

Self-oscillations and predictability in climate dynamics— periodic forcing and phase locking

By C. NICOLIS, *Institut d'Aéronomie Spatiale de Belgique, Avenue Circulaire 3, B-1180 Bruxelles, Belgium*

(Manuscript received August 28; in final form December 12, 1983)

ABSTRACT

The coupling between a climate model giving rise to an oscillatory behavior and an external periodic forcing is analyzed. It is shown that under appropriate resonance conditions, phase locking takes place leading to a well-defined value of the phase of the oscillator, contrary to the erratic behavior characterizing the phase in the unforced case. The implications of the results for the problem of predictability and the interpretation of the climatic records are pointed out. It is suggested that one of the mechanisms ensuring sharp periodicities in the climatic record is the interaction between an autonomous oscillator and an external periodic forcing, under conditions of fundamental or harmonic resonance.

1. Introduction

Climate models predicting self-oscillations are attracting an increasing amount of attention in view of their potential importance in the interpretation of the cyclic character of many known climatic changes. In recent publications (Nicolis, 1983, 1984) we have analyzed two typical families of such models and arrived at the conclusion that an autonomous oscillator subject to its own internal fluctuations or to a noisy environment is bound to behave eventually in an erratic way. Specifically, its dynamics can be decomposed into a "radial" and an "angular" part. The radial variable has strong stability properties and tends for long times to a well-defined stable limiting value. In contrast, the phase variable does not reach any well-defined limit and for this reason it becomes completely deregulated by the fluctuations. A destructive phase interference will thus result which will wipe out any trace of oscillatory behavior after a sufficiently long lapse of time. We have pointed out that this quite general property may be at the origin of a progressive loss of predictability, and we illustrated the general ideas using Saltzman's oscillator (coupling between sea-ice and mean ocean surface temperature (Saltzman, 1978; Saltzman et al.,

1982)) and Ghil's oscillator (coupling between land-ice and mean planetary surface temperature (Ghil and Tavantzis, 1983; Källén et al., 1979)).

Yet, many climatic episodes show a clearcut average periodicity, as revealed for instance by the ice core and deep ocean core data of quaternary glaciations (see for instance Berger (1981) and references therein). Somehow, in the course of these changes, the climatic system has therefore been capable of fixing its phase in a more-or-less sharp and predictable fashion. One is tempted to correlate this with the fact that in all known examples of pronounced cyclic changes, the climatic system was submitted to systematic external forcings acting on the same time scale. It is the purpose of the present work to show that the coupling of an oscillator with an external forcing can indeed, under certain conditions, lead to a sharply defined, stable value of its phase relative to the phase of the forcing, thus ensuring predictable behavior. We have already expressed this conjecture recently (Nicolis, 1983), but here we go much further and analyze in detail the mechanisms by which phase locking and concomitantly, predictability, can be ensured.

In Section 2 we recall, using the Saltzman oscillator as illustrative example, the basic ideas of the theory of *normal forms*, which allows one to cast the dynamics of an oscillator in a universal form from which the different roles of “radial” and “angular” variables becomes transparent. We also show how an external periodic forcing can be accommodated in the description.

In Section 3 we undertake a qualitative exploration of the various regimes that can be expected from the oscillator-forcing coupling by means of a straightforward perturbation expansion of the solution in powers of the strength of the forcing (frequently referred to in applied mathematics as “the outer expansion”, see e.g. Cole (1968)). We show that phase locking and predictability cannot be expected, unless certain appropriate *resonance conditions* are satisfied.

A more quantitative approach is outlined in Section 4 in the range of parameter values for which there is resonance. We use a singular perturbation technique, frequently referred to as “the inner expansion” (Rosenblat and Cohen, 1981) and derive a set of equations for both the amplitude and the phase of the oscillator. For Saltzman’s model, we find that, depending on the parameters, the response can be periodic (entrainment) or quasi-periodic, and that there may even be coexistence between two different stable regimes. These predictions are compared to the results of numerical simulations in Section 5.

The implications of the results on the problem of predictability and the possible effect of more complex external forcings are discussed in Section 6.

2. Normal form of a periodically forced oscillator. Illustration on Saltzman’s model

According to the qualitative theory of ordinary differential equations (Arnold, 1980), a dynamical system operating in the vicinity of a *Hopf bifurcation*, leading from a stable steady state to a self-oscillation of the limit cycle type, can be cast in a universal, *normal form*:

$$\frac{dz}{dt} = (\beta + i\omega_0)z - cz|z|^2. \tag{1}$$

Here t is a dimensionless time, z a suitable linear (generally complex-valued) combination of the initial variables, β the bifurcation parameter, ω_0 the frequency of the linearized motion, and $c = u + iv$ a combination of the other parameters occurring in the initial equations. Switching to radial and angular variables through

$$z = r e^{i\phi}, \tag{2}$$

we can separate the evolution into a radial and an angular part,

$$\frac{dr}{dt} = \beta r - ur^3, \tag{3a}$$

$$\frac{d\phi}{dt} = \omega_0 - vr^2. \tag{3b}$$

The first equation describes a relaxation of r towards a well-defined steady state, which is $r_0 = 0$ (i.e. the steady state) if $\beta/u < 0$, and $r_0 = (\beta/u)^{1/2}$ (i.e. the limit cycle) if $\beta/u > 0$. The second equation on the other hand does not yield a well-defined limit for the phase, as it determines a whole family of solutions differing from each other by an arbitrary constant.

