

The plasmopause as a plasma sheath: a minimum thickness

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Abstract—We consider the plasmopause as a stationary boundary layer (a sheath) separating two types of plasmas characterized by different temperatures and densities: on one side, the hot trapped particles imbedded in the cold exospheric plasma of ionospheric origin, and on the other side the plasmatrough including the hot ring current particles. The structure of this layer is described by a kinetic model using the Vlasov–Maxwell's equations for the charged particles and fields. In absence of any collisional effects or wave-particle interactions a minimum value for the thickness of the plasmopause is deduced which is of the order of 5 times the Larmor radius of the cold ions.

1. INTRODUCTION

We define the plasmopause as a boundary layer separating two regions, the plasmasphere and the plasmatrough, each of which having distinctly different properties. Cold plasma of approximately 1 eV evaporated from the top of the ionosphere is the main constituent of the plasmasphere. In the plasmatrough, hot ring current particles with characteristic energy 0.1–50 keV, of magnetospheric origin, become the dominant particles.

The purpose of this paper is to present a microscopic and basic description of the structure of this boundary. It is assumed that a change in the macroscopic velocity occurs across the plasma sheath: the plasmasphere corotates with the Earth, while the plasmatrough has a larger drift velocity. For describing the structure of this sheath we use a model based on an extension of the work of SESTERO (1964, 1966) on contact discontinuities in collisionless plasmas. This model is described in Section 2. Boundary conditions on each side of the plasmopause are described in Section 3. In Section 4, these boundary conditions are used to calculate some characteristic lengths and velocities, and to determine numerically the variations of the most important physical parameters through the transition. Conclusions are summarized in Section 5.

2. DESCRIPTION OF THE MODEL

Figure 1 represents the geometry of the model. It is a local representation of the plasmopause in the night-side sector. The x - y plane is the plane of the Earth Equator and the y - z plane represents the boundary surface. We make the assumption of steady-state and we assume that every physical quantity depends on one space coordinate, the x coordinate. The magnetic field \mathbf{B} points along the z -axis while the electric field \mathbf{E} is directed along the

x -axis. The current \mathbf{J} and the vector potential \mathbf{a} are in the y -direction. The plasma moves in the y -direction with different velocities on both sides of the boundary layer. In the plasmasphere and plasmatrough regions, far enough from the plasmopause, the plasma is assumed to be uniform and Maxwellian. The macroscopic velocities are denoted by \mathbf{V}_1 in the plasmasphere and by \mathbf{V}_2 in the plasmatrough. Cold and hot plasmas contain only hydrogen ions.

A microscopic description of this sheath is made by using Vlasov's equation for the distribution functions of the charged particles coupled with the Maxwell's equations for the electromagnetic fields. Any function of the constants of motion is a solution of Vlasov's equation. The constants of motion are the total energy and the generalized momentum p along the y -axis:

$$\varepsilon = \frac{1}{2} m_{\pm} (u^2 + v^2) \pm e\phi(x) \quad (1)$$

$$p = m_{\pm} v \pm ea(x). \quad (2)$$

The subscripts $+$ and $-$ are respectively for ions and electrons. Here u and v are the components of velocities along the x and y axes, e is the charge of the proton, m_{\pm} are the masses of the charged particles, \mathbf{a} and ϕ are the magnetic vector potential and the scalar electric potential satisfying Maxwell's equations:

$$\frac{1}{\mu} \frac{d^2 a}{dx^2} = -J \quad (3)$$

$$\kappa \frac{d^2 \phi}{dx^2} = -q \quad (4)$$

J and q are the current and charge densities, κ and μ being the permittivity and permeability of the vacuum in rationalized M.K.S. units.

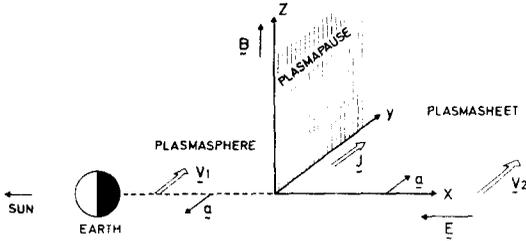


Fig. 1. Geometry of the model.

