Stochastic aspects of climatic transitions–
Additive fluctuations

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ABSTRACT

The Fokker-Planck equation corresponding to a zero-dimensional climatic model showing bistable behavior is analyzed. A *climatic potential* function is introduced, whose variational properties determine the most probable states of the stationary probability distribution. A study of the time-dependent properties leads to the identification of the characteristic time scales of evolution.

1. Introduction

The variability of the climatic system, associated both with the almost-intransitive character of atmospheric and oceanic processes as well as with the variability of solar output, has recently been pointed out by several authors (Hasselmann, 1976; Lemke, 1977; Robock, 1978; Nicolis and Nicolis, 1979). It has been realized that in order to account properly for this phenomenon it is necessary to set up a *stochastic study* of climate, incorporating the effect of statistical fluctuations around the deterministic evolution. Within this framework, Hasselmann (1976) and Fraedrich (1978) analyzed some aspects of the linear response of the system associated with small excursions around the present-day climate and in particular they evaluated the Fourier transform of the auto-correlation function (power spectrum) of the pertinent state variables.

Now, in addition to the short-term variability associated with the day-to-day changes of the "weather" component of climate, the possibility of long-term changes associated with *climatic transitions* is nowadays fully realized (e.g. North, 1975). If a linear response theory is clearly sufficient to investigate the variability of the first kind, it is equally clear that for long-term climatic transitions a *nonlinear response* analysis of

fluctuations is needed. Indeed, in a potentially unstable system even relatively small random fluctuations will sooner or later drive the system to a new regime. In a deterministic description the same system would not evolve, unless a finite disturbance exceeding some threshold value is applied to the reference state.

The purpose of the present paper is to develop such a nonlinear response analysis of climatic fluctuations. Preliminary results concerning the asymptotic behavior of a simple climate model under the effect of *multiplicative* fluctuations such as those associated with the variability of the solar output have been reported elsewhere (Nicolis and Nicolis, 1979). Here we perform a comprehensive study of both the static and time-dependent behavior and focus our attention on the simpler case of additive fluctuations. The methods we use are inspired from recent studies of bifurcation and transition phenomena in physical and chemical systems far from thermodynamic equilibrium, where the influence of fluctuations was indeed shown to be decisive (Nicolis and Prigogine, 1977; Haken, 1977; Nicolis and Turner, 1979).

The paper is organized as follows. In Section 2 we outline a general formulation of nonlinear response to fluctuations which is independent of the details of the climatic model, and derive an exact steady-state solution of the *Fokker-Planck*

equation for the probability distribution of the fluctuations valid for arbitrarily large fluctuations. In Section 3 the properties of this solution are illustrated on a simple zero-dimensional model involving two stable climatic states separated by an unstable one. This analysis leads us to introduce the notion of *climatic potential*, which plays here a role analogous to that of free energy density in thermodynamics. Section 4 is devoted to the time-dependent properties of the fluctuations. As the Fokker-Planck equation *cannot be solved* exactly, most of the analysis is based here on numerical simulations, following some special methods developed in the context of plasma physics (Chang and Cooper, 1970). On the other hand for the late stages of evolution one can apply a phenomenological theory due to Kramers, in which the dynamics of the fluctuations is viewed as a problem of diffusion over a potential barrier (see e.g. Wax, 1954). This allows us to identify the characteristic time scales of evolution. Some conclusions and future perspectives are summarized in the final Section 5.

2. General formulation

Let $\bar{x}$ denote a climatic variable obeying an autonomous equation of evolution. A typical example is the surface temperature T averaged over space coordinates. In the absence of fluctuations $\bar{x}$ is supposed to obey to the following "zero-dimensional" equation:

$$\frac{d\bar{x}}{dt} = f(\bar{x}, \lambda) \tag{2.1}$$

Here f is a nonlinear function describing the physics of the system, and λ stands for a set of characteristic, prescribed parameters such as albedo, emissivity and so forth. Of special interest for our work are cases in which the steady-state solutions

$$f(\bar{x}_s, \lambda) = 0 \tag{2.2}$$

are multiple, and in which their stability properties change as the parameters λ take different values.

As discussed in the Introduction, in order to analyze the mechanism of the spontaneous transition between these states it is necessary to consider the effect of fluctuations. In this paper we limit ourselves to additive fluctuations, associated

with random imbalances between the various transport and radiative mechanisms involved in the rate function $f(\bar{x}, \lambda)$. We denote their effect by a random force $F(t)$ which is assumed to be x-independent and define a *white noise* (Wax, 1954):

$$\langle F(t) \rangle = 0$$

$$\langle F(t) F(t') \rangle = q^2 \delta(t - t') \tag{2.3}$$

Here $\langle \ \rangle$ denotes the expectation operator over the ensemble of possible realizations.

