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MODEL OF THE POLAR ION-EXOSPHERE

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Abstract—A model of a polar ion-exosphere in which the geomagnetic field lines are open is developed. The electrostatic field in this region has been calculated by taking into account two fundamental conditions: (1) quasi-neutrality has to be satisfied everywhere in the exosphere; (2) escape fluxes of electrons and ions have to be equal. The density distributions are calculated for different evaporative models in the case of two (O^+ and e) and three (O^+, H^+ and e) constituents. In addition, the mean velocity, pressure and temperature distributions of the several constituents are derived.

1. INTRODUCTION

Recently, the model ion-exosphere of Eviatar, Lenchek and Singer (1964) for a non-rotating planet has been generalized by Hartle (1969), by permitting the density and temperature to vary over the baropause. In both papers the density, pressure and temperature distributions of the thermal particles are calculated in the *stable* trapping region of a centered-dipole magnetic field, i.e. along *closed* lines of force. Moreover, they assumed that all the charged particles of the exosphere have emerged from the barosphere, and that the electrostatic polarization potential and field are given by the Pannekoek (1922) or Rosseland (1924) formula:

$$\Phi_E = -\frac{m^+ - m^-}{2e}\,\Phi_g(r), \tag{1}$$

with $\mathbf{E} = -\nabla\Phi_E$ and $\mathbf{g} = -\nabla\Phi_g$.

In formula (1) m^- and m^+ are respectively the electron and mean ion masses, $\mathbf{g}$ is the gravitational acceleration, Φ_E and Φ_g are the electric and gravitational potentials.

In this paper we consider the case of the polar ion-exosphere where the magnetic field lines are *open* and connected with the magnetotail. It is clear that for such a model not all charged particles are trapped and some can escape if their kinetic energy is high enough. In order to calculate the electric potential Φ_E in the exosphere we use the quasi-neutrality condition and require that the escape fluxes of positive and negative charged particles are equal. We consider three models:

(1) an 'untrapped-model' (*UT*) where it is assumed that all exospheric particles emerge from the baropause, and can escape along the open field lines.

(2) a 'trapped-model' (*T*) where it is assumed that the exosphere is also populated by trapped particles, which are in thermal equilibrium with those emerging from the barosphere.

(3) the well-known barometric model (*B*) where there is no restriction on the energy values nor on the pitch angles of the exospheric particles.

For each model we determine the density n, the escape flux F, the mean velocity w, the pressure tensor components $p_{\parallel}$, $p_{\perp}$, and the temperatures $T_{\parallel}$ and $T_{\perp}$.

Moreover, in order to simplify the problem we consider that the transition layer separating the barosphere (where the mean free path $\bar{l}$ of the particles is small compared with the electron density scale height H_e) and the exosphere (where $\bar{l} \gg H_e$) is reduced to a spherical surface. Experimental electron density profiles in the polar ionosphere show that such a spherical surface (the baropause) may be taken about 2000 km above the Summer polar cap (Chan *et al.*, 1966; Thomas *et al.*, 1966; Donley, 1968; Thomas and Andrews, 1969).

2. THE GRAVITATIONAL AND ELECTROSTATIC POTENTIALS

It is well known that the gravitational potential Φ_g satisfies Poisson's equation

$$\Delta\Phi_g = 4\pi G \sum_k n_k m_k, \tag{2}$$

where G is the gravitational constant, and the summation in the right hand side runs over all kinds of particles with mass m_k and density n_k. The solution of (2) is

$$\Phi_g(r) = \Phi(r) y, \tag{3}$$

where

$$y = \frac{r_0}{r}; \qquad \Phi(r) = -G\frac{M(r)}{r_0}, \tag{4}$$

with $M(r)$ the total mass inside a sphere of radius r.

As the mass density is very small in the atmosphere, Equation (2) can in a first approximation be reduced to the Laplace equation. In this case $M(r)$ is the Earth's mass and $\Phi(r)$ is a constant equal to $\Phi_g(r_0)$. The gravitational acceleration is then given by

$$g(r) = -\frac{\mathrm{d}\Phi_g(r)}{\mathrm{d}r} = \Phi_g(r_0)\frac{y^2}{r_0}. \tag{5}$$

In the exosphere where the collision frequency is very small the electric potential can no longer be given by the Pannekoek's (1922) or Rosseland's (1924) formula (1), since such an approximation is only valid in the barosphere where the particles subject to collisions are in hydrostatic equilibrium. If the diffusion equilibrium electric potential (1) is used for oxygen ions of mass m_{O^+}, the escape flux of the thermal electrons of mass m_e is at least $(m_{O^+}/m_e)^{1/2} = 168$ times larger than the ion evaporative flux. Hence, a positively charged layer at the baropause would appear, which would increase the electric field in the exosphere. In any case, this process would necessarily reduce the electron flux and would lead to the stationary state in which the flux of electrons is equal to the flux of the positive ions.

The electrostatic potential Φ_E is a solution of Poisson's equation

$$\Delta\Phi_E = -4\pi e \sum_k Z_k n_k, \tag{6}$$

where Z_k is an integer: positive and equal to the degree of ionization for the ions and -1 for the electrons. The summation runs over all kinds of particles with density n_k.

As the baropause is a discontinuity surface separating the barosphere (collision dominated region) and the exosphere (collision free region) there can appear single and double layer electric potentials Φ_1 and Φ_2 (Lemaire and Scherer, 1969). The solution of Equation (6) is therefore given by

$$\Phi_E(\mathbf{r}) = [\Phi_0 + \Phi_1 U(r - r_0)]\frac{r_0}{r} + \Phi_2(r - r_0) + \Phi_3, \tag{7}$$

where r_0 is the radial distance to the baropause and $U(x)$ is the well-known Heaviside step-function defined by: $U(x < 0) = 0$, and $U(x > 0) = 1$; Φ_3 is an arbitrary constant; $\Phi_0 + \Phi_1 + \Phi_2 + \Phi_3$ is the electric potential at the base of the exosphere and $\Phi_0 + \Phi_3$ at the top of the barosphere. In Appendix A it is shown that for an isothermal ion-barosphere in hydrostatic equilibrium, Φ_0 is given by

$$\Phi_0 = -\frac{\sum_j Z_j m_j n_j(r_0)/kT_j(r_0)}{\sum_j Z_j^2 n_j(r_0)/kT_j(r_0)} \cdot \frac{\Phi_g(r_0)}{e}. \tag{8}$$

Finally Φ_1 is related to the electric space charge in the exosphere. Indeed from (6) and (7) one obtains for $r > r_0$,

$$4\pi e \sum_k Z_k n_k = -\frac{r_0}{r}\left(\Delta\Phi_1 - \frac{2\mathbf{r}}{r^2}\cdot\nabla\Phi_1\right). \tag{9}$$

As the charge excess, $\Sigma_k Z_k n_k$, although minute, is not strictly zero, Φ_1 is a non-linear function of the radial distance, in the same way as $\Phi(r)$ varies in Equations (3) and (4) when $\Sigma_k n_k m_k \neq 0$, i.e. when $M(r)$ is an increasing function of r.

The electric field corresponding to the potential distribution (7) is given by

$$\mathbf{E}(\mathbf{r}) = \left[[\Phi_0 + \Phi_1 U(r - r_0)]\frac{r_0}{r^2} - (\Phi_1 + \Phi_2)\,\delta(r - r_0)\right]\frac{\mathbf{r}}{r} - \frac{r_0}{r}\,U(r - r_0)\,\nabla\Phi_1 \tag{10}$$

where $\delta(x)$ is the Dirac δ-function.