In the situation considered here, we have a mixture of cold and hot non-collisionless plasmas interacting in the transition layer separating two uniform regions with distinctly different properties and including a change in the macroscopic velocity of the two regions. The simplest model describing such a transition (SESTERO, 1964, 1966) uses, for each type of plasma, distribution functions for the charged particles $F_{\pm}^i(p, \varepsilon)$, which are the sum of two different displaced Maxwellians (the superscripts i denotes the cold (c) or the hot (h) plasma):

$$F_{\pm}^i(p, \varepsilon) = F_{1\pm}^i(p, \varepsilon) + F_{2\pm}^i(p, \varepsilon) \quad (5)$$

with

$$F_{1\pm}^i(p, \varepsilon) = \begin{cases} \frac{N_{\pm}^i m_{\pm}}{2\pi k T_{1\pm}^i} \cdot \exp\left(-\frac{m_{\pm} V_1^2}{2kT_{1\pm}^i}\right) \cdot \exp\left(\frac{V_1 p}{kT_{1\pm}^i}\right) \\ \quad \times \exp\left(-\frac{\varepsilon}{kT_{1\pm}^i}\right) \\ \quad \text{for } p \text{ in }]-\infty, m_{\pm} V_1] \\ 0 \\ \quad \text{for } p \text{ in }]m_{\pm} V_1, \pm\infty[\end{cases}$$

$$F_{2\pm}^i(p, \varepsilon) = \begin{cases} 0 \\ \quad \text{for } p \text{ in }]-\infty, m_{\pm} V_2] \\ \frac{C_{\pm}^i N_{\pm}^i m_{\pm}}{2\pi k T_{2\pm}^i} \cdot \exp\left(-\frac{m_{\pm} V_2^2}{2kT_{2\pm}^i}\right) \cdot \exp\left(\frac{V_2 p}{kT_{2\pm}^i}\right) \\ \quad \times \exp\left(-\frac{\varepsilon}{kT_{2\pm}^i}\right) \\ \quad \text{for } p \text{ in }]m_{\pm} V_2, \pm\infty[\end{cases}$$

For each type (i) of plasma, $T_{1\pm}^i$ and N_{\pm}^i are the temperatures and densities of the corresponding species in the plasmasphere, $T_{2\pm}^i$ are the temperatures in the plasmatrough. The quantities C_{\pm}^i are constants ($C_{\pm}^i > 0$) related to the densities and temperatures in the plasmatrough and to the electric potential difference $[\phi - aV]_{-\infty}^{+\infty}$ through the sheath. As the magnetic field remains always positive, it is seen that the distribution functions of the

ions and electrons possess the desired asymptotic properties. Indeed, when x goes to $-\infty$, the vector potential acquires large negative values; therefore, the contribution of $F_{1\pm}^i$ becomes predominant while the contribution of $F_{2\pm}^i$ tends to zero. It follows that the velocity distributions of the plasma approach asymptotically the displaced Maxwellians given by the non-vanishing parts of $F_{1\pm}^i$. (When x goes to $+\infty$, the same argument holds by interverting the subscripts 1 and 2, the vector potential taking large positive values).

However, the state of the plasma at both ends of the transition region does not uniquely determine the transition profile (SESTERO, 1964, 1966). In the sheath, the particular forms of the distribution functions determine only the shape of the transition. The solutions of the Vlasov's equation used here correspond to one among many other possible solutions satisfying the same conditions at infinity, i.e. deep in the plasmasphere and plasmatrough regions.