In a previous work (Nicolis, 1984) we illustrated the process of reduction to a normal form in Saltzman’s model oscillator (Saltzman, 1978; Saltzman et al., 1982) describing the interaction between sea-ice extent and mean ocean surface temperature:

$$\begin{aligned} \frac{d\bar{\eta}}{d\bar{t}} &= -\phi_2 \bar{\eta} + \phi_1 \bar{\theta}, \\ \frac{d\bar{\theta}}{d\bar{t}} &= -\psi_1 \bar{\eta} + \psi_2 \bar{\theta} - \psi_3 \bar{\eta}^2 \bar{\theta}. \end{aligned} \tag{4}$$

Here ϕ_i, ψ_i are positive parameters, $\bar{\eta}$ is the deviation of the sine of the latitude of the sea-ice extent from the steady state and $\bar{\theta}$ is the excess mean ocean surface temperature. By performing the scaling transformation

$$\begin{aligned} \bar{\eta} &= \left(\frac{\phi_2}{\psi_3}\right)^{1/2} \eta, \quad \bar{\theta} = \frac{\phi_2^{3/2}}{\psi_3^{1/2} \phi_1} \theta, \\ \bar{t} &= \frac{1}{\phi_2} t, \end{aligned} \tag{5}$$

we can cast eqs. (4) in a form displaying two dimensionless parameters:

$$\frac{d\eta}{dt} = -\eta + \theta, \tag{6a}$$

$$\frac{d\theta}{dt} = -a\eta + b\theta - \eta^2\theta, \tag{6b}$$

with

$$a = \frac{\psi_1 \phi_1}{\phi_2^2}, \quad b = \frac{\psi_2}{\phi_2}.$$

The procedure transforming eqs. (6) to the normal form can now be outlined. We first compute the characteristic roots of the linearized stability problem. Straightforward algebra gives:

$$\beta \pm i\omega_0 = \frac{1}{2}\{b - 1 \pm [(b - 1)^2 - 4(a - b)]^{1/2}\}. \tag{7a}$$

We next perform a transformation to a new representation diagonalizing the linear part of eqs. (6). The transformation matrix turns out to be:

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ \beta + i\omega_0 + 1 & \beta - i\omega_0 + 1 \end{pmatrix}. \tag{7b}$$

We thus obtain, to the dominant order in β , the following equation for the transformed variable z :

$$\frac{dz}{dt} = (\beta + i\omega_0)z + \frac{i}{2\omega_0} (3 + i\omega_0)z|z|^2, \tag{8a}$$

giving rise to a radial and angular part of the form (3a) and (3b) respectively, with $u = \frac{1}{2}$ and $v = -3/2\omega_0$. The new variable z is related to the original ones, η, θ through

$$\eta = 2 \operatorname{Re} z, \tag{8b}$$

$$\theta = 2 [\operatorname{Re} z - \omega_0 \operatorname{Im} z].$$

We now want to extend this procedure to include an external periodic forcing. Within the range of validity of eq. (8a), $\beta/\omega_0 \ll 1$, and $|z|$ is relatively small. The most important contributions of the coupling to the external forcing in the right-hand side of eq. (1) will therefore be of the form

$$qf_1(t) + pzf_2(t),$$

with $f_i(t) = f_i(t + T_e)$, T_e being the external periodicity.

In what follows we shall restrict ourselves to the *additive part*. The reason is that according to recent work by Saltzman et al. (1983), the forcing arising from changes in the solar constant caused by orbital variations is to be incorporated in the original eqs. (4) in an additive fashion. In the simple case of a sinusoidal forcing one thus has

$$\frac{d\bar{\eta}}{dt} = -\phi_2 \bar{\eta} + \phi_1 \bar{\theta} + A_n \cos \bar{\omega}_e \bar{t},$$

$$\frac{d\bar{\theta}}{dt} = -\psi_1 \bar{\eta} + \psi_2 \bar{\theta} - \psi_3 \bar{\eta}^2 \bar{\theta} + A_\theta \cos \bar{\omega}_e \bar{t}, \tag{9}$$

or, in our dimensionless variables:

$$\frac{d\eta}{dt} = -\eta + \theta + q_n \cos \omega_e t$$

$$\frac{d\theta}{dt} = -a\eta + b\theta - \eta^2\theta + q_\theta \cos \omega_e t, \tag{10a}$$

with

$$q_n = A_n \frac{\psi_3^{1/2}}{\phi_2^{3/2}}, \quad q_\theta = A_\theta \frac{\psi_3^{1/2} \phi_1}{\phi_2^{5/2}},$$

$$\omega_e = \bar{\omega}_e \phi_2. \tag{10b}$$

By applying the same procedure as used before to get eq. (8a), we can now obtain the normal form of the Saltzman oscillator in the presence of the orbital forcing. After some manipulations we get:

$$\frac{dz}{dt} = (\beta + i\omega_0)z + \frac{i}{2\omega_0} (3 + i\omega_0)z|z|^2$$

$$+ \frac{i}{2\omega_0} [(\beta - i\omega_0 + 1)q_n - q_\theta] \cos \omega_e t. \tag{11}$$

The analysis of this equation in various types of regimes constitutes the main theme of the present article.

3. Qualitative approach

We first explore some general properties of the solutions of eq. (11), assuming that the amplitude of the periodic forcing is small. We express this by the scaling

$$q_n = \varepsilon \bar{q}_n,$$

$$q_\theta = \varepsilon \bar{q}_\theta, \quad \varepsilon \ll 1, \tag{12}$$

and seek for solutions of eq. (11) of the form

$$z = z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + \dots \tag{13}$$

Two cases can now be considered.

3.1. *The system operates below the bifurcation point (damped oscillations)*

In the absence of the forcing, we have a unique stable solution, $z_0 = 0$, representing the steady-state regime. From eqs. (13) and (11), we thus obtain to the first order in ε the following simple result:

$$\frac{dz_1}{dt} = (\beta + i\omega_0) z_1 + \bar{q} \cos \omega_c t, \tag{14}$$

where we have set

$$\bar{q} = \frac{i}{2\omega_0} [(\beta - i\omega_0 + 1) \bar{q}_n - \bar{q}_\theta], \tag{15}$$

and considered that the bifurcation parameter β has a fixed negative value independent of ε .