3. THE EQUATION OF MOTION OF A CHARGED PARTICLE IN THE EXOSPHERE

Following Eviatar *et al.* (1964), the equation of motion can be determined by writing down the law of conservation of the total energy in the static magnetic field and by using the first invariant in the guiding center approximation.

Hence we have:

$$v^2(\mathbf{r}) = v^2(\mathbf{r}_0) + 2\Phi_g(r_0)\left[1 + \alpha U(1 - y) - (1 + \beta)y + \frac{Ze(\Phi_1 y + \Phi_0)}{m\Phi_g(r_0)}\,U(y - 1)\right], \tag{11}$$

$$\sin^2\theta = \eta\,\frac{v^2(\mathbf{r}_0)}{v^2(\mathbf{r})}\,\sin^2\theta_0, \tag{12}$$

where θ and θ_0 are respectively the pitch angles at radial distances r and r_0, η is the relative magnetic field intensity along a line of force crossing the baropause at geomagnetic latitude λ_0:

$$\eta = \frac{B(r, \lambda)}{B(r_0, \lambda_0)}. \tag{13}$$

Moreover, we have used the following parameters

$$\alpha = \frac{Ze}{m\Phi_g(r_0)}\,(\Phi_0 - \Phi_2), \tag{14}$$

$$\beta = \frac{Ze}{m\Phi_g(r_0)}\,(\Phi_0 + \Phi_1), \tag{15}$$

which will be called the first and second reduced electric potential energies of the particle (Z, m).

In order to calculate the density and flux at a given altitude in the exosphere one has to classify the charged particles in several groups. According to their velocity and pitch angle there are trapped, incoming, escaping, and ballistic particles. All these different classes of particles are summarized in Tables 1(a) and (b).

TABLE 1(a). CLASSES OF PARTICLES WHICH HAVE A DECREASING VELOCITY WITH ALTITUDE $(1 + \alpha > (1 + \beta)y > 0)$.*

Class	$v(r_0)$	$\theta(r_0)$	$v(r)$	$\theta(r)$	Properties
a1	$[v_\infty(r_0), \infty]$ [(a)]	$\left[0, \frac{\pi}{2}\right]$	$[v_\infty(r), \infty]$ [(e)]	$[0, \theta_m(r)]$ [(g)]	Escaping particles.
a2	$[v_y(r_0), v_\infty(r_0)]$ [(b)]	$[0, \pi]$	$[v_y(r), v_\infty(r)]$ [(f)]	$[0, \theta_m(r)]$, $[\pi - \theta_m(r), \pi]$	Ballistic particles reaching level r with $\theta < \theta(r_0)$.
a3	$[v_x(r_0), v_y(r_0)]$ [(c)]	$[0, \theta_m(r_0)]$ [(d)], $[\pi - \theta_m(r_0), \pi]$	$[0, v_y(r)]$	$[0, \pi]$	Ballistic particles reaching level r with $\theta > \theta(r_0)$.
a4	$[0, v_x(r_0)]$	$[0, \pi]$	—	—	Ballistic particles not reaching level r.
a5	$[v_x(r_0), v_y(r_0)]$	$[\theta_m(r_0), \pi - \theta_m(r_0)]$	—	—	
a6	$[v_\infty(r_0), \infty]$	$\left[\frac{\pi}{2}, \pi\right]$	$[v_\infty(r), \infty]$	$[\pi - \theta_m(r), \pi]$	Incoming particles reaching the baropause.
a7	—	—	$[v_\infty(r), \infty]$	$[\theta_m(r), \pi - \theta_m(r)]$	Incoming particles never reaching the baropause.
a8	—	—	$[v_y(r), v_\infty(r)]$	$[\theta_m(r), \pi - \theta_m(r)]$	Trapped particles.

(a) $v_\infty^2(r_0) = -2(1+\alpha)\Phi_g(r_0)$; (b) $v_y^2(r_0) = \frac{R}{1-\eta}$; (c) $v_x^2(r_0) = R$; (d) $\sin^2\theta_m(r_0) = \frac{v^2(r_0) - R}{\eta v^2(r_0)}$; (e) $v_\infty^2(r) = -2(1+\beta)y\Phi_g(r_0)$;
(f) $v_y^2(r) = \frac{R\eta}{1-\eta}$; (g) $\sin^2\theta_m(r) = \eta\frac{v^2(r) + R}{v^2(r)}$; $R = -2[1 + \alpha - (1+\beta)y]\Phi_g(r_0)$.

* $\beta \geqslant (1+\alpha)\frac{\eta}{y} - 1$, for $v_y(r) \leqslant v_\infty(r)$.

TABLE 1(b). CLASSES OF PARTICLES FOR WHICH THE VELOCITY IS AN INCREASING FUNCTION OF ALTITUDE $(1 + \alpha < (1 + \beta)\, y < 0)$

Class	$v(r_0)$	$\theta(r_0)$	$v(r)$	$\theta(r)$	Properties
b1	$[0, \infty]$	$\left[0, \frac{\pi}{2}\right]$	$[v_z(r), \infty]^{(a)}$	$[0, \theta_m(r)]^{(b)}$	Escaping particles.
b2	$[0, \infty]$	$\left[\frac{\pi}{2}, \pi\right]$	$[v_z(r), \infty]$	$[\pi - \theta_m(r), \pi]$	Incoming particles reaching the baropause.
b3	—	—	$[v_z(r), \infty]$	$[\theta_m(r), \pi - \theta_m(r)]$	Incoming particles never reaching the baropause.
b4	—	—	$[0, v_z(r)]$	$[0, \pi]$	

(a) $v_z^2(r) = -R$; (b) $\sin^2 \theta_m(r) = \eta \dfrac{v^2(r) + R}{v^2(r)}$; $R = -2\,[1 + \alpha - (1 + \beta)y]\, \Phi_g(r_0)$.

In the case (a) (Table 1(a)) the velocity v is a decreasing function of the altitude $(1 + \alpha > (1 + \beta)y > 0)$. This occurs for the heavy oxygen ions which are bound to the Earth by the gravitational force. In Table 1(b) on the contrary, the velocity v is increasing with altitude. This is the case of protons accelerated outwards in the exosphere by an electric force which for these particles is larger than the gravitational force. Any ion gas for which $1 + \alpha < (1 + \beta)y < 0$ is therefore blown out of the ionosphere by the exospheric electric field.

4. VELOCITY DISTRIBUTION FUNCTION

In the assumption that above the baropause there are no collisions, Liouville's theorem can be applied to obtain the velocity distribution of the thermal particles in the exosphere (Herring and Kyle, 1961). Moreover, since in the barosphere the collision frequency is expected to be large enough to obtain rapid redistribution of particles, we will assume a Maxwellian velocity distribution function in this region. Hence the velocity distribution in the exosphere is given by

$$f(\mathbf{v}, r, \lambda) = n(r_0, \lambda_0) \left(\frac{m}{2\pi k T(r_0, \lambda_0)}\right)^{3/2} \exp\left[-q - \frac{mv^2}{2kT(r_0, \lambda_0)}\right] I(v, \theta), \qquad (16)$$

where for $y < 1$,

$$\left.\begin{aligned} q &= \Lambda[1 + \alpha - (1 + \beta)y], \\ \Lambda &= -\frac{m\Phi_g(r_0)}{kT(r_0, \lambda_0)}. \end{aligned}\right\} \qquad (17)$$

Furthermore, in the formula (16), $I(v, \theta)$ is a function which depends on the population of the different classes of particles at a level r in the exosphere. Under the assumption that the incoming particles which reach the baropause have the same Maxwellian velocity distribution as the barospheric particles, we introduced a parameter ζ such that $[n(r_0, \lambda_0)]_{\text{incoming}} = \zeta n(r_0, \lambda_0)$. If $\zeta = 0$ there are no incoming particles in the exosphere, and for $\zeta = 1$ one has a model exosphere in which the incoming particles are in thermal equilibrium with those escaping from the barosphere.

