundary x_s, as described in the original coordinate system, by (see Fig. 2a)

$$\frac{dl}{dt} = -\frac{R}{(1-x_s^2)^{1/2}}\frac{dx_s}{dt}, \tag{10a}$$
$$l = R\left(\frac{\pi}{2} - \phi_s\right)$$

ϕ_s being the latitude in radians ($\phi_s = \arcsin x_s$).

We next consider the case of a circumpolar ring of ice. Equation (7) remains unaltered provided that l is now interpreted as the half width. Thus,

$$\frac{dV}{dt} = 2(\lambda l)^{1/2}\frac{dl}{dt}. \tag{9b}$$

The connection between this expression and x_s becomes now more involved. From Fig. 2(b) we have

$$\frac{dl}{dt} = -\frac{1}{2}\frac{R}{(1-x_s)^{1/2}}\frac{dx_s}{dt}, \tag{10b}$$
$$l = \frac{1}{2}(l_M - l_s) = \frac{R}{2}(\phi_M - \phi_s)$$

where ϕ_M is the latitude of the poleward tip of the sheet. Using equations (6)–(10) we may finally write the

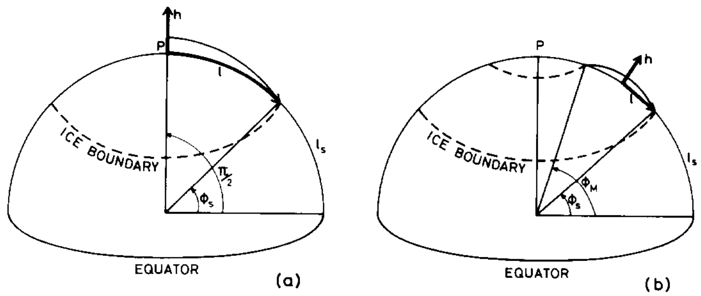

FIG. 2.—Schematic representation of hemispherical ice sheet models (a) full ice cap, (b) a circumpolar ring of ice.

energy balance equation taking ice melting or advance into account in the form

$$c\frac{\partial T}{\partial t} = QS(x)\,a(x, x_s) - [A + BT(x)]$$

$$+\frac{\partial}{\partial x}\left[(1-x^2)\,D\frac{\partial T(x)}{\partial x}\right] - L\rho\, h_{\text{eff}}\frac{\mathrm{d}x_s}{\mathrm{d}t}\,\delta(x-x_s), \quad (11)$$

where we introduce an "effective height" h_{eff}. Comparison with equations (9a) to (10b):

$$h_{\text{eff}} = \left[\lambda R\left(\frac{\pi}{2} - \phi_s\right)\right]^{1/2} \quad \text{for a full ice cap} \quad (12a)$$

$$h_{\text{eff}} = \left[\lambda\frac{R}{2}(\phi_M - \phi_s)\right]^{1/2} \quad \text{for a ring of ice.} \quad (12b)$$

On integrating both sides of equation (11) over a small slice around x_s and on assuming continuity of T one finds the free boundary condition, equation (6).

From the point of view of thermodynamics equation (11) can also be interpreted as follows. In a two phase system the internal energy E depends upon both temperature T and relative composition. Measuring the latter by the length l_s of a meridian between the equator and the ice boundary we have (at constant pressure)

$$E = E(T, l_s)$$

or

$$\mathrm{d}E = c\,\mathrm{d}T + \left(\frac{\partial E}{\partial l_s}\right)_{\text{TP}}\mathrm{d}l_s\,\delta(l - l_s) \quad (13)$$

where from now on l will denote the length along a meridian measured from the equator.

According to thermodynamics

$$\left(\frac{\partial E}{\partial l_s}\right)_{\text{TP}} = -P\left(\frac{\partial V}{\partial l_s}\right)_{\text{TP}} - r_{TP} \quad (14)$$

where $-r_{TP}$ is the heat of melting,

$$-r_{TP} = L\rho. \quad (15)$$

Neglecting the variation of volume with respect to l_s and substituting $\partial E/\partial l_s$ from the above expressions we obtain the extra term of equation (11) associated with the phase transition.