A particular solution of eq. (14) is easily found to be

$$z_{1p} = \frac{\bar{q}}{2} \left\{ \frac{1}{i(\omega_c - \omega_0) - \beta} e^{i\omega_c t} - \frac{1}{i(\omega_c + \omega_0) + \beta} e^{-i\omega_c t} \right\}. \tag{16}$$

This solution is well-behaved for all values of ω_c provided that $\beta \neq 0$. If, however, the bifurcation point is approached, $\beta \rightarrow 0$, the amplitude of z_{1p} will grow unboundedly when $\omega_0 \rightarrow \omega_c$. This is the phenomenon of *resonance*. It is easy to show that the higher-order terms in the expansion of eq. (13) give a similar result; they remain bounded, except when $\beta \rightarrow 0$ and ω_0 tends to some harmonic of the forcing, $\omega_0 \rightarrow k\omega_c$. Obviously, the response of the system to the forcing will be enhanced at the point of resonance with the fundamental and, to a lesser extent at the points of resonance with the harmonics of the forcing. At the same time, however, the perturbative method we use will break down. We therefore exclude resonance for the time being, postponing a deeper analysis until Section 4.

The general solution of eq. (14) is given by the particular solution (eq. 16)), to which the general solution of the homogeneous equation is added. As the homogeneous solution is simply the damped oscillator problem, we finally obtain:

$$z_1 = K e^{(\beta+i\omega_0)t} + z_{1p},$$

where K is an undertermined constant. In the limit of long times, the first term on the right-hand side will vanish ($\beta < 0$), and therefore the indeterminacy of K will become irrelevant. In other words, we obtain to the dominant order:

$$z = \varepsilon z_{1p} = \varepsilon \frac{\bar{q}}{2} \left\{ \frac{1}{i(\omega_c - \omega_0) - \beta} e^{i\omega_c t} - \frac{1}{i(\omega_c + \omega_0) + \beta} e^{-i\omega_c t} \right\}. \tag{17}$$

This solution represents a regime of entrainment of the system by the forcing. The phase of the response is perfectly well-defined, and the question of predictability simply does not arise.

3.2. *The system operates above the bifurcation (sustained oscillations)*

In the absence of the forcing we have a unique stable limit cycle,

$$z_0 = r_0 e^{i\phi_0},$$

where r_0, ϕ_0 are given from eqs. (3a), (3b) and (8a):

$$r_0 = (2\beta)^{1/2}, \quad \frac{d\phi_0}{dt} = \omega_0 + \frac{3}{2\omega_0} r_0^2. \tag{18}$$

To study the effect of perturbation on this new reference state, we express (11) in the radial and angular variables r and ϕ :

$$\left(\frac{dr}{dt} + i \frac{d\phi}{dt} r \right) e^{i\phi} = (\beta + i\omega_0) r e^{i\phi} + \frac{1}{2} \left(\frac{3i}{\omega_0} - 1 \right) r^3 e^{i\phi} + \varepsilon \bar{q} \cos \omega_c t.$$

Separating real and imaginary parts after setting (cf. eq. (15))

$$\bar{q} = \sigma e^{i\phi'}, \tag{19}$$

we obtain

$$\begin{aligned} \frac{dr}{dt} &= \beta r - \frac{1}{2} r^3 + \varepsilon \sigma \cos \omega_c t \cos(\phi' - \phi), \\ \frac{d\phi}{dt} &= \omega_0 + \frac{3}{2\omega_0} r^2 + \varepsilon \frac{\sigma}{r} \cos \omega_c t \sin(\phi' - \phi). \end{aligned} \tag{20}$$

We can now expand r around r_0 in a way similar to eq. (13). We obtain, to leading order in ϵ :

$$\begin{aligned} \frac{dr_1}{dt} &= -2\beta r_1 + \sigma \cos \omega_c t \cos(\phi' - \phi_0), \\ \frac{d\phi_0}{dt} &= \omega_0 + \frac{3\beta}{\omega_0}. \end{aligned} \tag{21}$$

The second eq. (21) is easily integrated to yield:

$$\phi_0 = \phi_0(0) + \left(\omega_0 + \frac{3\beta}{\omega_0} \right) t. \tag{22}$$

Substituting into the first equation we obtain:

$$\begin{aligned} \frac{dr_1}{dt} &= -2\beta r_1 + \sigma \cos \omega_c t \\ &\cos \left[\phi' - \phi_0(0) - \left(\omega_0 + \frac{3\beta}{\omega_0} \right) t \right]. \end{aligned} \tag{23}$$

As in Subsection 3.1, the general solution of this equation will be the sum of a particular solution and of the general solution of the homogeneous equation. As $\beta > 0$, the latter will simply be a decaying exponential, $K e^{-\beta t}$, whose influence will vanish as $t \rightarrow \infty$. We are thus left with the particular solution. After a straightforward integration of eq. (23) we obtain, for $t \rightarrow \infty$:

$$\begin{aligned} r_1 &= \frac{\sigma}{4} \left\{ \frac{1}{i(\omega_c - \omega_0 - 3\beta/\omega_0) + 2\beta} \right. \\ &\times \exp \{ i[\phi' - \phi_0(0) + (\omega_c - \omega_0 - 3\beta/\omega_0)t] \} \\ &- \frac{1}{i(\omega_c + \omega_0 + 3\beta/\omega_0) - 2\beta} \\ &\times \exp \{ i[\phi' - \phi_0(0) - (\omega_c + \omega_0 + 3\beta/\omega_0)t] \} \\ &\left. + \text{complex conjugate} \right\}. \end{aligned} \tag{24}$$