In a quite similar way we introduce a parameter ξ, in order to take into account the particles which are trapped above the baropause, $[n(r_0, \lambda_0)]_{\text{trapped}} = \xi n(r_0, \lambda_0)$. When $\xi = 0$, we consider that no trapped particles are present above the baropause (*UT*), and if $\xi = 1$, on the contrary, we assume that trapped particles are in thermal equilibrium with those emerging from the baropause (*T*). If $\xi = \zeta = 1$, all classes of particles are present and they are in thermal equilibrium, i.e. $I(v, \theta) \equiv 1$; we have then a barometric model (*B*).

It is easy to show that in the case (a), $I(v, \theta)$ is given by

$$I(v, \theta) = U(v - v_\infty) \,.\, U(\theta_m - \theta) + U(v_y - v) + U(v - v_y) \,.\, U(v_\infty - v)[U(\theta_m - \theta) + U(\theta + \theta_m - \pi)] + \zeta U(v - v_\infty) \,.\, U(\theta + \theta_m - \pi) + \xi U(v_\infty - v) \,.\, U(v - v_y) \,.\, U(\theta - \theta_m) \,.\, U(\pi - \theta_m - \theta), \quad (18)$$

where v_∞, v_y and θ_m are defined in Table 1(a).

On the other hand, in the case (b), i.e. when $1 + \alpha < (1 + \beta)y < 0$, we have

$$I(v, \theta) = U(v - v_z) \,.\, U(\theta_m - \theta) + \zeta[U(v_z - v) + U(v - v_z) \,.\, U(\theta - \theta_m)], \quad (19)$$

where v_z is defined in Table 1(b). If the incoming particles are in thermal equilibrium with the barospheric ones, $\zeta = 1$, hence $I(v, \theta) \equiv 1$ and we have once more a barometric model (B).

5. ESCAPE FLUX

The number of particles flowing each second through a unit surface normal to the magnetic field lines is given by

$$F(r, \lambda) = \int v_{\|} f(\mathbf{v}, \mathbf{r}) \, d^3 v. \quad (20)$$

In the case (a) where $1 + \alpha > (1 + \beta)y > 0$, using the expressions (16) and (18) we obtain

$$F(r, \lambda) = \tfrac{1}{4} n(r_0, \lambda_0) c_0 \eta (1 - \zeta)[1 + \Lambda(1 + \alpha)] \exp[-\Lambda(1 + \alpha)], \quad (21)$$

with

$$c_0 = \left[\frac{8kT(r_0, \lambda_0)}{\pi m}\right]^{1/2}. \quad (22)$$

Using the expressions (16) and (19), however, we obtain the escape flux in the case (b), where $1 + \alpha < (1 + \beta)y < 0$

$$F(r, \lambda) = \tfrac{1}{4} n(r_0, \lambda_0) c_0 \eta (1 - \zeta). \quad (23)$$

It is worthwhile noting that F is independent on the ξ-value, i.e. on the population parameter of the trapped particles, and that in both cases (a) and (b), the evaporative flux vanishes in the barometeric model, i.e. when $\zeta = 1$.

To avoid any steady electric charge accumulation a necessary condition is given by the equality of the electron flux and the total positive ion flux:

$$\textstyle\sum_j F_j(r, \lambda) = F_e(r, \lambda). \quad (24)$$

This equality must be satisfied at every point (r, λ) in the exosphere. The condition (24) yields a relation between α_e and all the other α_i and will fix the value of $(\Phi_0 - \Phi_2)$.

Moreover as can easily be seen from the definitions (14) and (17), the α_i and Λ_i for each kind of ion are related to α_e and Λ_e by

$$\alpha_i = -\alpha_e \frac{Z_i m_e}{m_i}; \qquad \Lambda_i = \Lambda_e \frac{m_i T_e}{m_e T_i}; \qquad \Lambda_e = \frac{GMm_e}{r_0 k T_e}. \quad (25)$$

Hence for fixed values of $n_i(r_0, \lambda_0)$, $n_e(r_0, \lambda_0)$, $T_i(r_0, \lambda_0)$ and $T_e(r_0, \lambda_0)$ at the baropause, the value of α_e can be calculated by means of (24).

Considering an exosphere in which only O^+ and H^+ ions are present, numerical calculations show that $1 + \alpha_{O^+}$ and $1 + \alpha_e$ are positive, i.e. the exospheric oxygen plasma is bounded to the Earth, and the evaporative fluxes F_{O^+} and F_e are given by formula (21) (case (a)). For the protons, however, $1 + \alpha_{H^+}$ is generally negative. Hence F_{H^+} is given by (23) (case (b)) and the protons are all blown out of the exosphere by the large polarization electric field. Therefore condition (24) applied to an exosphere with O^+ and H^+ ions gives the following equation:

$$\frac{n_{O^+}(r_0, \lambda_0)}{n_e(r_0, \lambda_0)} \left[\frac{m_e T_{O^+}(r_0, \lambda_0)}{m_{O^+} T_e(r_0, \lambda_0)}\right]^{1/2} [1 + (1 + \alpha_{O^+})\Lambda_{O^+}] \exp [-(1 + \alpha_{O^+})\Lambda_{O^+}]$$
$$+ \frac{n_{H^+}(r_0, \lambda_0)}{n_e(r_0, \lambda_0)} \left[\frac{m_e T_{H^+}(r_0, \lambda_0)}{n_{H^+} T_e(r_0, \lambda_0)}\right]^{1/2} = [1 + (1 + \alpha_e)\Lambda_e] \exp [-(1 + \alpha_e)\Lambda_e]. \quad (26)$$

Table 2 shows the values of $1 + \alpha_j$ as a function of the relative concentration $n_{H^+}(r_0, \lambda_0)/n_e(r_0, \lambda_0)$ for $r_0 = (6371 + 2000)$ km and $T_e(r_0, \lambda_0) = T_{O^+}(r_0, \lambda_0) = T_{H^+}(r_0, \lambda_0) = 3000°K$.

TABLE 2. VALUES OF $(1 + \alpha_j)$ IN AN $(O^+ - H^+ - e)$ EXOSPHERE, AS A FUNCTION OF THE RATIO $n_{H^+}(r_0, \lambda_0)$ $n_e(r_0, \lambda_0)$; BAROPAUSE ALTITUDE IS 2000 km AND $T_e(r_0, \lambda_0) = T_{O^+}(r_0, \lambda_0) = 3000°K$

$n_{H^+}(r_0, \lambda_0)/n_e(r_0, \lambda_0)$	$1 + \alpha_e$	$1 + \alpha_{O^+}$	$1 + \alpha_{H^+}$	$n_{O^+}(r_0, \lambda_0)/n_e(r_0, \lambda_0)$
0·000	1·733 10^4	0·410	−8·440	1·000
0·001	1·285 10^4	0·563	−5·998	0·999
0·010	1·044 10^4	0·644	−4·688	0·990
0·100	7·992 10^3	0·728	−3·352	0·900
0·500	6·229 10^3	0·788	−2·392	0·500
1·000	5·450 10^3	0·814	−1·968	0·000

As can be seen from this table the exospheric electric field depends very strongly on the concentration of the hydrogen ions even if they form only a minor constituent at the baropause. Hence an increase of the number of light ions (i.e. an increase of the total ion-escape-flux) diminishes the absolute value of the exospheric electrostatic potential $|\Phi_2|$.