The problem we now face is to solve equation (11) subject to the additional boundary conditions mentioned in Section 2, namely

$$J_q(0) = J_q(1) = 0, \quad (16a)$$

$$T(x_s - \varepsilon) = T(x_s + \varepsilon) = T_{\text{ice}}. \quad (16b)$$

4. LINEARIZATION AND MODE TRUNCATION

In view of the complexity of the full problem we carry out a *linear analysis* around the present day temperature distribution $T = T^*(x)$ and position of the ice-sheets, $x_s = x_s^*$. To this end we set

$$T(x, t) = T^*(x) + \theta(x, t), \quad (17a)$$

$$x_s(t) = x_s^* + \xi(t). \quad (17b)$$

Keeping dominant terms in θ and ξ one finds from equations (11) and (16):

$$C\frac{\partial\theta}{\partial t} = -QS(x)\,[\alpha(x, x_s) - \alpha(x, x_s^*)]$$

$$-B\theta + D\frac{\partial}{\partial x}(1 - x^2)\frac{\partial\theta}{\partial x}$$

$$-L\rho h_{\text{eff}}\frac{\mathrm{d}\xi}{\mathrm{d}t}\,\delta(x - x_s), \quad (18a)$$

$$\left(\frac{\mathrm{d}T^*}{\mathrm{d}x}\right)_{x_s^*}\xi(t) + \theta(x_s^*) = 0. \quad (18b)$$

Let θ_n denote the nth Legendre moment of θ:

$$\theta(x) = \sum_{\substack{n=0\\ \text{even}}}^{\infty}\theta_n P_n \quad (19)$$

$$\theta_n = \int_0^1 (2n + 1)\,\theta(x)\,P_n(x)\,\mathrm{d}x.$$

The following equations for θ_n are easily obtained from equations (18), after a complete linearization in ξ and θ is made.

$$c\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = -[B + n(n+1)D]\,\theta_n - (2n+1)$$

$$\times\left[Q\Delta_n(x_s^*)\,\xi + L\rho h P_n(x_s^*)\frac{\mathrm{d}\xi}{\mathrm{d}t}\right], \quad (20a)$$

$$\left.\begin{aligned}\xi &= -\left(\frac{dT^*}{dx}\right)^{-1}_{x_s^*}\theta(x_s^*)\\ &= -\left(\frac{dT^*}{dx}\right)^{-1}_{x_s^*}\sum_{\substack{n=0\\ \text{even}}}^{\infty}\theta_n P_n(x_s^*)\end{aligned}\right\}\quad(20\text{b})$$

where

$$\Delta_n = \left[\frac{d}{dx}\int_0^1 S(x)\,\alpha(x, x_s)\,P_n\,dx\right]_{x_s^*}.$$

From these relations one can obtain the characteristic equation for the problem which, as is well known, determines the temporal evolution in the vicinity of the reference state (T^*, x_s^*). Setting

$$\theta_n = \hat{\theta}_n e^{\omega t} \text{ and } \xi = \hat{\xi} e^{\omega t} \quad (21)$$

we find

$$\left(\frac{dT^*}{dx}\right)^{-1}\sum_{\substack{n=0\\ \text{even}}}^{\infty}\frac{(2n+1)P_n(x_s^*)}{\omega' + \left[1 + n(n+1)\frac{D}{B}\right]}$$
$$\times\left[\frac{1}{B}Q\Delta_n(x_s^*) + \frac{L\rho h_{\text{eff}}}{c}P_n(x_s^*)\,\omega'\right] = 1 \quad (22\text{a})$$

where we introduced the dimensionless variable

$$\omega' = \frac{\omega c}{B}. \quad (22\text{b})$$

An important feature of equation (22a) is to contain the effect of the discontinuity of the flux at the ice boundary in each term of the series. For this reason a truncation to the first few terms, which is necessary if explicit values of ω' are to be obtained, can be envisaged without contradicting the free boundary condition, equation (6).