Several conclusions follow straightforwardly from this result. Firstly, as in Subsection 3.1, the response remains bounded unless the system is near bifurcation ($\beta \rightarrow 0$) and near resonance ($\omega_0 \rightarrow \omega_c$). Secondly, the response shows two independent frequencies,

$$\begin{aligned} \Omega_- &= \omega_c - \omega_0 - \frac{3\beta}{\omega_0} \quad (\text{slow modulation}), \\ \Omega_+ &= \omega_c + \omega_0 + \frac{3\beta}{\omega_0} \quad (\text{fast motion}). \end{aligned}$$

Thirdly, and most importantly, both the phase (eq. (22)) and the amplitude (eq. (24)) of the response contain an undetermined phase $\phi_0(0)$. In other words, the problem of unpredictability of the phase of the free oscillator is *not* solved by applying the external forcing.

One way out of this difficulty could be that the procedure leading to the above results breaks down. This is exactly what happens near the bifurcation and resonance points (including points of resonance with higher harmonics of the forcing). In Section 4, we analyze in detail the regime of resonance and show how predictability, in this case, can be secured through a phase-locking phenomenon.

4. Resonance as a mechanism of phase locking and predictability

In view of the subtleties arising in the handling of the phase, it will be convenient to work with the equations for the real and imaginary parts of our variable z . Setting

$$z = x + iy,$$

we obtain from eq. (11):

$$\begin{aligned} \frac{dx}{dt} &= \beta x - \omega_0 y - \frac{1}{2}(x^2 + y^2) \\ &\times \left(x + \frac{3}{\omega_0} y \right) + \epsilon \bar{q}_x \cos \omega_c t, \\ \frac{dy}{dt} &= \omega_0 x + \beta y - \frac{1}{2}(x^2 + y^2) \\ &\times \left(y - \frac{3}{\omega_0} x \right) + \epsilon \bar{q}_y \cos \omega_c t, \end{aligned} \tag{25}$$

where we have set

$$\bar{q}_x = \frac{\bar{q}_n}{2}, \quad \bar{q}_y = \frac{1}{2\omega_0} [(\beta + 1)\bar{q}_n - \bar{q}_\theta]. \tag{26}$$

In order to avoid the singular denominators arising near resonance in the standard perturbation approach of Section 3, we perform here an altogether different expansion of the solutions. The main point is that near resonance a new, slow time scale, related to the inverse of the difference $\omega_c - \omega_0$, appears in the system, in addition to the scale

defined by the period of the external forcing. We express this fact by scaling the distance from resonance by a suitable power of the smallness parameter ε :

$$\omega_c - \omega_0 = \bar{\omega}\varepsilon^\alpha, \quad \alpha > 0. \tag{27a}$$

The existence of the fast scale, $T = \omega_c t$ and the slow one, τ will then imply a decomposition of the time derivative as follows:

$$\frac{d}{dt} = \omega_c \frac{d}{dT} + \varepsilon^\alpha \frac{d}{d\tau}. \tag{27b}$$

Finally, the smallness of the distance from bifurcation will be introduced by the scaling of β ,

$$\beta = \bar{\beta}\varepsilon^\gamma, \quad \gamma > 0. \tag{27c}$$

Eqs. (25) become:

$$\begin{aligned} \omega_c \frac{\partial}{\partial T} x + \varepsilon^\alpha \frac{\partial}{\partial \tau} x &= \bar{\beta}\varepsilon^\gamma x + (\bar{\omega}\varepsilon^\alpha - \omega_c)y \\ &- \frac{1}{2}(x^2 + y^2) \left[x + \frac{3}{\omega_c - \bar{\omega}\varepsilon^\alpha} y \right] + \varepsilon \bar{q}_x \cos T, \\ \omega_c \frac{\partial}{\partial T} y + \varepsilon^\alpha \frac{\partial}{\partial \tau} y &= (\omega_c - \bar{\omega}\varepsilon^\alpha)x + \bar{\beta}\varepsilon^\gamma y \\ &- \frac{1}{2}(x^2 + y^2) \left[y - \frac{3}{\omega_c - \bar{\omega}\varepsilon^\alpha} x \right] \\ &+ \varepsilon \bar{q}_y \cos T. \end{aligned} \tag{28}$$

We now have to expand the solution (x, y) in terms of ε . The choice of the expansion will be guided by the requirement that one avoid secular terms, which arise frequently in this context (see below), and incorporate all relevant terms in the first few orders of the perturbation. The following development satisfies these requirements (Rosenblat and Cohen, 1981):

$$\begin{aligned} x &= \varepsilon^{1/3} x_1 + \varepsilon^{2/3} x_2 + \varepsilon x_3 + \dots, \\ y &= \varepsilon^{1/3} y_1 + \varepsilon^{2/3} y_2 + \varepsilon y_3 + \dots \end{aligned} \tag{29a}$$

with the choice

$$\alpha = \gamma = \frac{2}{3}. \tag{29b}$$

Substituting into eqs. (28), we obtain, to order $\varepsilon^{1/3}$, a homogeneous system of equations:

$$\frac{\partial x_1}{\partial T} = -y_1, \quad \frac{\partial y_1}{\partial T} = x_1. \tag{30}$$

The solution of this problem is the harmonic oscillator:

$$\begin{aligned} x_1 &= A(\tau) \cos T + B(\tau) \sin T, \\ y_1 &= A(\tau) \sin T - B(\tau) \cos T. \end{aligned} \tag{31}$$

The coefficients A, B remain undetermined at this stage and are expected to depend, in general, on the slow time scale τ which has not entered in eqs. (30).