Moreover, as shown in Table 3, which gives the $1 + \alpha_j$ values as a function of the exospheric temperature for a constant ion-composition $n_{H^+}(r_0, \lambda_0)/n_e(r_0, \lambda_0) = 0{\cdot}10$, the reduced first electric potential energy for the electrons α_e and the electric potential $|\Phi_2|$ decrease with decreasing temperature.

6. DENSITY DISTRIBUTION

By definition the density distribution of the particles is given by

$$n(r, \lambda) = \int f(\mathbf{r}, \mathbf{v})\, d^3v. \quad (27)$$

TABLE 3. VALUES OF $(1 + \alpha_j)$ AS A FUNCTION OF THE EXOSPHERIC TEMPERATURES AT THE BAROPAUSE FOR $n_{H^+}(r_0, \lambda_0)/n_e(r_0, \lambda_0) = 0{\cdot}10$ AND $r_0 = 8371$ km

$T_e(r_0, \lambda_0)$ (°K)	$T_{O^+}(r_0, \lambda_0) = T_{H^+}(r_0, \lambda_0)$ (°K)	$1 + \alpha_e$	$1 + \alpha_{O^+}$	$1 + \alpha_{H^+}$
1000	1000	2·663 10^3	0·909	−0·450
2000	2000	5·327 10^3	0·818	−1·901
3000	3000	7·991 10^3	0·728	−3·352
4000	4000	1·065 10^4	0·637	−4·803
5000	5000	1·332 10^4	0·546	−6·253
3000	2000	8·210 10^4	0·720	−3·471

Taking into account (16) and (18) we find for the case (a), when $1 + \alpha > (1 + \beta)y > 0$

$$n(r, \lambda) = n(r_0, \lambda_0)\Big\{(1 + \zeta)K_2(\infty) + (1 - \zeta)K_2(V_\infty) - (1 - \eta)^{1/2}[(1 - \zeta)K_2(\infty) + (1 + \zeta - 2\xi)K_2(X_\infty)] \exp\left(-\frac{\eta_q}{1 - \eta}\right)\Big\} \exp(-q) \quad (28)$$

where

$$V_\infty{}^2 = \Lambda(1 + \beta)y; \qquad X_\infty{}^2 = \Lambda \frac{(1 + \beta)y - (1 + \alpha)\eta}{1 - \eta}. \quad (29)$$

The functions $K_m(x)$ are defined in Appendix B. They can be expressed explicitly in terms of exponential functions and the error function erf (x).

On the other hand, if $1 + \alpha < (1 + \beta)y < 0$, (case (b)) substitution of formula (16) and (19) in (27) yields

$$n(r, \lambda) = n(r_0, \lambda_0)\Big\{(1 + \zeta)K_2(\infty) - (1 - \zeta)K_2[(-q)^{1/2}] - (1 + \zeta)(1 - \eta)^{1/2}\left[K_2(\infty) - K_2\left[\left(\frac{-q}{1 - \eta}\right)^{1/2}\right]\right] \exp\left(-\frac{\eta q}{1 - \eta}\right)\Big\} \exp(-q). \quad (30)$$

It may be noted that for $\zeta = \xi = 1$ both formula (28) and (30) reduce to a barometric model (B).

The condition of quasi-neutrality in the exosphere implies that the relation

$$\textstyle\sum_k Z_k n_k(r, \lambda) \simeq 0, \quad (31)$$

is satisfied.

As the density distribution $n_j(r, \lambda)$ of each constituent depends on $\beta_j(r, \lambda)$, the Equation (31) enables us to calculate the value of $\beta_e(r, \lambda)$ as the β_j are related to β_e by

$$\beta_j = -\frac{Z_j m_e}{m_j}\beta_e. \quad (32)$$

For instance, in an $(O^+ - e)$-exosphere, condition (31) taken at the level r yields

$$n_{O^+}(r, \lambda, \alpha_{O^+} \beta_{O^+}) = n_e(r, \lambda, \alpha_e, \beta_e). \quad (33)$$

Hence using Equations (32) and (33) we can calculate β_e the reduced second electric potential energy, for the electrons. In order to maintain the quasi-neutrality, β_e must be slowly varying functions of the radial distance r and the geomagnetic latitude λ.

In Fig. 1 we have plotted the value $\beta_e(r)$ as a function of altitude at a geomagnetic latitude of 90°, for an $(O^+ - e)$-exosphere and for $r_0 = (6371 + 2000)$ km, $T_e = T_{O^+} =$ 3000°K. Two models have been considered: The 'untrapped' model (UT) for which $\zeta = \xi = 0$ and the 'trapped' model (T) for which $\zeta = 0$, $\xi = 1$.

As β_e is approximately a linear function of r, it can be seen from Equation (15) that

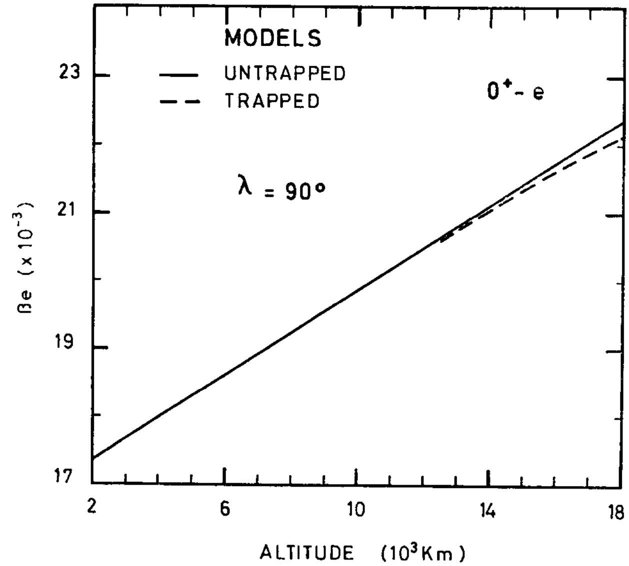

FIG. 1. VALUES OF THE SECOND REDUCED ELECTRICAL POTENTIAL β_e IN AN (O^+ − e) EXOSPHERE VS. ALTITUDE ABOVE THE GEOMAGNETIC POLE ($\lambda = 90°$) FOR THE TRAPPED AND UNTRAPPED MODELS. The altitude of the baropause is 2000 km; $T_e(r_0, \lambda_0) = T_{O^+}(r_0, \lambda_0) = 3000°K$; $\alpha_e = 1{\cdot}733 \times 10^4$ (cf. Table 2); $\beta_{O^+} = -\beta_e m_e/m_{O^+}$.

$\Phi_1(r)$ is also a quasi-linear function of the altitude. Therefore the electric field in an (O^+ − e)-exosphere, given by Equation (10), or by

$$\mathbf{E}(\mathbf{r}) = \frac{mg(r)}{Ze}\left(\beta\frac{\mathbf{r}}{r} - r\nabla\beta\right) \tag{34}$$

is practically equal to $(m_e - m_{O^+})\,\mathbf{g}/2e$.