The results are highly dependent upon the numerical value of h_{eff} as defined by equations (12a) and (12b) for the two ice sheet models respectively. Adopting $\lambda \sim 7$ m, which gives reasonable central thicknesses as compared to those characterizing Antarctica and Greenland today, we have for a full ice cap: $\pi/2 - \phi_s \sim 16°$ and $h_{\text{eff}} \sim 3500$ m. On the other hand for a ring of ice: $\phi_M - \phi_s \sim 6°$ and $h_{\text{eff}} \sim 1500$ m.

For usually accepted values of the thermal inertia coefficient, associated with a mixed ocean layer of a few metres (e.g. $c = 4.6 \times 10^7$ J m^{-2} K) and for the values of L, ρ given by thermodynamics, $L\rho h_{\text{eff}}/c$ is of the order of 2.3×10^4 and 1×10^4 respectively.

We solved equation (22a) when truncation to $n = 0$ and 2 was successively performed, and found that the solutions ω' were always real and negative. This implies the absence of oscillations and the stability of the present-day climatic regime with respect to small ice sheet disturbances. The first two columns of Table 1 summarise the results concerning that solution ω' which corresponds to the longest characteristic time scale, $\tau \sim \omega^{-1} = c/B\omega'$, for various values of h_{eff}. We see that ω' becomes as small as $\sim 10^{-3}$ for values of $L\rho h_{\text{eff}}/c$ of the order of 10^4 which is precisely the order of magnitude suggested by the two model ice sheets. Now $\omega' \sim 1$ corresponds (see equation 22b) to a relaxation rate of the order of Bc^{-1}, characteristic of usual energy balance models. Such values are indeed found from the solution of the characteristic equation (22a). In addition to them however the results given in Table 1 show that we have been able to generate, *self-consistently*, the long time scale characterizing the interaction between atmosphere, hydrosphere and cryosphere. The appearance of such long scales reflects the enhanced inertia gained by the system as a result of the presence of ice sheets. In this respect from the estimations we made earlier it becomes obvious that the full ice cap gives rise to a greater inertia than the circumpolar one.

5. SOLUTION BY GALERKIN'S METHOD

The mode truncation obtained in the preceding section gave rise to a characteristic equation containing explicitly the effect of the ice sheets. On the other hand any truncation to a finite number of modes implies (see equations 18) that the discontinuity of the flux across the ice boundary will be smeared out. In order to remove this deficiency we analyze in this section the linearized problem using Galerkin's method. We start from expression:

$$\theta = \sum_{\substack{n=0\\ \text{even}}}^{N}\theta_n(t)P_n(x) + u(x) \quad (23)$$

Table 1. Dependence on the slowest ω', solution of equation (22a) for various values of $L\rho h_{\text{eff}}/c$

$L\rho h_{\text{eff}}/c$	One mode	Two modes	Galerkin's method
1	-0.610	-0.235	-0.234
10	-0.550	-0.175	-0.172
100	-0.277	-0.481×10^{-1}	-0.457×10^{-1}
1000	-0.464×10^{-1}	-0.579×10^{-2}	-0.544×10^{-2}
1000	-0.498×10^{-2}	-0.591×10^{-3}	-0.554×10^{-3}

Two mode truncation (first 2 columns) compared to the results obtained by Galerkin's method (3rd column). Numerical values of the parameters used: $Q = 340$ Wm^{-2}; $A = 214.2$ Wm^{-2}; $B = 1.575$ Wm^{-2} K^{-1}; $D = 0.591$ Wm^{-2}; $S(x) = 1 - 0.477\,P_2(x)$; $1 - \alpha(x, x_s) = 0.697 - 0.0779\,P_2(x)$ for $x < x_s$ and $1 - \alpha(x, x_s) = 0.38$ for $x > x_s$.