The next order, $\varepsilon^{2/3}$, leads to equations identical to (30), and therefore adds nothing new. To the order ε , on the other hand, we obtain:

$$\begin{aligned} \omega_c \left(\frac{\partial x_3}{\partial T} + y_3 \right) &= -\frac{\partial x_1}{\partial \tau} + \bar{\beta}x_1 + \bar{\omega}y_1 \\ &- \frac{1}{2}(x_1^2 + y_1^2) \left(x_1 + \frac{3}{\omega_0} y_1 \right) + \bar{q}_x \cos T, \\ \omega_c \left(\frac{\partial y_3}{\partial T} - x_3 \right) &= -\frac{\partial y_1}{\partial \tau} + \bar{\beta}y_1 - \bar{\omega}x_1 \\ &- \frac{1}{2}(x_1^2 + y_1^2) \left(y_1 - \frac{3}{\omega_0} x_1 \right) + \bar{q}_y \cos T. \end{aligned} \tag{32}$$

This is an inhomogeneous system of equations for x_3, y_3 . It admits a solution only if a *solvability condition* expressing the absence of terms growing unboundedly in time, is satisfied. Such terms may arise by the following mechanism. To obtain (x_3, y_3) , we have to "divide" the right-hand side of eqs. (32) by the differential matrix operator

$$\omega_c \begin{pmatrix} \frac{\partial}{\partial T} & 1 \\ -1 & \frac{\partial}{\partial T} \end{pmatrix}; \tag{33}$$

but this is precisely the operator appearing in eqs. (30). According to this latter equation and eqs. (31), it possesses a non-trivial *null space*, that is, non-trivial eigenvectors corresponding to a zero eigenvalue. By dividing with such an operator, one may therefore introduce singularities, if the right-hand sides of eqs. (32) contain contributions lying in this nul space. The solvability condition (Sattinger, 1973) allows one to rule out this possibility, by requiring that the right-hand sides of eqs. (32), viewed as a vector, be orthogonal to the eigen-

vectors of the operator (33). The latter are (cf. eq. (31)):

$$(\cos T, \sin T) \quad \text{and} \quad (\sin T, -\cos T). \quad (34)$$

The scalar product to be used is the conventional scalar product of vector analysis, supplemented by an averaging over T . The point is that we will dispose of two solvability conditions which will provide us with two equations for the coefficients $A(\tau)$, $B(\tau)$ appearing in the first order of the perturbative development (eqs. (31)). After a lengthy calculation one finds

$$\begin{aligned} \frac{dA}{d\tau} &= \bar{\beta}A - \bar{\omega}B - \frac{1}{2}(A^2 + B^2) \left(A - \frac{3}{\omega_e} B \right) + \frac{\bar{q}_x}{2}, \\ \frac{dB}{d\tau} &= \bar{\beta}B + \bar{\omega}A - \frac{1}{2}(A^2 + B^2) \left(B + \frac{3}{\omega_e} A \right) - \frac{\bar{q}_y}{2} \end{aligned} \quad (35)$$

Let us discuss, successively, the steady state solutions of these equations and their stability properties. Note that, in view of eqs. (31), the steady states of (35) correspond to periodic solutions of the original problem, that is, solutions entrained to the periodicity of the external forcing.

4.1 Steady states

Introducing a transformation to (R, ψ) through

$$A = R \cos \psi, \quad B = R \sin \psi, \quad (36)$$

we obtain

$$\begin{aligned} \frac{1}{2} R^3 - \bar{\beta}R &= \frac{\bar{q}_x}{2} \cos \psi - \frac{\bar{q}_y}{2} \sin \psi, \\ \frac{3}{2\omega_e} R^3 - \bar{\omega}R &= -\frac{\bar{q}_x}{2} \sin \psi - \frac{\bar{q}_y}{2} \cos \psi. \end{aligned} \quad (37)$$

Taking the sum of the squares of these two relations one can eliminate ψ and obtain a closed equation

$$\begin{aligned} \xi^3 \frac{1}{4} \left(1 + \frac{9}{\omega_e^2} \right) - \xi^2 \left(\bar{\beta} + \frac{3}{\omega_e} \bar{\omega} \right) \\ + \xi(\bar{\omega}^2 + \bar{\beta}^2) - \rho^2 = 0, \end{aligned} \quad (38a)$$

where

$$\xi = R^2, \quad \rho^2 = \frac{\bar{q}_x^2 + \bar{q}_y^2}{4}. \quad (38b)$$

Fig. 1 describes the result of the numerical solution of eq. (38a) as a function of the distance from resonance, $\bar{\omega}$, obtained for $\bar{\beta} = 1$, $\bar{q}_x^2 + \bar{q}_y^2 = 1$, and a typical value of $\omega_0 \sim \omega_e = 2.32$ (Saltzman et al., 1982, 1983). We see that we have two regions of unique solution interrupted by a region of three solutions, associated with the appearance of an isolated branch of states limited by two folds (points Q and Q'). This phenomenon is referred to in bifurcation theory as *isola formation*. Note that by varying the forcing amplitude ϵ (while keeping it small) we obtain a set of diagrams which differ from each other only through the scaling given by eqs. (27) and (29). In other words the results reported in Fig. 1 are not tied to specific parameter values: they apply to whole classes of situations, among which the values used by Saltzman et al. (1982, 1983) constitute a particular example.