It can also be verified that $\mathrm{d}^2\Phi_1/\mathrm{d}r^2$ is extremely small so that Equations (9) and (31) are consistent to a high degree of accuracy.

7. THE MEAN EXPANSION VELOCITY

The mean velocity w of the particles is parallel to the magnetic field lines and is defined by the relation

$$w(r, \lambda) = F(r, \lambda)/n(r, \lambda). \tag{35}$$

Moreover the mean mass motion in the exosphere which is given by

$$u(r, \lambda) = \textstyle\sum_k n_k m_k w_k / \sum_k n_k m_k,$$

is equal to zero for $\zeta = 1$. In the trapped and untrapped models, w and u are increasing functions of r. The mean velocities w and u vanish in a barometric model exosphere.

8. PRESSURE TENSORS AND TEMPERATURES

The longitudinal and transverse momentum flux tensors $P_\parallel$ and $P_\perp$ are respectively defined by

$$\left.\begin{aligned} P_\parallel &= m \int v_\parallel^2 f(\mathbf{v}, \mathbf{r})\, \mathrm{d}^3 v \\ P_\perp &= m \int v_\perp^2 f(\mathbf{v}, \mathbf{r})\, \mathrm{d}^3 v \end{aligned}\right\} \tag{36}$$

and in the case (a), when $1 + \alpha > (1 + \beta) y > 0$, one finds,

$$P_{\|}(r, \lambda) = \tfrac{2}{3}p(r, \lambda)\Big\{(1 + \zeta) \,.\, K_4(\infty) + (1 - \zeta) \,.\, K_4(V_\infty) - [(1 - \zeta) \,.\, K_4(\infty) + (1 + \zeta - 2\xi) \,.\, K_4(X_\infty)] \,.\, (1 - \eta)^{3/2} \exp\left(-\frac{\eta q}{1 - \eta}\right)\Big\} \exp(-q), \tag{37}$$

$$P_{\perp}(r_0, \lambda_0) = \tfrac{4}{3}p(r_0, \lambda_0)\Big\{(1 + \zeta) \,.\, K_4(\infty) + (1 - \zeta) \,.\, K_4(V_\infty) - \left(1 + \frac{\eta}{2}\right)(1 - \eta)^{1/2} \times [(1 - \zeta) \,.\, K_4(\infty) + (1 + \zeta - 2\xi) \,.\, K_4(X_\infty)] \exp\left(-\frac{\eta q}{1 - \eta}\right) - \tfrac{3}{2}\,\eta q(1 - \eta)^{-1/2} \times [(1 - \zeta) \,.\, K_2(\infty) + (1 + \zeta - 2\xi) \,.\, K_2(X_\infty)] \exp\left(-\frac{\eta q}{1 - \eta}\right)\Big\} \exp(-q), \tag{38}$$

where $p(r_0, \lambda_0) = n(r_0, \lambda_0)\, kT(r_0, \lambda_0)$.

If on the contrary, $1 + \alpha < (1 + \beta)y < 0$, we obtain

$$P_{\|}(r, \lambda) = \tfrac{2}{3}p(r_0, \lambda_0)\Big\{(1 + \zeta) \,.\, K_4(\infty) - (1 - \zeta) \,.\, K_4[(-q)^{1/2}] - (1 - \zeta)(1 - \eta)^{3/2}\left[K_4(\infty) - K_4\left[\left(\frac{-q}{1 - \eta}\right)^{1/2}\right]\right] \exp\left(-\frac{\eta q}{1 - \eta}\right)\Big\} \exp(-q), \tag{39}$$

$$P_{\perp}(r, \lambda) = \tfrac{4}{3}p(r_0, \lambda_0)\Big\{(1 + \zeta) \,.\, \mathrm{K}_4(\infty) - (1 - \zeta) \,.\, K_4[(-q)^{1/2}] - (1 - \zeta)\left(1 + \frac{\eta}{2}\right)(1 - \eta)^{1/2}\left[K_4(\infty) - K_4\left[\left(\frac{-q}{1 - \eta}\right)^{1/2}\right]\right] \exp\left(-\frac{\eta q}{1 - \eta}\right) - \tfrac{3}{2}\eta q(1 - \zeta)(1 - \eta)^{-1/2}\left[K_2(\infty) - K_2\left[\left(\frac{-q}{1 - \eta}\right)^{1/2}\right]\right] \exp\left(\frac{-\eta q}{1 - \eta}\right)\Big\} \exp(-q). \tag{40}$$

Using the results (37)–(40) the longitudinal and transverse pressures can be calculated by means of

$$p_{\|} = P_{\|} - nmw^2, \tag{41}$$

$$p_{\perp} = P_{\perp}. \tag{42}$$

The longitudinal and transverse temperatures are defined by

$$kT_{\|} = \frac{p_{\|}}{n}, \qquad kT_{\perp} = \frac{p_{\perp}}{n} \tag{43}$$

and may be calculated from (28), (37), (38), (39), (40), (41) and (42).

9. DISCUSSION

In all our formulas we have not yet specified the function $\eta(r, \lambda)$ which according to (13) is the ratio of the magnetic field strength taken at two different points of a field line. As the dipole configuration is a fairly good approximation up to a radial distance of about four Earth radii (Mead, 1964; Roederer, 1969) we have used this geometry to determine $\eta(r, \lambda)$ in the region between 2000 and 18,000 km. Hence

$$n(r, \lambda) = y^3 \frac{(4 - 3\cos^2 \lambda)^{1/2}}{(4 - 3\cos^2 \lambda_0)^{1/2}}. \tag{44}$$

To obtain the density and mean velocity along the radial direction corresponding to a geomagnetic latitude λ, λ_0 has to be defined by $\lambda_0 = \text{arcos}\,(y^{1/2} \cos \lambda)$.

In Fig. 2 we have plotted the relative radial density distributions in an $(O^+ - e)$-exosphere at a geomagnetic latitude of 90°. At the baropause which is situated at an altitude of 2000 km, we have assumed that the ion and electron temperatures are both equal to 3000°K. Moreover we considered three different models: (1) the untrapped model, Equation (28) where $\zeta = \xi = 0$, (*UT*-model); (2) the trapped model, Equation (28) where $\zeta = 0$ and $\xi = 1$ (*T*-model); (3) the barometric model, Equation (28) where $\zeta = \xi = 1$ (*B*-model).