where $u(x)$ is orthogonal to all Legendre polynomials P_0 to P_N, and view θ as a trial function, to be adequately parameterized. The simplest non-trivial case is

$$\theta = \theta_0(t) + \theta_2(t)P_2(x) + u(x, t). \quad (24)$$

In addition to being orthogonal to P_0 and P_2, the function $u(x, t)$ is taken to satisfy the boundary conditions (6) and (16b), including the flux discontinuity at the ice edge. The simplest x-dependence of $u(x)$ compatible with these requirements is:

$$u(x) = a_0 + a_2 \frac{x^2}{2} \qquad x < x_s, \quad (25)$$

$$u(x) = b_0 + b_2 x + c_2 \frac{x^2}{2} \quad x > x_s.$$

The orthogonality condition of $u(x)$ together with relations (6) and (16) constitute the following system of 4 equations with 5 unknowns:

$$a_0 + a_2 \frac{x_s^2}{2} - b_2 x_s - c_2 \frac{x_s^2}{2} = b_0,$$

$$a_0 x_s + a_2 \frac{x_s^3}{6} - \frac{b_2}{2}(x_s^2 - 1) - \frac{c_2}{6}(x_s^3 - 1) = b_0(x_s - 1),$$

$$a_0(x_s^3 - x_s) + \frac{a_2}{2}\left(\frac{3x_s^3}{5} - \frac{x_s^3}{3}\right) - \frac{b_2}{2}\left(\frac{3x_s^4}{2} - x_s^2 - \frac{1}{2}\right)$$

$$-c_2\left(\frac{3x_s^5}{5} - \frac{x_s^3}{3} - \frac{4}{15}\right) = b_0(x_s^3 - x_s),$$

$$-a_2 x_s + b_2 + c_2 x_s = \frac{L\rho h_{\text{eff}}}{D(1-x_s^2)} \frac{\mathrm{d}x_s}{\mathrm{d}t} \quad (26)$$

Leaving b_0 as a free parameter we want to derive the equations of evolution for this quantity as for θ_0 and θ_2. To this end, substituting the trial function equation (24) into the augmented energy balance equation, equation (11), multiplying successively by P_0, P_2 and P_4, integrating over the domain of x and linearizing, we arrive at the following expressions for θ_0, θ_2 and b_0:

$$c\frac{\partial \theta_0}{\partial t} + L\rho h_{\text{eff}} \frac{\mathrm{d}\xi}{\mathrm{d}t} = -(B\theta_0 + Q\Delta_0 \xi),$$

$$c\frac{\partial \theta_2}{\partial t} + 5\, L\rho h_{\text{eff}} P_2 \frac{\mathrm{d}\xi}{\mathrm{d}t} = -[(B+6D)\theta_2 + 5Q\Delta_2\xi],$$

$$c\frac{\partial b_0}{\partial t} F_1 + L\rho h_{\text{eff}}\left(F_2 \frac{\mathrm{d}\xi}{\mathrm{d}t} + F_3 \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}\right) = b_0 F_4 - Q\Delta_4 \xi \quad (27)$$

where F_1 to F_4 are cumbersome functions of x_s^* and the parameters arising from the solution of equation (26) and the integration of the different terms of the energy balance equation.

From the above system of linear differential equations (27) together with the ice boundary condition

$$\xi = -[\theta_0 + \theta_2 P_2(x_s^*) + u(x, t)]\left(\frac{\partial T^*}{\partial x}\right)^{-1}_{x_s^*} \quad (28)$$

one can again obtain a characteristic equation which is of 6th degree in ω. However it can be considerably simplified if one anticipates, in agreement with the previous section, the existence of solutions corresponding to a long relaxation time. Another simplification which yields similar results is to uncouple the first two equations from the third one by choosing $b_0 = 0$, but still keeping the influence of $u(x)$, which does not now contain any free parameters, in the first two equations.