The concentrations $n_{\pm}^i(x)$ and the current densities $J_{\pm}^i(x)$ can be analytically determined by evaluating moments of the distribution functions. In dimensionless variables, denoted with stars, we find:

$$n_{\pm}^{*i} = \rho_{\pm}^i \exp[\mp \alpha_{1\pm}^i (\phi^* - 2V_1^* a^*)] \\ \times [1 - \operatorname{erf}(\alpha_{1\pm}^{i/2} \gamma_{\pm} a^*)] + C_{\pm}^i \rho_{\pm}^i \\ \times \exp[\mp \alpha_{2\pm}^i (\phi^* - 2V_2^* a^*)] \cdot \operatorname{erf}(\alpha_{2\pm}^{i/2} \gamma_{\pm} a^*) \quad (6)$$

$$J_{\pm}^{*i} = \rho_{\pm}^i \exp[\mp \alpha_{1\pm}^i (\phi^* - 2V_1^* a^*)] \\ \times \left\{ \pm V_1^* [1 - \operatorname{erf}(\alpha_{1\pm}^{i/2} \gamma_{\pm} a^*)] - \frac{\gamma_{\pm}}{2(\pi \alpha_{1\pm}^i)^{1/2}} \right. \\ \times \exp(-\alpha_{1\pm}^i \gamma_{\pm}^2 a^{*2}) \left. \right\} + C_{\pm}^i \rho_{\pm}^i \\ \times \exp[\mp \alpha_{2\pm}^i (\phi^* - 2V_2^* a^*)] \cdot \left\{ \pm V_2^* \operatorname{erf}(\alpha_{2\pm}^{i/2} \gamma_{\pm} a^*) \right. \\ \left. + \frac{\gamma_{\pm}}{2(\pi \alpha_{2\pm}^i)^{1/2}} \exp(-\alpha_{2\pm}^i \gamma_{\pm}^2 a^{*2}) \right\} \quad (7)$$

where the error function is defined by:

$$\operatorname{erf}(x) = \frac{1}{\pi^{1/2}} \int_{-\infty}^x e^{-t^2} dt.$$

The following relations hold:

$$n_{\pm}^{*i}(x^*) = n_{\pm}^i(x) / N_{\pm}^c$$

$$J_{\pm}^{*i}(x^*) = \left[\left(\frac{m_{\pm}}{2kT_{1\pm}^c} \right)^{1/2} / eN_{\pm}^c \right] J_{\pm}^i(x)$$

$$x^* = e \left(\frac{\mu N_{\pm}^c}{m_{\pm}} \right)^{1/2} x$$

$$\rho_{\pm}^i = N_{\pm}^i / N_{\pm}^c$$

$$\alpha_{j\pm}^i = T_{1\pm}^c / T_{j\pm}^i, \quad j = 1, 2$$

$$\begin{aligned}\phi^* &= e\phi/kT_{1-}^c \\ V_j^* &= \left(\frac{m_-}{2kT_{1-}^c}\right)^{1/2} V_j, j = 1, 2 \\ a^* &= (e^2/2m_-kT_{1-}^c)^{1/2} a \\ \gamma_{\pm} &= \left(\frac{m_{\pm}}{m_-}\right)^{1/2}.\end{aligned}$$

For the charge densities q_{\pm}^i , we have:

$$q_{\pm}^{*i} = q_{\pm}^i/eN_-^c, \text{ so that } n_{\pm}^{*i} = \pm q_{\pm}^{*i}.$$

One can also deduces the following relations for the fields:

$$\begin{aligned}E^* &= \frac{1}{kT_{1-}^c} \left(\frac{m_-}{\mu N_-^c}\right)^{1/2} E \\ B^* &= (2\mu N_-^c kT_{1-}^c)^{-1/2} B.\end{aligned}$$

The basic equations (3) and (4) become in dimensionless form:

$$\begin{cases} \frac{d^2 a^*}{dx^{*2}} = -J^* & (8) \\ \frac{\kappa\mu kT_{1-}^c}{m_-} \frac{d^2 \phi^*}{dx^{*2}} = -q^* & (9) \end{cases}$$