We have also considered an exosphere build-up of electrons and oxygen and hydrogen ions $(O^+ - H^+ - e)$. In this case we calculated numerically the density and mean velocity for each constituent along a given field line, i.e. for $\lambda = \text{arcos}\,(y^{-1/2} \cos \lambda_0)$. Note that if λ_0 the geomagnetic latitude at the baropause is smaller than the lower limit, $\lambda_{\text{min.}}$, ($\lambda_{\text{min.}} \approx 60°$), the field lines of the magnetosphere are closed and all the particles are trapped (Thomas

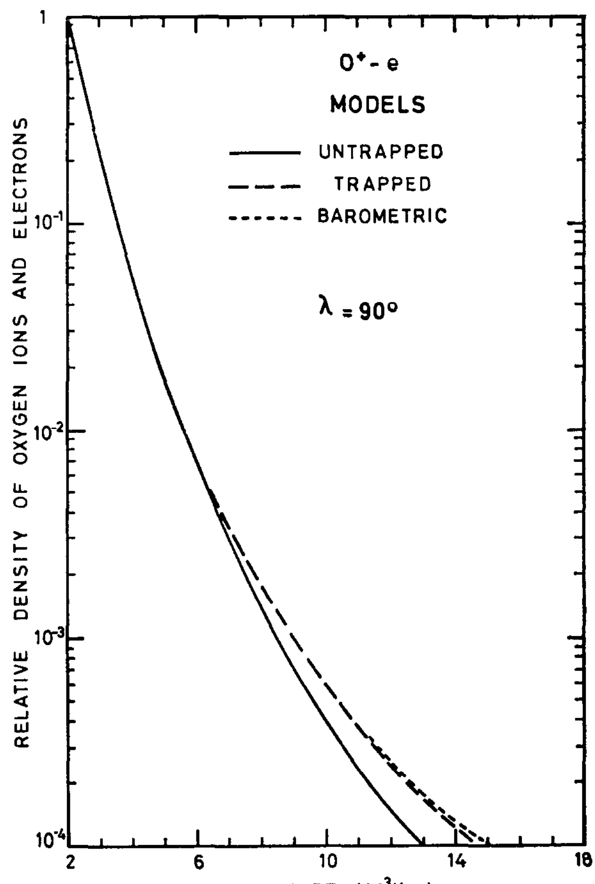

FIG. 2. DENSITY RATIO $n_{O^+}(r,\lambda)/n_{O^+}(r_0, \lambda_0)$ VS. ALTITUDE ALONG A MAGNETIC FIELD LINE CROSSING THE BAROPAUSE AT 2000 km AND AT 90° LATITUDE ($\lambda = \lambda_0 = 90°$) FOR THE TRAPPED, 'UNTRAPPED' AND BAROMETRIC MODELS.
The baropause temperatures are $T_{O^+} = T_e = 3000°$K.

and Andrews, 1969). In this case one must use the model ion-exosphere proposed by Eviatar *et al.* (1964) or by Hartle (1969).

In Fig. 3 we have plotted the density distributions along a line of force crossing the baropause at the geomagnetic latitude $\lambda_0 = 80°$. Furthermore, considering the untrapped and trapped models we made the calculations for the following concentrations: $n_e(r_0, \lambda_0) = 10^3$ cm^{-3}, $n_{O^+}(r_0, \lambda_0) = 9\ 10^2$ cm^{-3}, $n_{H^+}(r_0, \lambda_0) = 10^2$ cm^{-3}, with the temperatures $T_e(r_0, \lambda_0) = T_{O^+}(r_0, \lambda_0) = T_{H^+}(r_0, \lambda_0) = 3000$°K. From Table 2, we immediately find $\alpha_e = 7{\cdot}991\ 10^3$; $\alpha_{O^+} = -0{\cdot}272$; $\alpha_{H^+} = -4{\cdot}352$.

To obtain at each level r the appropriate value of β_e such that the quasi-neutrality

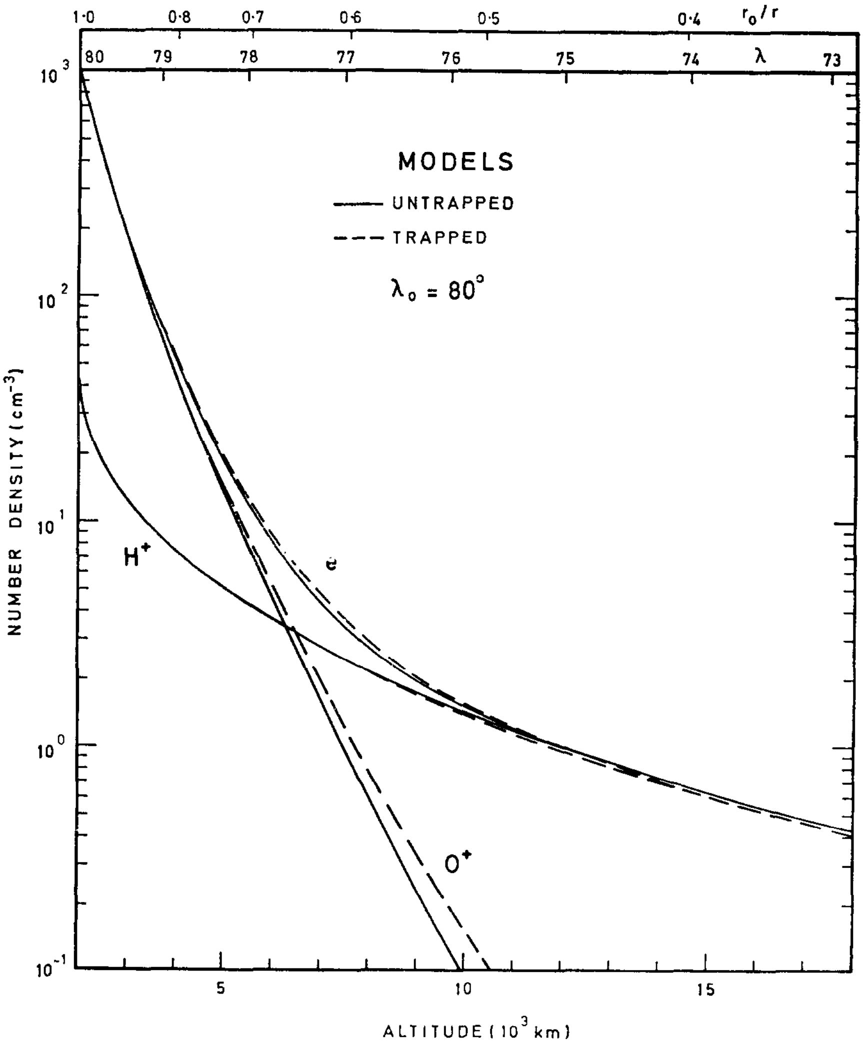

FIG. 3. DENSITY DISTRIBUTIONS IN AN ($O^+ - H^+ - e$)- EXOSPHERE VS. ALTITUDE ALONG A DIPOLE MAGNETIC FIELD LINE CROSSING THE BAROPAUSE AT 80° LATITUDE.

The solid and dashed lines correspond to the 'untrapped' and 'trapped' models respectively. The baropause temperatures and concentrations are: $T_e = T_{O^+} = T_{H^+} = 3000$°K; $n_e{:}n_{O^+}{:}n_{H^+} = 10{:}9{:}1$.

