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FIRST AND SECOND ORDER APPROXIMATIONS OF THE FIRST ADIABATIC INVARIANT FOR A CHARGED PARTICLE INTERACTING WITH A LINEARLY POLARIZED HYDROMAGNETIC PLANE WAVE

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Abstract—In the present paper the effect of a sinusoidal modulation of an electromagnetic field on the invariance of the magnetic moment is studied. Such a generalized invariant plays an important role in problems concerning the motion of charged particles in the non-uniform magnetic field of the magnetosphere or the solar wind. In order to find an adiabatic invariant J, a canonical transformation is introduced, and J is expanded in an asymptotic series in the relative modulation amplitude. We are studying the first and second order terms of this expansion. It is further shown that the curves $J =$ constant closely fit the results obtained by a numerical integration of the system of differential equations governing the motion of the particles.

1. INTRODUCTION

Since Alfvén (1950) introduced the adiabatic theory, much attention has been devoted to the study of the three adiabatic invariants (the magnetic moment, the longitudinal adiabatic invariant and the flux adiabatic invariant). This continuous interest is justified by the role these invariants play in problems regarding the motion of particles in the magnetosphere and in the interplanetary space. Interactions with hydromagnetic waves, and collisions with particles can be held responsible for the violation of the invariance.

In this paper, the behaviour of the magnetic moment of a collisionless charged particle, interacting with a linearly polarized hydromagnetic plane wave, is studied. The methods used here are similar to those employed by Dunnett *et al.* (1968), Dunnett and Jones (1972), and Roth (1974), for square wave and sine wave modulation.

In a reference system (X, Y, Z) we consider the motion of a charged particle under the influence of an electromagnetic field. The magnetic field is the sum of a uniform component, parallel to the Z-axis, and a plane wave, linearly polarized along the Y-axis. The electromagnetic field is given by the equations:

$$\vec{B} = B_0\vec{e}_z + hB_0 \cos\,(kZ - \Omega t)\vec{e}_y \tag{1}$$

$$\vec{E} = h\frac{\Omega}{k}B_0 \cos\,(kZ - \Omega t)\vec{e}_x. \tag{2}$$

The relative modulation amplitude is h, while the phase velocity U equals Ω/k. We perform a Lorentz transformation, which under the assumption that U/c is much smaller than unity, reduces to

$$X \to X \tag{3}$$

$$Y \to Y \tag{4}$$

$$Z \to Z - Ut \tag{5}$$

$$t \to t. \tag{6}$$

In the new cartesian coordinate system, we have:

$$\vec{B} = B_0\vec{e}_z + hB_0 \cos\,(kZ)\vec{e}_y \tag{7}$$

$$\vec{E} = 0. \tag{8}$$

The variables defining the position of the particle $(\vec{R})$, its velocity $(\vec{V})$, its linear momentum $(\vec{P})$, and its Hamiltonian $(\mathcal{H})$, were replaced by dimensionless variables, using the following definitions:

$$\vec{R} = \lambda\vec{r}, \quad \vec{V} = \tfrac{1}{2}\omega\lambda\vec{v}, \quad \vec{P} = \tfrac{1}{2}m\omega\lambda\vec{p}, \quad \mathcal{H} = \tfrac{1}{4}m^2\omega^2\lambda^2 H. \tag{9}$$

The time t was replaced by

$$t = \frac{2}{\omega}\tau. \tag{10}$$

In these definitions $\omega = qB_0/m$ is the gyration frequency in the field $B_0\vec{e}_z$, and $\lambda = 2\pi/k$ is the modulation wavelength.

The Hamiltonian H can then be written as :

$$H = H_0 + hH_1 + h^2H_2, \tag{11}$$

with

$$H_0 = \tfrac{1}{2}[(p_x + y)^2 + (p_y - x)^2 + {p_z}^2] \tag{12}$$

$$H_1 = -\frac{1}{\pi}(p_x + y)\sin 2\pi z \tag{13}$$

$$H_2 = \frac{1}{2\pi^2}\sin^2 2\pi z. \tag{14}$$

A canonical tranformation (Roth, 1974) is introduced:

$$(x, y, z; p_x, p_y, p_z) \to (Q_1, Q_2, Q_3; P_1, P_2, P_3),$$

defined by

$$x = P_1 + P_2^{1/2} \sin Q_2 \quad (15)$$

$$y = Q_1 + P_2^{1/2} \cos Q_2 \quad (16)$$

$$z = Q_3 \quad (17)$$

$$p_x = -Q_1 + P_2^{1/2} \cos Q_2 \quad (18)$$

$$p_y = P_1 - P_2^{1/2} \sin Q_2 \quad (19)$$

$$p_z = P_3. \quad (20)$$

Figure 1 shows the zero order representation of the particle motion. It can be seen that P_1 and Q_1 represent the cartesian coordinates of the guiding center in the (x, y)-plane, while P_2 and Q_2 are respectively the square of the Larmor radius and the Larmor phase angle. H_0, H_1, H_2 can then be written as

$$H_0 = 2P_2 + \tfrac{1}{2}P_3^2 \quad (21)$$

$$H_1 = -\frac{2}{\pi} P_2^{1/2} \cos Q_2 \sin 2\pi Q_3 \quad (22)$$

$$H_2 = \frac{1}{2\pi^2} \sin^2 2\pi Q_3. \quad (23)$$

The equations of Hamilton can now be deduced and integrated numerically. We refer to these

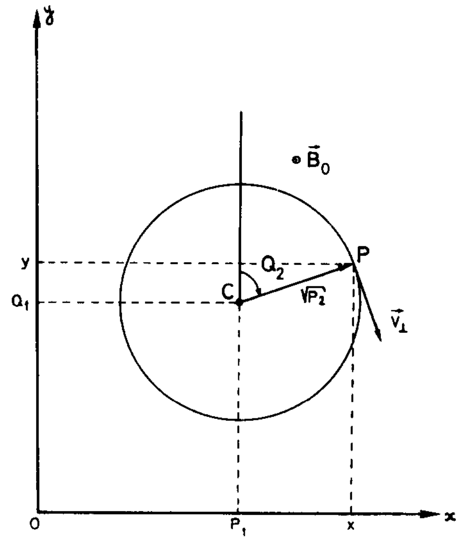

FIG. 1. ZERO ORDER REPRESENTATION OF THE PARTICLE GYROMOTION.

results later on, in comparing them with the results of the adiabatic theory.

Any generalized invariant J can be found as a solution of

$$[J, H] = 0, \quad (24)$$

where [ ] is the Poisson bracket. Writing the function J as a power series expansion:

$$J = \sum_{n=0}^{\infty} h^n J_n, \quad (25)$$

leads to a set of an infinite number of recursion relations:

$$[J_0, H_0] = 0 \quad (26)$$

$$[J_1, H_0] + [J_0, H_1] = 0 \quad (27)$$

$$[J_2, H_0] + [J_1, H_1] + [J_0, H_2] = 0 \quad (28)$$

$$[J_3, H_0] + [J_2, H_1] + [J_1, H_2] = 0. \ldots \quad (29)$$

The result of these formulae is a system of differential equations, determining J_i, $i = 0, 1, 2, \ldots$. The aim of the present work is to solve these equations for J_0, J_1 and J_2. It will be shown that J_0 must satisfy certain conditions in order to obtain physically possible solutions.

2. SOLUTIONS FOR THE FIRST AND SECOND ORDER PROBLEMS

Introducing the transformation

$$Q_2 = 2(s + z)/P_3 \quad (30)$$

$$Q_3 = z \quad (31)$$

enables us to write equation (26) as

$$\frac{\partial J_0}{\partial z} = 0. \quad (32)$$

As a consequence, J_0 must be an arbitrary function of P_1, P_2, P_3, Q_1 and

$$s = \frac{P_3 Q_2 - 2Q_3}{2}. \quad (33)$$

J_0 will be chosen to depend only upon P_3. Defining $u = Q_2$, $w = 2\pi Q_3$, equations (27) and (28) can be replaced by :

$$P_3 \frac{\partial J_1}{\partial z} = -4P_2^{1/2} \cos u \cos w \frac{dJ_0}{dP_3} \quad (34)$$

$$P_3 \frac{\partial J_2}{\partial z} = \frac{1}{\pi P_2^{1/2}} \cos u \sin w \frac{\partial J_1}{\partial u} + \frac{2P_2^{1/2}}{\pi} \sin u \sin w \frac{\partial J_1}{\partial P_2}$$

$$-4P_2^{1/2} \cos u \cos w \frac{\partial J_1}{\partial P_3} + \frac{1}{\pi} \sin 2w \frac{dJ_0}{dP_3}. \quad (35)$$