The third column of Table 1 gives the slowest mode ω' in terms of $L\rho h_{\text{eff}}/c$, as determined from the above-described Galerkin procedure. We see that the agreement with the two mode truncation is excellent. We are therefore confident that we have indeed determined an intrinsically long time scale of the climatic system.

6. THERMODYNAMIC ASPECTS. CONCLUDING REMARKS

In this section we further discuss the origin of the enhanced inertia arising from the presence of the ice sheets. To this end, we construct the entropy balance equation and then analyse the stability properties in terms of the excess entropy production.

We first write the energy balance equation (11) in the form

$$c\frac{\partial T}{\partial t} = R + r_{TP}\frac{\mathrm{d}l}{\mathrm{d}t}\delta(l - l_s) \quad (29)$$

where R stands for all terms except those related to the movement of the ice boundary.

According to the discussion at the end of Section 3, the entropy of the system in the presence of the moving boundary is (at constant pressure):

$$S = S(T, l) = \int s\,\mathrm{d}l \quad (30)$$

where the integration extends from the equator to the pole and

$$\frac{\partial S}{\partial t} = \int \mathrm{d}l\left[\frac{c}{T}\frac{\partial T}{\partial t} + \left(\frac{\partial S}{\partial l}\right)_{TP}\frac{\mathrm{d}l}{\mathrm{d}t}\delta(l - l_s)\right]. \quad (31)$$

According to chemical thermodynamics [11]

$$\left(\frac{\partial S}{\partial l}\right)_{TP} = \frac{A - r_{TP}}{T} \quad (32)$$

where A is the affinity of the phase transformation. Combining equations (29)–(32) we obtain

$$\frac{\partial S}{\partial t} = \int \mathrm{d}l \frac{1}{T} R + \frac{A}{T}\frac{\mathrm{d}l_s}{\mathrm{d}t}. \quad (33)$$

The first term of the right hand side has been studied in detail in a recent paper [12]. The second term is specific to the ice boundary. It has the familiar bilinear form [13] of a thermodynamic force, A/T, multiplied by the flux, $\mathrm{d}l_s/\mathrm{d}t$, of an irreversible process.

The next step is to construct the balance equation for the *excess entropy* of the system. The main motivation behind this calculation is the

Glansdorff–Prigogine theory [13], according to which excess entropy is a convenient Lyapounov functional governing stability of a reference state. Specifically, if $\delta^2 S$ is the second differential of entropy evaluated at the present day climate, stability of thermodynamic equilibrium implies that

$$\delta^2 S \leqslant 0. \tag{34a}$$

Thus, if

$$\frac{\partial}{\partial t}\delta^2 S \geqslant 0 \tag{34b}$$

by Lyapounov's theorem, then the reference state will be neutrally stable [if the equality sign prevails in (34b)] or asymptotically stable [if the inequality sign prevails in (34b)].

The first differential of entropy density is [cf. equations (30)–(32) and the notation of Sections 4 and 5]:

$$\delta s = \frac{c}{T}\theta + \frac{A - r_{TP}}{T}\xi\delta(l-l_s). \tag{35}$$

It follows that

$$\frac{1}{2}\delta^2 s = -\frac{1}{2}\frac{c}{T}\theta^2 + \left\{\frac{1}{2}\left[\frac{\partial\left(\frac{A-r_{TP}}{T}\right)}{\partial l}\right]_{TP}\xi^2 + \left[\frac{\partial\left(\frac{A-r_{TP}}{T}\right)}{\partial T}\right]_{Pl}\theta\xi\right\}\delta(l-l_s) \tag{36}$$

where we neglected the variation of c on l and of r_{TP} on T.

The differential of affinity at the ice boundary (at constant pressure) is given by

$$\delta\left(\frac{A}{T}\right) = -\frac{r_{TP}}{T^2}\theta - \frac{G_{TP}}{T}\xi \tag{37a}$$

where

$$G_{TP} = -\left(\frac{\partial A}{\partial l}\right)_{TP} = \left(\frac{\partial^2 G}{\partial l^2}\right)_{TP}, \tag{37b}$$

G being Gibb's free energy.