4.2 Stability

To check the stability of the above states, we linearize eqs. (35) successively around each of the solutions and study the roots of the characteristic equation, whose form is

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0, \quad (39)$$

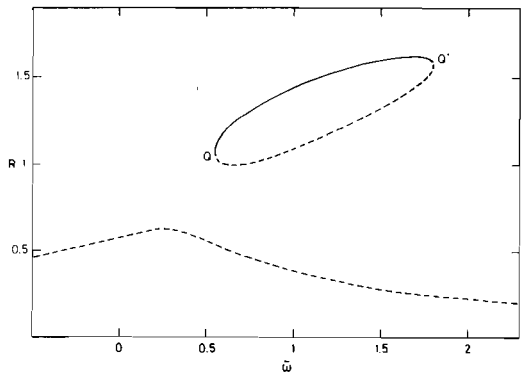


Fig. 1. Stationary solution branches of eqs. (35) as given by numerical solution of eq. (38a). Full and dotted lines represent stable and unstable states, respectively. Notice the occurrence of the isolated branch limited by the two folds Q and Q'. Parameter values used $\bar{\beta} = 1$, $\omega_0 = 2.32$ and $\rho^2 = \frac{1}{4}$.

with

$$\begin{aligned}
 a_{11} &= \bar{\beta} - \frac{1}{2} (A_0^2 + B_0^2) - A_0 \left(A_0 - \frac{3}{\omega_e} B_0 \right), \\
 a_{12} &= -\bar{\omega} + \frac{3}{2\omega_e} (A_0^2 + B_0^2) - B_0 \left(A_0 - \frac{3}{\omega_e} B_0 \right), \\
 a_{21} &= \bar{\omega} - \frac{3}{2\omega_e} (A_0^2 + B_0^2) - A_0 \left(B_0 + \frac{3}{\omega_e} A_0 \right), \\
 a_{22} &= \bar{\beta} - \frac{1}{2} (A_0^2 + B_0^2) - B_0 \left(B_0 + \frac{3}{\omega_e} A_0 \right). \tag{40}
 \end{aligned}$$

The results are indicated again in Fig. 1, where full and dotted lines represent stable and unstable solutions, respectively. In all cases, the states on the lower branch turn out to be unstable ($\text{Re } \lambda > 0$ on these states, and for most of the parameter values considered, the perturbations around them grow in an oscillatory fashion ($\text{Im } \lambda \neq 0$). On the other hand the upper part of the isola branch corresponds to stable states ($\text{Re } \lambda < 0$), whereas the lowest one behaves like a saddle point (two real roots λ of opposite sign).

These results imply the following behaviour for the full non-linear eqs. (35). In the region beyond the two limit points of the isola, there is no stable steady-state regime, and the system is expected to evolve to a periodic solution of the limit cycle type. This is indeed verified by numerical solution of eqs. (35) in this range. On the other hand, in the region between the two limit points, the system should be able to reach the stable steady state on the isola branch, thereby enhancing its response to the forcing. Again, numerical solution of eqs. (35) confirms this trend.

Translated in terms of the original Saltzman model (eqs. (25), (10a) or (9)) these results mean that in the above-mentioned two regions, the system should exhibit *quasi-periodic behavior* and *periodic behavior*, respectively. According to eqs. (35), in the quasi-periodic case, one of the two dominant frequencies should be the frequency of the external forcing, whereas the other should be the intrinsic frequency of the oscillator described by eqs. (31). The latter is equal to $\varepsilon^{3/2} \bar{\omega}$, plus corrections related to the distance from the bifurcation, and corresponds therefore to a slow modulation of the oscillation imposed by the external forcing. On the other hand, in the periodic case, A and B are constant in eqs. (31) and as a

result the system is completely entrained by the external periodicity.

Obviously, the periodic regime is characterized by a well-defined value of the phase, since it is free of any indeterminacy in the values of the amplitudes A and B . It should be therefore qualified as "predictable". In fact, it is a regime of complete *phase locking*, in which the phase difference of the system with respect to the forcing is fixed. On the other hand, in the quasi-periodic case, the phase difference is a complicated function of time giving rise to a rather loose predictability.

The analysis of this section was limited to the case of resonance with the fundamental frequency of the forcing. A similar, though technically more involved calculation, can be carried out for resonances with higher harmonics, following a procedure recently developed by Rosenblat and Cohen (1981). We do not pursue this matter further in the present paper.

5. Numerical simulations

Let us now confront these theoretical predictions with the numerical simulations of the full forced Saltzman equations, (eqs. (10)). We first choose the strength of the forcing ε in such a way that the parameters in the original equations are those determined by Saltzman et al. (1982, 1983). Thus, keeping $a = 6.4$, choosing $\bar{\beta} = 1$ in the calculations of Section 4 and requiring that the distance from bifurcation be equal to $\beta = 0.5$ (corresponding to $b = 2$), we obtain from eq. (27c) and (29b), $\varepsilon^{2/3} = 0.5$, or $\varepsilon = 0.35$. Under these conditions $\omega_e - \omega_0 = 0.5 \bar{\omega}$. Moreover, choosing $\rho = 1/2$ in eq. (38) and requiring a ratio of forcing amplitudes q_θ/q_η between about 10 and 80 as in Saltzman et al. (1983), we arrive, taking for instance $q_\theta/q_\eta \sim 24$, at the values $q_\eta = 0.07$, $q_\theta = 1.7$. In view of the comment made after eq. (38b), Fig. 1 should still qualitatively describe the behavior of the system in this range. Quantitatively, we do expect some deformation, arising from the fact that the value of ε is not very small, as it should be in a perturbative calculation.

We now describe the results of the numerical integration of eqs. (10) for different values of the distance from resonance, $\bar{\omega}$. Fig. 2 gives two typical time series obtained for $\bar{\omega} = 1$ and $\bar{\omega} = -0.1$. We

observe, periodic and aperiodic behavior, respectively, in agreement with the theoretical predictions summarized in Fig. 1. It would therefore appear that a *strict* resonance condition is not indispensable for the validity of the general results of Section 4.