From these equations and from the particular choice of J_0, it follows that J_1 and J_2 can be written as functions of the zero order Larmor radius in the following manner:

$$J_1 = A_{1,1} P_2^{1/2} \tag{36}$$

$$J_2 = A_{2,0} + A_{2,2} P_2. \tag{37}$$

The unknown coefficients $A_{1,1}$, $A_{2,0}$ and $A_{2,2}$ are functions of u, w and P_3 alone. By defining $A_{0,0} = J_0$, the original system of differential equations can be replaced by:

$$P_3 \frac{\partial A_{j,j}}{\partial z} = -4 \cos u \cos w \frac{\partial A_{j-1,j-1}}{\partial P_3} \quad j = 1, 2 \tag{38}$$

$$P_3 \frac{\partial A_{2,0}}{\partial z} = \frac{1}{\pi} \cos u \sin w \frac{\partial A_{1,1}}{\partial u} + \frac{1}{\pi} \sin u \sin w A_{1,1} + \frac{1}{\pi} \sin 2w \frac{dA_{0,0}}{dP_3}. \tag{39}$$

This system can be easily integrated:

$$A_{1,1} = -\pi \left[\frac{1}{A} \sin(u+w) + \frac{1}{B} \sin(u-w) \right] J_0' \tag{40}$$

$$A_{2,0} = -\frac{1}{8C} \left(2 - \frac{1}{A} - \frac{1}{B} \right) \cos 2w J_0' \tag{41}$$

$$A_{2,2} = -\frac{\pi^2}{4} \left[\frac{K}{A^2} \cos(2u+2w) + \frac{L}{B^2} \cos(2u-2w) + \frac{1}{C} \left(\frac{K}{A} - \frac{L}{B} \right) \cos 2w + \left(\frac{K}{A} + \frac{L}{B} \right) \cos 2u \right], \tag{42}$$

with

$$\beta = \pi P_3; \quad A = 1 + \beta; \quad B = 1 - \beta; \quad C = \beta$$

and

$$J_0' = \frac{dJ_0}{d\beta}; \quad J_0'' = \frac{d^2 J_0}{d\beta^2};$$
$$K = -\frac{1}{A} J_0' + J_0''; \quad L = \frac{1}{B} J_0' + J_0''. \tag{43}$$

In our discussion the magnetic moment $\frac{1}{2} m v_\perp^2 / B$ will be evaluated at points corresponding to successive periods of the magnetic field modulation. For these points we have $Z = n\lambda$, or $z = n$, n being an integer. P_2 and P_3 can then be written as functions of

$$\xi = \frac{v_\perp^2}{v^2} \tag{44}$$

We have

$$P_2 = \tfrac{1}{4} \xi v^2 \tag{45}$$

$$P_3 = \pm v(1-\xi)^{1/2}; \quad \beta = \pm \pi v (1-\xi)^{1/2}. \tag{46}$$

For the points where $z = n$, the magnetic field always has the same value and ξ is proportional to the magnetic moment. The particle motion will consequently be studied in a (u, ξ)-plane. In the zero order approximation the magnetic moment is conserved and we therefore see that ξ is independent of the phase angle u. At $w = 2\pi n$, equations (40), (41) and (42) can be simplified to:

$$A_{1,1} = -\pi \left(\frac{1}{A} + \frac{1}{B} \right) J_0' \sin u \tag{47}$$

$$A_{2,0} = -\frac{1}{8C} \left(2 - \frac{1}{A} - \frac{1}{B} \right) J_0' \tag{48}$$

$$A_{2,2} = -\frac{\pi^2}{4} \left\{ \left[\left(-\frac{1}{A^3} + \frac{1}{B^3} - \frac{1}{A^2} + \frac{1}{B^2} \right) J_0' + \left(\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{A} + \frac{1}{B} \right) J_0'' \right] \cos 2u - \frac{1}{C} \left(\frac{1}{A^2} + \frac{1}{B^2} \right) J_0' + \frac{1}{C} \left(\frac{1}{A} - \frac{1}{B} \right) J_0'' \right\}. \tag{49}$$

Conditions which J_0 must satisfy will result from the fact that all zeros in the denominators of $A_{1,1}$, $A_{2,2}$ and $A_{2,0}$ must be eliminated.

Another condition will be explained in the following sections.