These equations can be integrated numerically to determine the distribution of the vector and electric potentials. The total current J^* is the sum: $J_+^{*c} + J_-^{*c} + J_+^{*h} + J_-^{*h}$ of the partial current densities due to the cold (c) and hot (h) ions and electrons. At this point, however, it is suitable to make the simplifying approximation of charge-neutrality. This approximation replaces the Poisson's, equation (9) for the electric potential by the algebraic equation of charge neutrality everywhere, i.e.

$$q^* = q_+^{*c} + q_-^{*c} + q_+^{*h} + q_-^{*h} = 0. \quad (10)$$

The small quantity $\kappa\mu kT_{1-}^c/m_-$ is the square of the ratio of the mean thermal speed of the cold electrons of the plasmasphere to the speed of light. For a typical value of 1 eV for kT_{1-}^c , this quantity is equal to 1.95×10^{-6} . It is expected, therefore, that, with a electric potential solution of (10), the first member of equation (9) remains everywhere a small quantity, so that this potential is self-consistent. In the Section 4, it will indeed be shown, that the charge neutral approximation holds everywhere in the sheath.

3. BOUNDARY CONDITIONS

From the expressions (6) giving the concentrations of the charged particles, it is easy to give a

physical meaning to the parameters C_{\pm}^i . Indeed, if the potential $\phi^* - 2V_1^* a^*$ is normalized to be 0 in the plasmasphere (at $x^* = -\infty$), we deduce from (6):

$$\begin{aligned}\phi^*(+\infty) - 2V_2^* a^*(+\infty) &= \frac{1}{\alpha_{2-}^c} \ln X \\ C_{\pm}^i &= \frac{n_{\pm}^{*i}(+\infty)}{\rho_{\pm}^i} X^{\pm(\alpha_{2\pm}^i/\alpha_{2-}^c)} \quad (C_{\pm}^i \neq C_-^c)\end{aligned}$$

where

$$X = \frac{n_-^{*c}(+\infty)}{C_-^c \rho_-^c}.$$

It is seen that the potential $\phi^* - 2V_2^* a^*$ and the number densities n_{\pm}^{*i} in the plasmatrough ($x^* = +\infty$) are simply related to the constants C_{\pm}^i . Assuming a value for C_-^c , the other constants C_+^c , C_-^h , C_+^h and the potential at $x^* = +\infty$ are determined as functions of the densities and temperatures in the plasmatrough. For given values of the densities and temperatures at both ends of the sheath, a family of solutions can be obtained depending on the value of the free parameter C_-^c which physically determines the potential difference across the sheath.

By taking $C_-^c = 1$, the distribution of the cold electrons is a full Maxwellian and we expect therefore that the transition for the cold plasma will occur over a scale length proper to the cold ions.

Table 1 gives an example of a set of boundary conditions we have used to integrate the basic equations (8) and (9).

These boundary conditions have been chosen in order to show a characteristic behavior of the plasmas across the transition.

From Table 1, it can be seen that the two types of plasma are separately neutral in the plasmasphere while it is not the case in the plasmatrough. For each type of plasma there are no variations in

Table 1: Boundary conditions.

	Plasmasphere ($x^* = -\infty$)	Plasmatrough ($x^* = +\infty$)
N_-^c (cm ⁻³)	300	1
N_+^c (cm ⁻³)	300	5
N_-^h (cm ⁻³)	0.5	7
N_+^h (cm ⁻³)	0.5	3
T_-^c (eV)	1	1
T_+^c (eV)	1	1
T_-^h (KeV)	5	5
T_+^h (KeV)	50	50

the temperatures across the transition. The unitary value of C_-^c corresponds to a potential difference $[\phi - 2aV]_{-\infty}^{+\infty}$ equal to -5.70 V. The other constants C are found to be: $C_+^c = 0.556 \times 10^{-4}$, $C_-^h = 14.016$ and $C_+^h = 5.999$.

4. NUMERICAL RESULTS

(a) Some representative lengths and velocities

Before presenting the structure of the sheath it is interesting to give values for some representative lengths and velocities characterizing the problem under consideration.