Fig. 3 represents the power spectra corresponding to the two cases. As expected, in the periodic case, we obtain a sharp peak at a single preferred frequency and its harmonics (the latter are not visible in the range of values considered in the figure). In the aperiodic case, on the other hand, we detect two preferred frequencies, as well as less important peaks centered on their linear combina-

tions. It therefore seems that the aperiodic regime is of the quasi-periodic type with two incommensurate frequencies.

Perhaps the most unambiguous way to characterize the type of regime displayed by a dynamical system is to perform a *Poincaré surface of section* (Baesens and Nicolis, 1983). Remember that we are dealing with a forced system involving the two variables η and θ . Effectively, such a dynamics takes place in a three-dimensional space, since one can always express the forcing through $q \cos \chi_e$, $d\chi_e/dt = \omega_e$, thereby introducing its phase χ_e as a third variable. One can now map the original continuous dynamical system into a

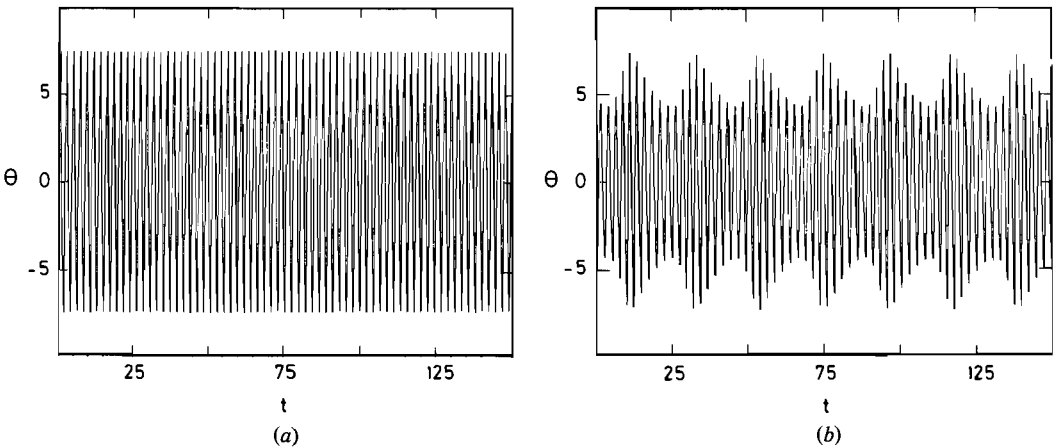


Fig. 2. Time evolution of temperature θ in the forced Saltzman oscillator. (a) Periodic behavior obtained for $\hat{\omega} = 1$. (b) Quasi-periodic behavior obtained for $\hat{\omega} = -0.1$.

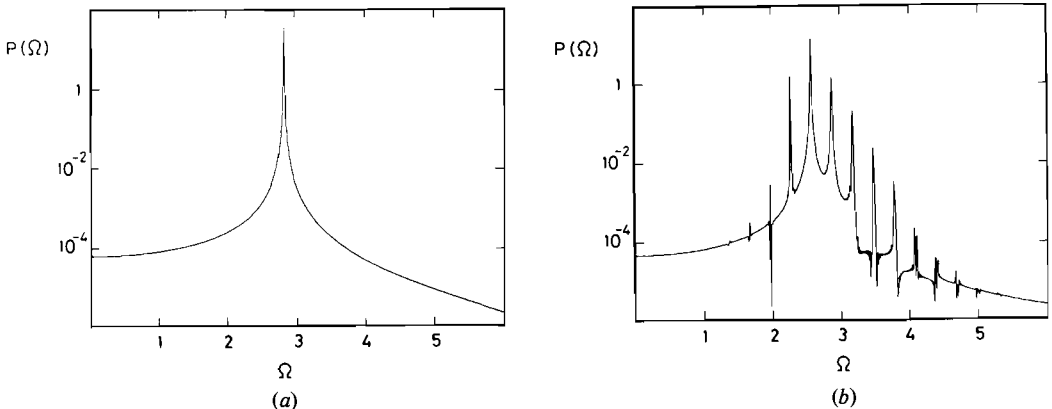


Fig. 3. Power spectra corresponding to cases (a) and (b) of Fig. 2.

discrete time system by following the points at which the trajectories cross (with a slope of prescribed sign) the plane $\cos \chi_c = C$, corresponding to a given value of the forcing. One obtains in this way a *Poincaré map*, (Fig. 4), that is to say, a recurrence relation

$$\begin{aligned} \eta_{n+1} &= f(\eta_n, \theta_n), \\ \theta_{n+1} &= g(\eta_n, \theta_n), \end{aligned} \tag{41}$$

where n labels the successive intersections.

Suppose now that the trajectories of the continuous time flow tend, as $t \rightarrow \infty$, to an asymptotic regime. In the three-dimensional state space, this regime will be characterized by an invariant object, the *attractor*. The signature of this object on the surface of the section will obviously be an attractor of the discrete dynamical system, eq. (41). Conversely, from the existence of an attractor on the surface of the section, we can infer the properties of the underlying continuous time flow.