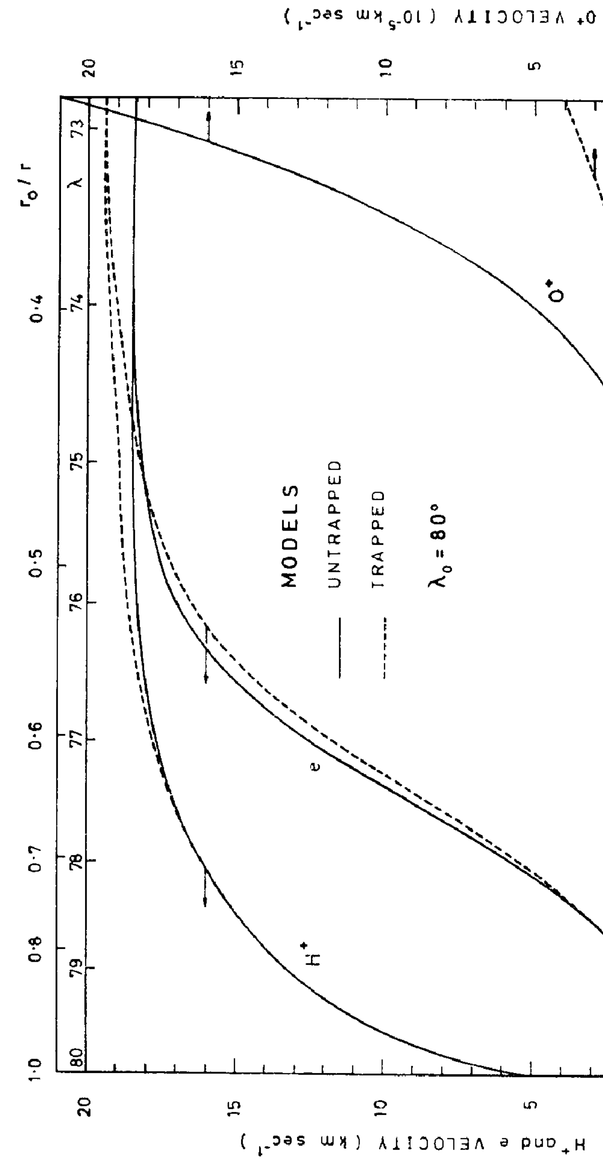

FIG. 4. MEAN VELOCITIES IN AN ($O^+ - H^+ - e$)- EXOSPHERE VS. ALTITUDE, ALONG A DIPOLE MAGNETIC FIELD LINE CROSSING THE BAROPAUSE AT 80° LATITUDE.

The solid and dashed lines correspond to the 'untrapped' and 'trapped' models respectively. In both cases, the baropause temperatures and concentrations are: $T_e = T_{O^+} = T_{H^+} = 3000°K$; $n_e:n_{O^+}:n_{H^+} = 10:9:1$.

condition is satisfied, we calculated the solutions of Equation (31) by means of an iterative process. The density distributions are respectively given by (28) for the electrons and oxygen ions, and by (30) for the protons.

It can be seen that the O^+ concentration decreases much more rapidly when light ions are present in the exosphere. Note that the trapped particles do not contribute much to increase the O^+ density below an altitude of 6000 km. For the barospheric conditions we used in this paper, the O^+ ions remain the most abundant constituent up to an altitude of 6300 km.

At large distances the electronic density decreases as r^{-3} due to the dipolar geometry of the magnetic field.

Finally it is worthwhile mentioning that the density distributions along an open field line do not differ significantly with the choice of the field line ($75° < \lambda_0 < 90°$); e.g. n_{H^+} (17,000 km, 75°) is only 4 per cent smaller than n_{H^+} (17,000 km, 90°).

Figure 4 shows the mean velocities of O^+, H^+ and e under the same conditions as for Fig. 3. It can be seen that w_{O^+} remains quite small (< 20 cm sec^{-1}) comparatively to the mean velocity of the protons which are accelerated by the electric field to supersonic velocities reaching 20 km sec^{-1} at large distances. At an altitude of 3000 km, w_{H^+} is equal to 11 km sec^{-1} in both of our untrapped and trapped models. These results are in good agreement with the 10 km sec^{-1} value reported by Dessler and Cloutier (1969).

In both models the fluxes at the baropause are $F_e = 2{\cdot}0 \times 10^7$ cm^{-2} sec^{-1}, $F_{O^+} = 2{\cdot}4 \times 10^{-1}$ cm^{-2} sec^{-1} and $F_{H^+} = 2{\cdot}0 \times 10^7$ cm^{-2} sec^{-1}.

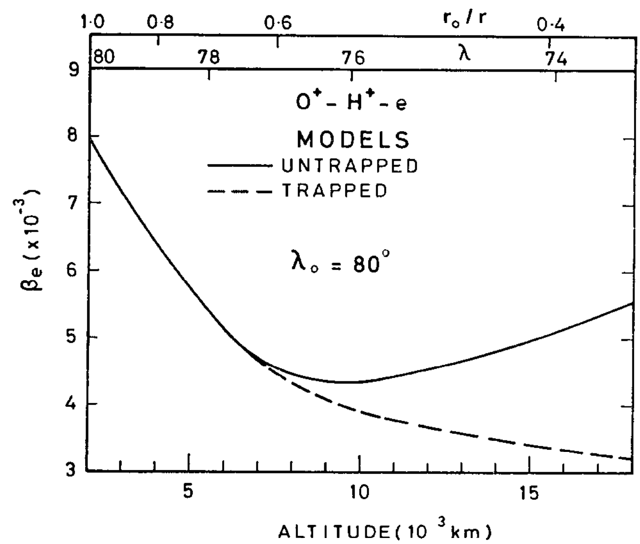

FIG. 5. ALTITUDE DISTRIBUTION OF β_e IN AN ($O^+ - H^+ - e$)- EXOSPHERE, FOR THE 'TRAPPED' AND 'UNTRAPPED' MODELS UNDER THE SAME CONDITIONS AS IN FIGS. 3 AND 4.

Figure 5 gives the altitude dependence of β_e for the three components ion-exosphere. Comparison with Fig. 1 shows that an amount of only 10 per cent hydrogen ions at the baropause level reduces by a large factor the value of β_e and the electric potential Φ_1.

In Fig. 6 we compare the intensity of the electric field in two 'untrapped models' along the field line, $\lambda_0 = 90°$, firstly in the case of an (O^+-e) exosphere and secondly for an