It is well known that the natural unit of length (ξ_-) in a plasma sheath is the electron skin depth defined by:

$$\xi_- = \left(\frac{m_-}{\mu e^2 n_-} \right)^{1/2} = \frac{c}{\omega_p} \tag{11}$$

It is the ratio of the speed of light (c) to the plasma frequency (ω_p). This length represents an estimation of the thickness of a boundary layer between a vacuum magnetic field and a cold plasma of zero temperature. In this case, the magnetic pressure of the field on one side is just equal to the kinetic pressure of the plasma on the other side. Then, it can be shown, (LONGMIRE, 1963) that the electron skin depth represents the geometric mean of the electron and ion Larmor radii (these Larmor radii refer to particles of energy equal to the corresponding thermal energy in the constant magnetic field of the vacuum side). For the cold plasma of the plasmasphere formula (11) gives $\xi_-^c(-\infty) = 306$ m.

The transition occurs near $x^* = 0$ where the magnetic field has a value of 340γ (corresponding to the dipole field at $L = 4.5$). Values of the bulk velocities V_1 and V_2 are taken at large distances from the sheath (at $x^* = \mp 1000$) and correspond to drift velocities equal, respectively to 1 and 1.5 times the angular velocity of the Earth.

Other lengths of interest are the Larmor radii (R_{\pm}^i), which will be evaluated at $x^* = 0$. Mean thermal velocities (U_{\pm}^i) are also of interest in the discussion of the stability of the transition. These values are listed in Table 2.

From expression (6) giving the densities, it can be seen that variations in the concentrations are linked to variations of the quantities $\alpha_{i\pm}^{i/2} \gamma_{\pm} a^*$, arguments of the error function. However, the dimensioned variations of these quantities represent the inverses of the corresponding Larmor radii. Therefore, the characteristic lengths quoted in Table 2 are some scale lengths characterizing the variations of the densities along the x -axis. In particular, in

Table 2. Characteristic lengths and velocities

$\xi_-^c(-\infty)$ (m)	3.0×10^2
R_-^c (m)	10^1
R_+^c (m)	4.25×10^2
R_-^h (m)	7.0×10^2
R_+^h (m)	9.5×10^4
U_-^c (m.s ⁻¹)	6.0×10^5
U_+^c (m.s ⁻¹)	1.4×10^4
U_-^h (m.s ⁻¹)	4.2×10^7
U_+^h (m.s ⁻¹)	3.1×10^6

the case considered here ($C_-^c = 1$, $\alpha_{1\pm}^c = \alpha_{2\pm}^c = 1$), the concentrations of the cold plasma decrease with the electric potential, $\phi^* - 2a^*V^*$, on which they depend exponentially and the transition will occur over a scale length comparable to the cold ion Larmor radius (425 m).

(b) Structure of the sheath

Figure 2 shows the transition profile for the densities. As expected, the cold plasma (1 eV) scales according to the cold ion Larmor radius in the center of the sheath. Its density decreases from 300 particles per cm³ to 6 particles per cm³ for the ions and to 0.8 particle per cm³ for the electrons over a distance equal to 7 cold electron skin depths, which represents only 2.1 km. The thickness of this transition is equal to about 5 times the cold proton Larmor radius. At the edge of the sheath, the cold plasma scales according to the hot ion Larmor radius and the asymptotic values of its densities are reached at many times the hot proton Larmor radius. The hot (5 KeV) ring electron density reaches its asymptotic values across a transition of