3. CHOICE OF J_0 FOR THE FIRST ORDER PROBLEM

Several functions for J_0 may be chosen, provided that they eliminate the singularities in $A_{1,1}$ for the values $\beta = \pm 1$. It should be noted that the representations of the particle motion in the (u, ξ)-plane with each of these functions will not be identical. They must be considered as different first order approximations. As J_0' must be proportional to AB in order to remove the singularities, we choose

$$J_0 = A^a B^b + \text{arbitrary constant}, \tag{50}$$

with $a \geq 2$, $b \geq 2$.

Taking for instance $a = b = 2$, we have

$$J_0' = -4ABC, \tag{51}$$

and

$$A_{0,0} = \text{const} + A^2 B^2 \tag{52}$$

$$A_{1,1} = 8\pi C \sin u. \tag{53}$$

The expression for J can be written as

$$J = \text{const} + A^2 B^2 + 8\pi h C P_2^{1/2} \sin u. \tag{54}$$

We define

$$X(\xi)=A^2B^2 \quad (55)$$

$$Y(\xi,h)=8\pi hCP_2^{1/2}. \quad (56)$$

Selecting values for h and v, and considering a number of initial conditions (u_0, ξ_0), we obtain the adiabatic invariant curves from the equation

$$X(\xi)+Y(\xi,h)\sin u = X(\xi_0)+Y(\xi_0,h)\sin u_0. \quad (57)$$

In these and subsequent calculations, the positive value of β is taken into account. This corresponds to particles with positive initial velocities along the main magnetic field $B_0\vec{e}_z$. In Figs. 2 and 3 the curves resulting from equation (57) are displayed in the (u, ξ)-plane. ξ varies between 0 and 1, u varies between $-3\pi/2$ and $\pi/2$. The velocities in Figs. 2 and 3 are assigned the values $1/\pi$ and $2/\pi$ respectively. The relative amplitude of the field perturbation, h equals 0.025. These curves agree with the results obtained from numerical integration of the equations of Hamilton derived from (21), (22) and (23). This integration has been performed by Roth (1974) and the results of these numerical calculations are shown in Figs. 4 and 5. The curves in Figs. 2 and 3 demonstrate the manner in which the adiabatic theory predicts the existence of a functional relationship between the perpendicular kinetic energy ξ and the phase angle u. It is also clear that there exists a resonance point surrounded by closed curves. The abscissa of the resonance point is $-\pi/2$. The ordinate can be obtained from equation (57). Indeed, determining the derivative of $\sin u$ with respect to ξ, for $u=-\pi/2$, yields

$$\varphi(\xi)=\left(\frac{d\sin u}{d\xi}\right)_{u=-\pi/2}=\frac{-\frac{dX}{d\xi}+\frac{dY}{d\xi}}{Y}. \quad (58)$$

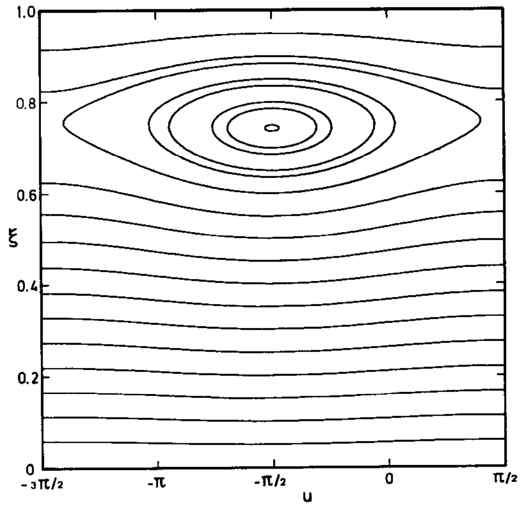

FIG. 3. INVARIANT CURVES RESULTING FROM FIRST ORDER THEORY, FOR $h=0.025$, $v=2/\pi$, $u_0=-\pi/2$, AND FOR DIFFERENT VALUES OF ξ_0, FROM 0.05 TO 0.95. $J_0=A^2B^2$.