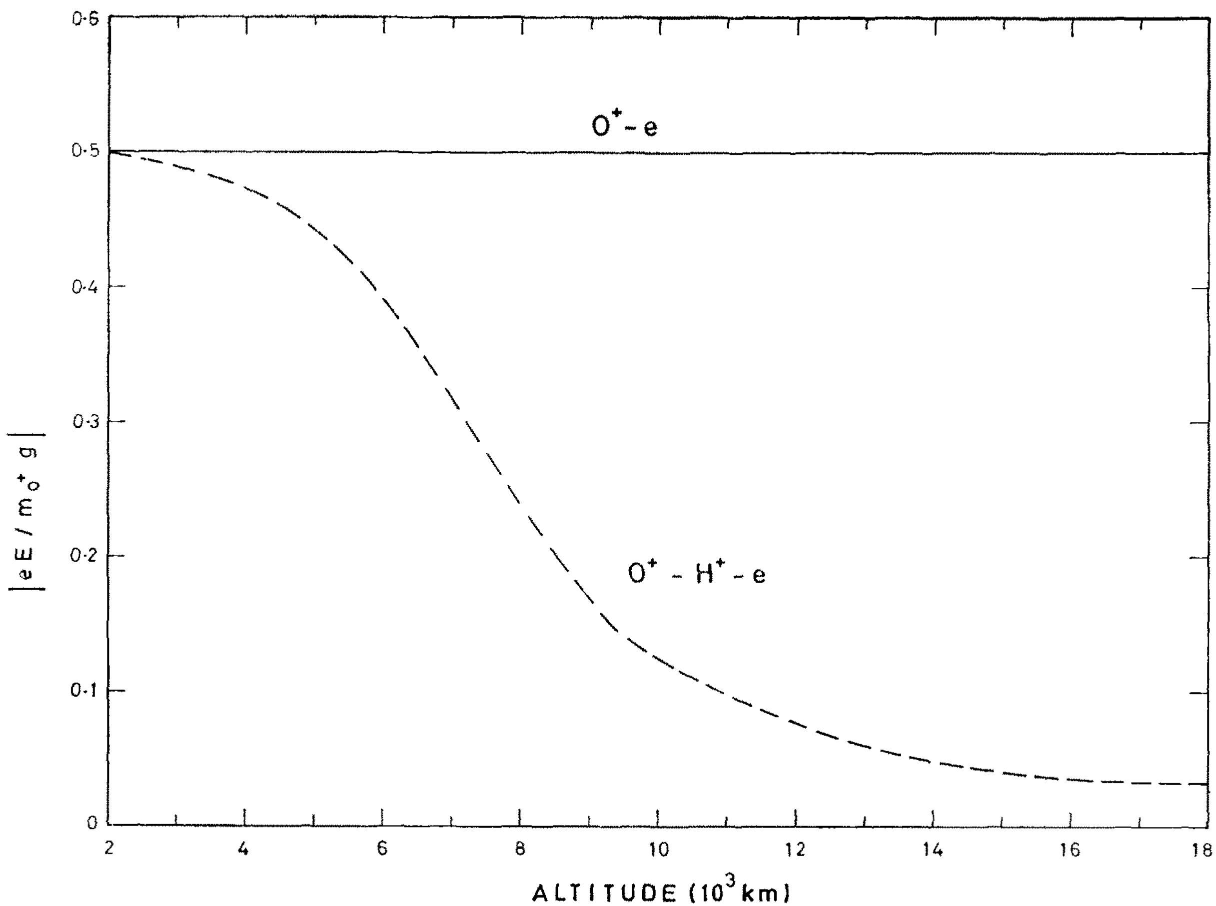

FIG. 6. RATIO OF THE ELECTRIC FORCE (eE) AND GRAVITATIONAL FORCE ($g\,m_{O^+}$) ACTING UPON AN O^+ ION VS. ALTITUDE ABOVE THE GEOMAGNETIC POLE ($\lambda = \lambda_0 = 90°$) FOR TWO 'UNTRAPPED' MODELS: ($O^+ - e$) AND ($O^+ - H^+ - e$) FOR THE SAME TEMPERATURE AND CONCENTRATION CONDITIONS AS IN FIGS. 1–5.

($O^+ - H^+ - e$) atmosphere. It can be seen that in the first case the ratio of the electric (eE) and gravitational ((gm_{O^+}) force is practically equal to $\frac{1}{2}$ as in Pannekoek's (1922) theory. However a small percentage of light ions at the baropause reduces $|eE/gm_{O^+}|$ to $m_{H^+}/2m_{H^+} = 0{,}03125$ at the high altitudes where H^+ becomes the most abundant constituent. Therefore at large distances, eE, becomes equal to $-\frac{1}{2}gm_{H^+}$.

10. CONCLUSION

The model ion-exosphere which we have proposed in this paper can be used for a magnetic field with open field lines. The electric potential and the corresponding radial electric field in this region have been obtained by requiring quasi-neutrality and equality between the electron and positive ion fluxes i.e. the zero electric current condition. Using a magnetic dipole field, we have made numerical calculations of the density distributions and mean velocities of the different constituents. The mean expansion velocity of the thermal protons is in our models of the order of 11 km sec^{-1} at an altitude of 3000 km.

Moreover by putting $Z = 0$ (leading to $\alpha = \beta = 0$) and $\eta(r, \lambda) = y^2$ we recover the well-known neutral exospheric models. Indeed with the above assumption our 'untrapped' model with $\zeta = \xi = 0$ reduces to the Öpik–Singer (1959, 1961) model where the satellite (or trapped) particles were neglected, and the 'trapped' model, where $\zeta = 0$ and $\xi = 1$, yields Chamberlain's model (1960, 1963) where these latter particles contribute to the density in the exosphere.

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APPENDIX A

In an isothermal ionized atmosphere which is in hydrostatic and diffusion equilibrium the density distributions satisfy the equation

$$kT_j\nabla n_j = m_j n_j \mathbf{g} + Z_j n_j e\mathbf{E}. \tag{A1}$$

The electrostatic field, $\mathbf{E}$, preventing a significant charge separation is given by

$$e\mathbf{E} = -\mu(r)\mathbf{g}, \tag{A2}$$

where

$$\mu(r) = (\textstyle\sum_j Z_j m_j n_j/kT_j)/(\sum_j Z_j^2 n_j/kT_j). \tag{A3}$$

Applying the differential operator ∇ to (A2) and using Poisson's Equations (2) and (4) we obtain

$$4\pi e^2 \textstyle\sum_j Z_j n_j = 4\pi G\mu(r)\sum_k m_k n_k - \mathbf{g}\cdot\nabla\left(\dfrac{\sum_j Z_j m_j n_j/kT_j}{\sum_j Z_j^2 n_j/kT_j}\right). \tag{A4}$$

Taking into account (A1) and (A2) we can calculate the second term in the right hand side of (A5). Finally the electric charge density is given by:

$$\textstyle\sum_j Z_j n_j = \dfrac{G}{e^2}\,\mu(r)\sum_j m_j n_j - \dfrac{g^2}{4\pi e^2}\sum\limits_j \dfrac{Z_j n_j}{(kT_j)^2}\cdot\dfrac{(m_j - Z_j\mu)^2}{\sum_i Z_i^2 n_i/kT_i}. \tag{A5}$$

In the case of a pure hydrogen isothermal atmosphere and with the assumption that $T_{\mathrm{H}^+} = T_e =$ constant, the charge excess due to gravitational effects is very small

$$\frac{n_{\mathrm{H}^+} - n_e}{n_e} \simeq \frac{Gm^2_{\mathrm{H}^+}}{2e^2} \simeq 4\ 10^{-37}.$$

In a pure O^+ isothermal atmosphere it would be 256, e.g. $(m_{\mathrm{O}^+}/m_{\mathrm{H}^+})^2$, times larger but still remain minute. It can be shown that the addition of a third kind of ions in diffusion equilibrium does not change the order of magnitude of this result and that quasi-neutrality in the barospheric region where Equation (A1) is valid is always satisfied, i.e.

$$\textstyle\sum_k Z_k n_k \simeq 0. \tag{A6}$$

In the exosphere where the Equations (A1), (A2), (A3) are not applicable, the charge excess is no longer given by Equation (A4). In this case (9) has to be used. From our models exospheres where $\Phi_1(r)$ has been calculated we have evaluated the excess of charge concentration. It comes out that $|\Sigma_k Z_k n_k/n_e|$ is smaller than 10^{-12} except in a small transition region close to the barosphere. Hence the quasi-neutrality condition (A6) is also satisfied in the exosphere to a very high degree of accuracy.

APPENDIX B

The functions $K_m(b)$ are defined by

$$K_m(b) = \frac{2}{\pi^{1/2}}\int_0^b x^m e^{-x^2}\,\mathrm{d}x. \tag{B1}$$

Partial integration yields immediately the recurrence formula

$$K_m(b) = \frac{m-1}{2} K_{m-2}(b) - \frac{b^{m-1}}{\pi^{1/2}} \exp(-b^2), \qquad \text{(B2)}$$

which enables calculation of $K_m(b)$ in terms of the well-known error function, and in terms of exponential functions.

Indeed, straightforward calculations yield

$$K_0(b) = \operatorname{erf}(b), \qquad \text{(B3)}$$

$$K_1(b) = \pi^{-1/2}[1 - \exp(-b^2)].$$

Note also that $K_m(\infty) = 1.3.5...(m-1)/2^{m/2}$ if m is even and $K_m(\infty) = 2.4.6...(m-1)/(2^{(m-1)/2}\pi^{1/2})$ if m is odd.