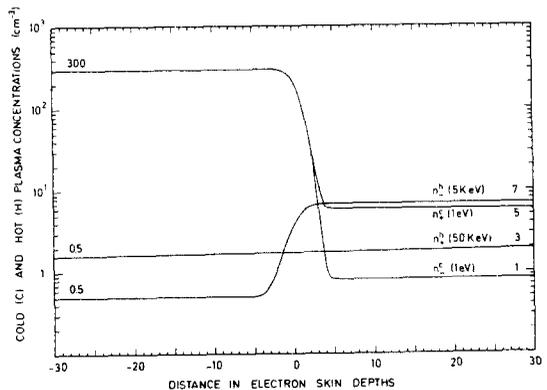


Fig. 2. Transition profiles for the densities. The subscripts + and - are for the ions and electrons. The superscripts c and h denote respectively the cold plasma and the hot plasma. Numbers at both ends of the curves are the asymptotic values (cm⁻³) of the densities. Numbers between parentheses are the temperatures of the corresponding species. The unit of length is the electron skin depth $\xi_-^c(-\infty)$ equal to 306 m.

about 10 units of length which represent 4 or 5 times the hot electron Larmor radius. As for the very hot (50 keV) ring proton density it scales also according to its own Larmor radius and reaches its asymptotic values across a transition of about 5 times this characteristic length.

The thickness of the transition for the cold plasma should be considered as a *minimum thickness* because we did not take into account any collisional or wave-particle interactions effects which tend to broaden this boundary layer. In Fig. 3, we have plotted the bulk velocities V_{\pm}^i of each species:

$$V_{\pm}^i = V_{D_{\pm}}^i - V_0$$

where $V_{D_{\pm}}^i$ are the total drift velocities:

$$V_{D_{\pm}}^i = \pm \frac{J_{\pm}^i}{n_{\pm}^i e}$$

and V_0 is the mass velocity:

$$V_0 = \frac{m_- n_-^c V_{D_-}^c + m_+ n_+^c V_{D_+}^c + m_- n_-^h V_{D_-}^h + m_+ n_+^h V_{D_+}^h}{m_+ (n_+^c + n_+^h) + m_- (n_-^c + n_-^h)}$$

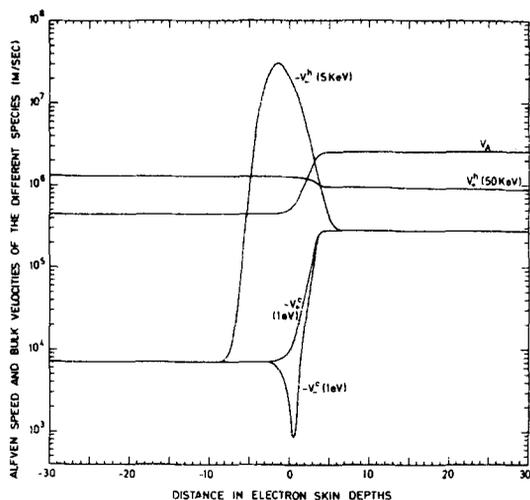


Fig. 3. Bulk velocities V_{\pm}^i of the different species relatively to the mass velocity. Also shown is the Alfvén speed V_A . The superscripts *c* and *h* denote respectively the cold plasma and the hot plasma. Numbers between parentheses are the corresponding temperatures. The unit of length is the electron skin depth $\xi_-^c(-\infty)$ equal to 306 m. Thermal ions speeds for the hot and cold particles are respectively equal to $3.1 \times 10^6 \text{ m.s}^{-1}$ and $1.4 \times 10^4 \text{ m.s}^{-1}$. An instability occurs when the relative velocity of the ions and electrons exceeds the thermal ions speed. For the cold plasma, this instability is electrostatic in nature for the relative velocity does not exceed the local Alfvén wave speed and occurs for $1 < x^* < 4$. For the hot plasma, the relative velocity exceeds the local Alfvén wave speed and the electromagnetic instability occurs for $-5 < x^* < 3.5$.