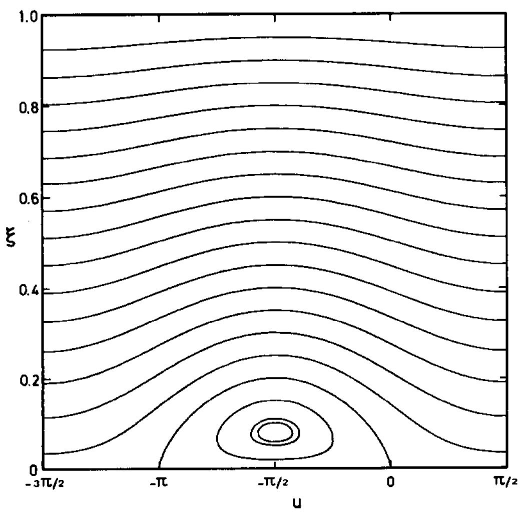

FIG. 2. INVARIANT CURVES RESULTING FROM FIRST ORDER THEORY, FOR $h=0.025$, $v=1/\pi$, $u_0=-\pi/2$, AND FOR DIFFERENT VALUES OF ξ_0, FROM 0.05 TO 0.95. $J_0=A^2B^2$.

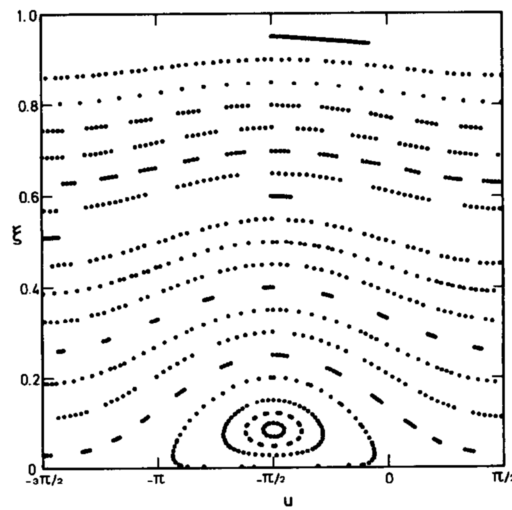

FIG. 4. INTEGRATED ORBITS, FOR $h=0.025$, $v=1/\pi$, $u_0=-\pi/2$, AND FOR DIFFERENT VALUES OF ξ_0, FROM 0.05 TO 0.95.

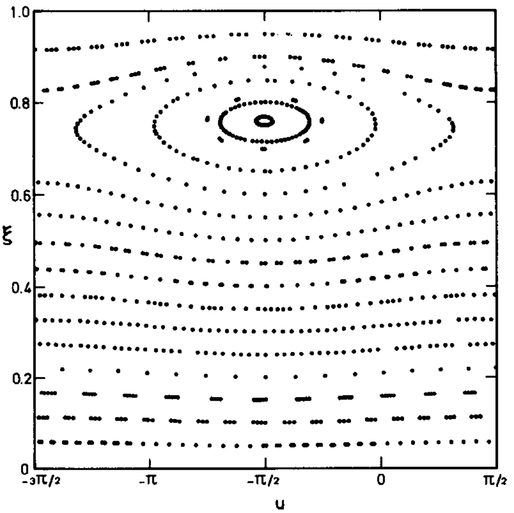

FIG. 5. INTEGRATED ORBITS, FOR $h = 0.025$, $v = 2/\pi$, $u_0 = -\pi/2$, AND FOR DIFFERENT VALUES OF ξ_0 FROM 0.05 TO 0.95.

A resonance point $R(-\pi/2, \xi_R)$ is given by

$$\varphi(\xi_R) = 0. \tag{59}$$

Making $a > 2$ or $b > 2$ in the above theory will inevitably give rise to the appearance of several resonance points. This is due to the fact that the corresponding choices of J_0 lead to functions Y, which have a number of zeros, and these induce a discontinuous behaviour of $\varphi(\xi)$, with several intersections of the ξ-axis.

As a conclusion we may say that as far as the first order is concerned we can use

$$J_0 = A^2B^2 + \text{arbitrary constant.} \tag{60}$$

This means that we cannot use analogous solutions for the second and higher order problems. Indeed, elimination of all singularities is not possible without higher powers of A and B. These give rise to several resonance points which do not exist in the exact numerical solutions.