Substituting into equation (36) we see that

$$\left[\frac{\partial\left(\frac{A-r_{TP}}{T}\right)}{\partial T}\right]_{Pl} = 0$$

and

$$\left[\frac{\partial\left(\frac{A-r_{TP}}{T}\right)}{\partial l}\right]_{TP} = -\frac{G_{TP}}{T}.$$

Thus

$$\frac{1}{2}\delta^2 S = -\frac{1}{2}\int dl\frac{c}{T^2}\theta^2 - \frac{1}{2}\frac{G_{TP}}{T_s}\xi^2, \tag{38}$$

where the last term is evaluated at the ice boundary.

Expression (38) is negative definite. Indeed, c and G_{TP} are non-negative and do not vanish simultaneously owing to the convexity of the Gibbs free energy.

We now evaluate the time derivative of $\delta^2 S$. Within the framework of a linearized stability analysis we discard the time variation of the coefficients of the quadratic form, which are to be evaluated at the (stationary) reference state. We thus have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\delta^2 S\right) = -\int dl\frac{c}{T^2}\theta\frac{\partial\theta}{\partial t} - \frac{G_{TP}}{T_s}\xi_s\frac{\partial\xi_s}{\partial t}. \tag{39}$$

On the other hand, the linearized form of equation (29) reads

$$c\frac{\partial\theta}{\partial t} = \delta R + r_{TP}\frac{d\xi_s}{dt}\delta(l-l_s). \tag{40}$$

Substituting into equation (39) we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\delta^2 S\right) = -\int dl\frac{\theta}{T^2}\delta R - \frac{1}{T_s}\left(G_{TP}\xi_s + \frac{r_{TP}}{T_s}\theta_s\right)\frac{d\xi_s}{dt}. \tag{41}$$

The first term of the right hand side has been analyzed in detail in [12]. The remaining terms are specific to the dynamics of the ice boundary. Utilizing equation (37a) one can easily see that they can be put in the form

$$\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\delta^2 S\right)\right]_{\text{bound}} = \delta\left(\frac{A}{T}\right)_s\frac{d\xi_s}{dt}. \tag{42}$$

This has the same structure as the second term of equation (33), except that the force and flux have been replaced, respectively, by their excess values around the reference state. We may therefore refer to this contribution as *excess entropy production.*

In our problem the reference state around which $\delta(A/T)_s$ is to be evaluated is a steady state. From the point of view of the phase transformation, it has to be considered as a state of equilibrium (zero affinity), since the two phase coexist under these conditions. It is therefore reasonable to evaluate $\delta(A/T)_s$ by adopting a linear law relating fluxes and forces

$$\frac{A}{T} = \mathscr{L}\frac{dl_s}{dt}, \tag{43}$$

where the Onsager coefficient $\mathscr{L}$ is positive.

Thanks to equation (43), expression (42) becomes

$$\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\delta^2 S\right)\right]_{\text{bound}} = \mathscr{L}\left(\frac{d\xi_s}{dt}\right)^2 > 0. \tag{44}$$

Thus according to Lyapounov's stability theorem, equation (34b), the presence of the ice boundary has a stabilizing effect. This result may seem unexpected at first hand, but can nevertheless be understood as follows. When ice melts the region around the ice boundary blocks a certain amount of energy. Thus, on the average the temperature around this region will have to drop, and this will tend to move the boundary back to its initial position. A similar negative feedback would obtain in case the ice front would tend to advance as a result of a perturbation. Note, however, that the coupling between ice boundary and bulk terms may have a destabilizing effect through the dependence of the albedo on the position of the ice boundary.