We have studied the Poincaré surface of section of eqs. (10) generated by the sequence $\chi_c = 2\pi n$, at which the forcing takes its maximum value ($\cos \chi_c = 1$). A typical result in the range of parameter values for which the theory predicts aperiodic solutions is given in Fig. 5. We obtain a closed curve and, remembering that such a curve is the section of the attractor of the initial system by the

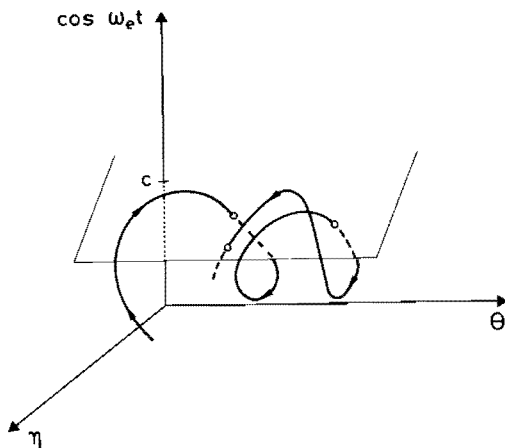


Fig. 4. Schematic representation of a Poincaré map for a forced system. As the system evolves, a representative trajectory cuts a plane of section C at discrete times t_n . The study of the dynamics on the surface of the section gives valuable information of the qualitative behavior of the initial system.

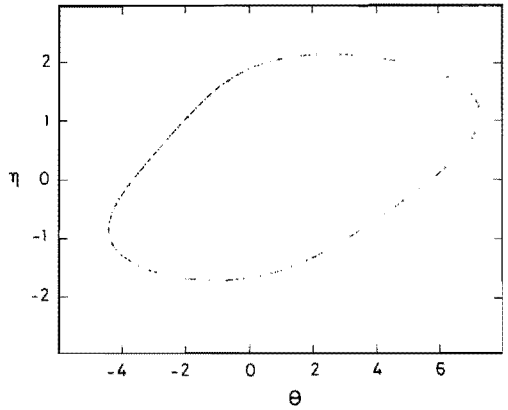


Fig. 5. Poincaré map of the forced Saltzman oscillator obtained with the parameters used in Fig. 2b and Fig. 3b.

plane $\cos \chi_c = 1$, we conclude that the attractor should be a *two-dimensional toroidal surface*. This proves that the behavior is quasi-periodic with two incommensurate frequencies.

6. Discussion

We have seen that self-oscillations in climate dynamics can display predictable behavior associated with the existence of a well-defined phase, as a result of resonant coupling with an external periodic forcing. Depending on the parameters two types of behavior were found: a perfect phase locking leading to a periodic response; and a complex phase variation, leading to a quasi-periodic response. Moreover, the coupling with the forcing could lead to an enhancement of the response through a mechanism of jumps between two different branches of solution (cf. Fig. 1). These theoretical predictions are in complete agreement with the results of numerical simulations of Saltzman's forced oscillator.

In view of these results, it is tempting to suggest that one of the origins of the periodicities found in the climatic record is in the non-linear interaction of an underlying autonomous oscillator of the limit cycle type and an external oscillatory forcing. Moreover, these two oscillators should interact under conditions close to fundamental or harmonic resonance.

A forced oscillator can in principle give rise to even more complicated regimes, such as higher

order quasi-periodic solutions or even chaotic dynamics. It should be possible to find such behaviors by changing such parameters as a , b etc. in the model. Moreover, a stochastic analysis incorporating the effect of both fluctuations and external periodic forcing, would provide a better understanding of the mechanism of phase locking and predictability.

Throughout this paper, we restricted ourselves to a sinusoidal external forcing. A more realistic case arising, for instance in astronomical forcing, is the superposition of several nearly periodic disturbances of different periods. The analysis of Subsection 3.2 applies straightforwardly to this case

and leads to an equation of a form similar to (24). Again, to ensure phase locking and predictability, a resonance condition with at least one of the external frequencies is required. However, the detailed quantitative treatment of the resonant region along the lines of Section 4 appears to be now much subtler. We intend to report on this problem in future work.

7. Acknowledgement

This work is supported, in part, by the EEC under contract no. CLI-027-B(G).

REFERENCES

- Arnold, V. 1980. *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*. Mir, Moscow, U.S.S.R.
- Baesens, C. and Nicolis, G. 1983. Complex bifurcations in a periodically forced normal form. *Z. Physik B52*, 345–354.
- Berger, A. 1981 (ed). *Climatic variations and variability: facts and theories*. Reidel.
- Cole, J. 1968. *Perturbation methods in applied mathematics* (ed. Blaisdell). Waltham, Mass., USA.
- Ghil, M. and Tavantzis, J. 1983. Global Hopf bifurcation in a simple climate model. *SIAM J. Appl. Math.* in press.
- Källén, E., Crafoord, C. and Ghil, M. 1979. Free oscillations in a climate model with ice-sheet dynamics. *J. Atmos. Sci.* 36, 2292–2303.
- Nicolis, C. 1983. Self-oscillations, external forcings and climate predictability. In: *Milankovitch and climate: understanding the response to orbital forcing*. (eds. A. Berger, J. Hays, J. Imbrie, G. Kukla and B. Saltzman). Dordrecht, Holland, in press.
- Nicolis, C. 1984. Self-oscillations and predictability in climate dynamics. *Tellus 36A*, 1–10.
- Rosenblat, S. and Cohen, D. S. 1981. Periodically perturbed bifurcation. (II). Hopf bifurcation. *Stad. Appl. Math.* 64, 143–175.
- Saltzman, B. 1978. A survey of statistical-dynamical models of the terrestrial climate. *Adv. Geophys.* 20, 183–304.
- Saltzman, B., Sutera, A. and Hansen, A. R. 1982. A possible marine mechanism for internally-generated long-period climate cycles. *J. Atmos. Sci.* 39, 2634–2637.
- Saltzman, B., Sutera, A. and Hansen, A. R. 1983. Earth-orbital eccentricity variations and climatic change. In: *Milankovitch and climate: understanding the response to orbital forcing* (eds. A. Berger, J. Hays, J. Imbrie, G. Kukla and B. Saltzman). Dordrecht, Holland, in press.
- Sattinger, D. 1973. *Topics in stability and bifurcation theory*. Springer-Verlag, Berlin.