4. CHOICE OF J_0 FOR THE SECOND ORDER PROBLEM

To solve the second order problem we have tried a function J_0, satisfying the differential equation:

$$J_0' = ABCf(\beta). \tag{61}$$

In this case the equations (47), (48) and (49) are reduced to:

$$A_{2,0} = \tfrac{1}{4}\beta^2 f(\beta) \tag{63}$$

$$A_{2,2} = -\frac{\pi^2}{4}\left\{\frac{[2(2+\beta^2)f(\beta) + 4\beta f'(\beta)]}{AB} \times \cos 2u - 2[2f(\beta) + \beta f'(\beta)]\right\}. \tag{64}$$

Elimination of the singularities in the coefficient of $\cos 2u$ is possible. We take a function $f(\beta)$ satisfying a linear differential equation, such as:

$$2\beta f'(\beta) + (2+\beta^2)f(\beta) = a\beta(1-\beta^2). \tag{65}$$

Integration leads to the result

$$f(\beta) = \frac{1}{\beta}\left[a(5-\beta^2) + be^{-(\beta^2/4)}\right], \tag{66}$$

where b is the integration constant. The calculation of J_0 follows from (61). Indeed integration (61) leads to:

$$J_0 = a\left(5\beta - 2\beta^3 + \frac{\beta^5}{5}\right) + b\left[-\pi^{1/2}\, erf\left(\frac{\beta}{2}\right) + 2\beta e^{-(\beta^2/4)}\right] + \text{arbitrary constant.} \tag{67}$$

The equation defining the adiabatic invariant curves is now given by

$$X(\xi, h) + Y(\xi, h)\sin u + Z(\xi, h)\cos 2u = X(\xi_0, h) + Y(\xi_0, h)\sin u_0 + Z(\xi_0, h)\cos 2u_0, \tag{68}$$

with

$$X(\xi, h) = J_0 + \frac{h^2}{4}\{\beta^2 f(\beta) + 2\pi^2 P_2[2f(\beta) + \beta f'(\beta)]\} \tag{69}$$

$$Y(\xi, h) = -2\pi hC \quad P_2^{1/2} f(\beta) \tag{70}$$

$$Z(\xi, h) = -\frac{\pi^2}{2} h^2 CP_2 a. \tag{71}$$

Equation (68) can be written as a quadratic equation in $\sin u$. This equation is homogeneous in a and b, so that the invariant curves will depend upon the parameter $m = a/b$. Difficulties similar to the ones described in section 3 may arise here. Indeed we have:

$$\varphi(\xi) = \left(\frac{d\sin u}{d\xi}\right)_{u=-\pi/2} = \frac{-\dfrac{dX}{d\xi} + \dfrac{dY}{d\xi} + \dfrac{dZ}{d\xi}}{Y + 4Z}. \tag{72}$$

The equation $\varphi(\xi) = 0$, may only have one solution, namely the ξ_R-value for the resonance point. Situations can be described where a particular choice of m gives rise to a second resonance point, due to the discontinuous behaviour of $\varphi(\xi)$. An important re-

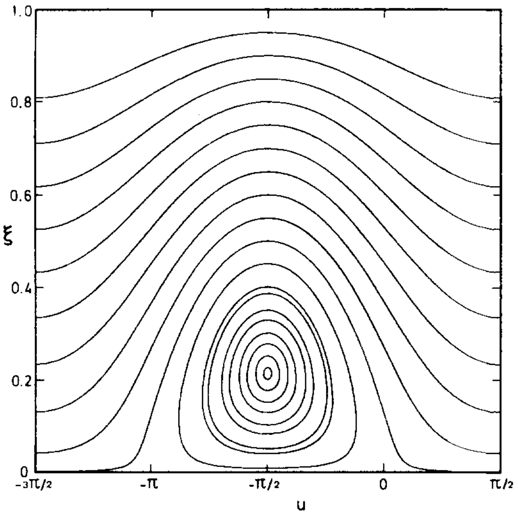

FIG. 6. INVARIANT CURVES RESULTING FROM SECOND ORDER THEORY, FOR $h=0.1$, $m=1$, $v=1/\pi$, $u_0=-\pi/2$, AND FOR DIFFERENT VALUES OF ξ_0 FROM 0.05 TO 0.95.