In summary, the analysis reported in this paper establishes the high inertia and the infinitesimal stability of the ice edge characterizing present-day climate as well as the absence of oscillations (even damped ones) in the time dependence of perturbations. The absence of oscillations implies that one cannot expect any resonance phenomena associated with a weak external periodic forcing. In the context of climate modelling, such forcings (associated with the earth's orbital variations) have been widely invoked [14] to explain the glaciation cycles. An explanation of these cycles based on a resonance mechanism has therefore to be ruled out in our model.

It would be interesting to extend some of the results reported in this paper by taking nonlinear effects into account. An intriguing possibility is the appearance of new bifurcations to time dependent solutions. Numerical experiments aiming to verify these points are in progress.

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UN PROBLEME DE FRONTIERE LIBRE RESULTANT DE LA DYNAMIQUE CLIMATIQUE

Résumé—On étudie le transfert de masse et de chaleur en présence de phases coexistantes dans un problème d'intérêt géophysique. Le système dynamique considéré est l'atmosphère–hydrosphère–cryosphère, où l'on a effectué des moyennes suivant les directions verticale et longitudinale. Le système réduit qui en résulte est stylisé par un modèle de bilan énergétique uni-dimensionel tenant compte du couplage entre les deux phases en présence (les calottes glaciaires et les océans), par l'intermédiaire d'une condition aux bords du type Stefan (problème de frontière libre). L'équation de bilan est résolu analytiquement dans l'approximation linéaire, par une procédure de troncature de modes, ainsi que par la méthode de Galerkin. On obtient ainsi des informations sur la stabilité du climat actuel par rapport à de petits déplacements de la limite de la glace. On insiste également sur les aspects thermodynamiques, ainsi que sur les échelles de temps caractéristiques de l'évolution.

EIN FREIES RANDWERTPROBLEM DER KLIMA-DYNAMIK

Zusammenfassung—Es wird ein Wärme- und Stoffübergangsproblem, das von geophysikalischem Interesse ist, behandelt. Das betrachtete dynamische System umfaßt Atmosphäre, Hydrosphäre und Kryosphäre, wobei die räumlichen Freiheitsgrade in vertikaler und horizontaler Richtung punktförmig konzentriert werden. Das reduzierte eindimensionale System wird durch ein einfaches jahreszeitlich gemitteltes Energiebilanzmodell simuliert unter Berücksichtigung der Kopplung der beiden vorhandenen Phasen: der Eisschichten und des Ozeans. Dieses erfolgt konsistent durch Einführung einer Stefan'schen Randbedingung an der Phasengrenzfläche. Die daraus resultierende Gleichgewichtsbedingung wird linearisiert und analytisch nach dem Galerkin-Verfahren gelöst. Die Untersuchung konzentriert sich auf die Stabilität der gegenwärtigen klimatischen Verhältnisse im Hinblick auf kleine Verschiebungen der Eisgrenze. Besonders hervorgehoben werden sowohl die thermodynamischen Aspekte als auch die für die Entwicklung charakteristischen Zeiträume.

ОДНА ЗАДАЧА СО СВОБОДНОЙ ГРАНИЦЕЙ, ВСТРЕЧАЮЩАЯСЯ В ДИНАМИКЕ КЛИМАТА

Аннотация — Проведено исследование интересной с геофизической точки зрения задачи о тепло- и массопереносе при наличии одновременно нескольких фаз. Рассматривается динамическая система, включающая атмосферу, гидросферу и криосферу, в которой пространственные координаты в вертикальном и горизонтальном направлениях представлены в виде одной обобщенной координаты. Полученная таким образом одномерная система моделируется простой, усредненной по годам, моделью баланса энергии, в которой учитывается взаимодействие двух фаз: ледяной покров и океан. Самосогласованность достигнута за счет использования стефановского граничного условия на границе раздела фаз. Полученное уравнение баланса линеаризовано и решено аналитически с помощью усечения мод и использования метода Галеркина. Анализируется стабильность современного климатического режима с учетом небольших перемещений ледовой границы. Особое внимание уделено термодинамическим аспектам, а также характеристическим временным масштабам рассматриваемых процессов.