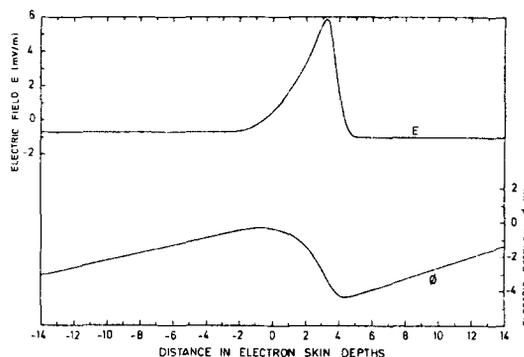


Fig. 4. Variations of the electric field (E) and electric potential (ϕ) through the sheath. The unit of length is the electron skin depth $\xi_-^c(-\infty)$ equal to 306 m.

The Alfvén wave speed given by

$$V_A = \frac{B}{\sqrt{\mu\rho}}$$

is also plotted for comparison.

As the relative velocity of the hot electrons and ions $V_+^h - V^h$ exceeds the thermal velocity of the ions ($U_+^h = 3.1 \times 10^6 \text{ m.s}^{-1}$) in the transition region between $x^* \sim -5$ and $x^* \sim 3.5$, a hydrodynamic beam instability occurs (PAPADOPOULOS, 1973). For the cold plasma the same instability is present between $x^* \sim 1$ and $x^* \sim 4$ ($U_+^c = 1.4 \times 10^4 \text{ m.s}^{-1}$) but is electrostatic in nature for the relative velocity is smaller than the Alfvén wave speed.

Figure 4 shows the variation of the electric potential and electric field. By calculating the slope of the E -variation, it is shown that the charge neutrality approximation remains valid everywhere through the sheath. (At $x^* \sim 4$, the relative excess charge $\Delta q/q_+$ calculated by evaluating the slope $d^2\phi/dx^2$ is maximum and takes a value of 2×10^{-4} .) However, it can be seen that even the small deviation from electric neutrality produces excessively large electric field. This field reaches a maximum value of 6 mV/m near $x^* \sim 3$.

Finally, the total diamagnetic current is principally due to the hot ring current particles and causes a large decrease of the magnetic field, with increasing x^* , of about 200γ . This variation scales with a length related to the hot ion Larmor radius and takes place over a distance of several hundreds of units of length.

5. CONCLUSIONS

In this paper we have described the structure of a plasma layer, the plasmopause, separating two regions of the magnetosphere, with distinctly different properties. This microscopic description uses the

Vlasov–Maxwell’s equations for the charged particles and fields. Using distribution functions similar to those used by SESTERO (1966) for the study of contact discontinuities in collisionless plasmas, we have computed transition profiles for the different species. For the cold plasma, the profile is ion-dominated (i.e., with negligible contribution of the electron current to the change in magnetic field). This is done by using a distribution function for the cold electrons which remains a total Maxwellian throughout the sheath. The density for the cold particles drops from 300 particles per cm^3 to a few particles per cm^3 over a distance equal to 5 times the cold ion Larmor radius. With the boundary conditions given in Section 3, this transition has a thickness of 2.1 km. However in Section 4, it is shown that, in a thin region of the sheath, the cold plasma is electrostatically unstable. This instability will destroy the steady-state transition and will lead to a broadening of the transition layer.

As the model does not take into account of any collisional effects or wave-particle interactions, the

thickness of the plasmopause we calculate, must therefore be considered as a minimum thickness, although it is also possible to construct a cold electron-dominated transition (probably more strongly unstable) with a characteristic thickness of at least a cold electron Larmor radius.

The presence of the hot plasma has a great influence on the variation of the magnetic field, as it furnishes the greatest contribution to the current density. In Section 4, it is shown that the hot plasma becomes unstable near the middle of the sheath. This electromagnetic instability will in turn allow for an increase of the layer thickness.

Finally, an increase of the temperature of the cold ions in the plasmatrough ($\alpha_{2+} < 1$) and the inclusion of finite integrated Pedersen and Hall conductivities would also lead to a broadening of the ideal layer discussed here.

Acknowledgements—The author is deeply grateful to Dr. J. LEMAIRE for his valuable comments in evaluating this paper. He is also indebted to Prof. M. NICOLET for useful advices and constant encouragement.

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