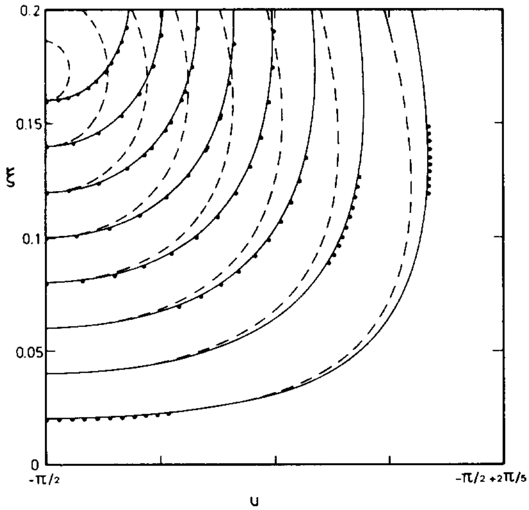

FIG. 8. COMPARISON OF INTEGRATED ORBITS (DOTS) TO INVARIANT CURVES RESULTING FROM (a) FIRST ORDER THEORY (DASHED LINES); (b) SECOND ORDER THEORY ($m=1$), (SOLID LINES), FOR DIFFERENT VALUES OF ξ_0 FROM 0.02 TO 0.16.

determine the value of m in such a way that a fixed point $(-\pi/2, \xi)$ is a resonance point. This is only acceptable when the condition $Y+4Z\neq 0$ is satisfied for any ξ between 0 and 1. In our calculations we obtained satisfactory results with the value

$$m=1. \tag{73}$$

In Figs. 6 and 7 we show the curves resulting from equation (68) for $h=0.1$ and $v=1/\pi$ and $2/\pi$ respectively. In Fig. 8 we compare the results obtained from numerical computation with the curves deduced (a) from the first order theory as explained in Section 3, and (b) from the second order theory as shown in Section 4, for $h=0.1$ and $v=1/\pi$. To display the differences clearly, we examine a region in the resonance domain.

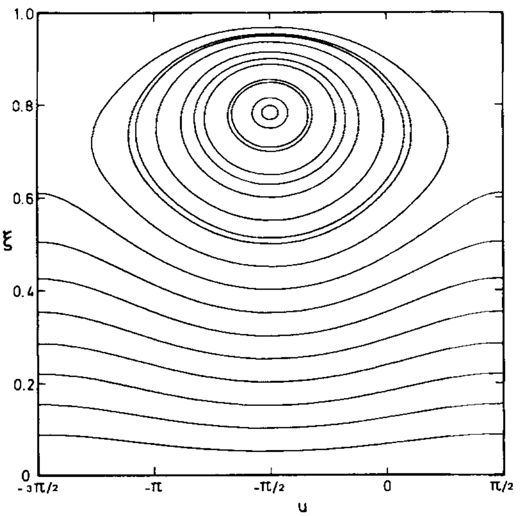

FIG. 7. INVARIANT CURVES RESULTING FFROM SECOND ORDER THEORY, FOR $h=0.1$, $m=1$, $v=2/\pi$, $u_0=-\pi/2$, AND FOR DIFFERENT VALUES OF ξ_0 FROM 0.05 TO 0.95.

5. OTHER POSSIBLE CHOICES OF J_0 FOR THE FIRST ORDER PROBLEM

In Section 3 it is shown that an essential condition J_0 must satisfy is that its first derivative be proportional to AB, so that we can write

$$J_0' = AB\, g(\beta) = (1-\beta^2)\, g(\beta). \tag{74}$$

The results obtained in that section correspond to

$$g(\beta) = -4\beta. \tag{75}$$

Roth (1974) found some remarkable results in using

$$g(\beta) = -(\pi^2 v^2 - \beta^2)^{-1/2}. \tag{76}$$

Finally we can obtain a first order solution by neglecting all terms of h^2 in the solution explained in Section 4. This corresponds to $g(\beta)=\beta f(\beta)$, $f(\beta)$ being defined by (66). The curves resulting from first order theory, but making use of a function J_0 satisfying second order requirements, proved to be better than those obtained from the other first order theories mentioned in this paper.

6. CONCLUSIONS

In the previous sections, it is shown that it is possible to find different approximations of the first adiabatic invariant. Visual evaluation helps us to conclude that for our choices of the parameters h and v, the second order approximation is the best one. The first order counterpart of this approximation proves to be better than any of the other first order solutions. This shows that the magnetic moment can be replaced by the first order truncation of the second order invariant. The accuracy is very high even in the resonance region and for larger values of the relative amplitude of the magnetic field perturbation. This first order truncation avoids tedious numerical integration of a system of differential equations. In fact, the introduction of this new approximate constant of motion, added to the constancy of the total energy permits in many practical cases the full description of the charged particle motion. This is important in determining the trajectories of cosmic ray particles in interplanetary space and the dispersion of the solar wind particles through the inhomogeneities of the interplanetary magnetic field.

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