Equation (2.1) is now to be replaced by the stochastic differential equation

$$\frac{dx}{dt} = f(x, \lambda) + F(t) \tag{2.4}$$

Equations (2.3)–(2.4) are equivalent to the following Fokker-Planck equation (e.g. Arnold, 1973) with nonlinear friction coefficient and constant diffusion coefficient:

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial x} f(x, \lambda) P(x, t) + \frac{q^2}{2} \frac{\partial^2 P(x, t)}{\partial x^2} \tag{2.5}$$

$P(x, t)$ is the probability density for having the value x of the state variable at time t.

It should be realized that eqs. (2.3) defining the properties of the random force F are in principle rather restrictive. Nevertheless, we expect them to describe satisfactorily the situation for roughly the same reason as in brownian motion and other problems in statistical mechanics: Namely, because of their *local character*, the fluctuations of various fluxes are expected to loose rapidly the memory of the state of the system which prevailed when they occurred and, partly as a result of this, to occur independently of each other. Further arguments in essentially the same direction have been developed by Hasselmann (1976).

For arbitrary nonlinear functions $f(x, \lambda)$, the full analysis of eq. (2.5) constitutes an unsolved problem. Let us therefore first focus on the steady-state solution, $\partial P_s / \partial t = 0$. Integrating once the right hand side with respect to x we get:

$$-J_{P,s}(x) \equiv -f(x, \lambda) P_s(x) + \frac{q^2}{2} \frac{\partial P_s}{\partial x} = \text{constant} \tag{2.6}$$

In all physically reasonable situations we expect that when x reaches the boundaries of the process

(e.g. 0 and ∞ if x is the temperature), P_s will tend very rapidly to zero, i.e. both $P_s = 0$ and $\partial P_s / \partial x = 0$ at the boundaries. We set therefore the *probability flux* $J_p(x)$ to zero at the steady state:

$$J_{P,s}(x) = 0 \qquad \text{for all } x \qquad (2.7)$$

Equation (2.7) is known as a generalized detailed balance condition (Haken, 1977) and leads to an exact solution for P_s in the form:

$$P_s(x) = Z^{-1} \exp\left[-\frac{2}{q^2} U(x) \right] \qquad (2.8)$$

where we defined the *potential $U(x)$*:

$$U(x) = -\int^x d\xi \, f(\xi, \lambda) \qquad (2.9)$$

The proportionality constant Z^{-1} is determined from the normalization of P_s

$$\int_D dx \, P_s(x) = 1$$

D being the domain of variation of x; thus

$$Z = \int_D dx \left[\exp -\frac{2}{q^2} U(x) \right] \qquad (2.10)$$

From eqs. (2.8)–(2.10) a number of general properties of $P_s(x)$ can already be deduced. First, the extrema of P_s are those of the potential $U(x)$ or, in terms of eq. (2.9):

$$f = -\frac{dU}{dx} = 0$$

In other words, P_s has an extremum for those values of x for which the *deterministic* balance equation admits stationary solutions. This is a consequence of the additive character of the fluctuations and of the constancy of the "diffusion coefficient" $\frac{1}{2}q^2$. Second, P_s has a maximum around a stable steady-state solution of eq. (2.1), and a minimum around an unstable one. To see this, we observe that from eq. (2.9):

$$\left. \frac{d^2 U}{dx^2} \right|_{\bar{x}_s} = -\left. \frac{df}{dx} \right|_{\bar{x}_s} \qquad (2.11)$$

On the other hand, from eq. (2.1), $(df/dx)_{\bar{x}_s}$ determines the linear stability of $\bar{x}_s$. Combining with eq. (2.11) we conclude that:

$$\left. \frac{d^2 U}{dx^2} \right|_{\bar{x}_s} < 0: P_s \text{ minimum}, \bar{x}_s \text{ unstable}$$

$$\left. \frac{d^2 U}{dx^2} \right|_{\bar{x}_s} > 0: P_s \text{ maximum}, \bar{x}_s \text{ stable} \qquad (2.12)$$

In the next section we illustrate the significance of these results on a simple zero-dimensional energy balance model.

3. A simple model: Climatic potential and coexistence curve

Suppose that $\bar{x}$ denotes the average surface temperature. The rate function f in eq. (2.1) is then the difference between the solar influx $Q(1 - a(\bar{x}))$ [Q being the solar constant divided by 4, taken to be $Q = 340$ W m^{-2}, and a the albedo] and the infrared cooling rate, $\varepsilon\sigma x^4$, [ε being the emissivity and σ the Stefan constant]. Equation (2.1) becomes:

$$\frac{d\bar{x}}{dt} = \frac{1}{C} [Q(1 - a(\bar{x})) - \varepsilon\sigma\bar{x}^4] \qquad (3.1)$$

where C is the thermal inertia coefficient. Hereafter we normalize the time scale so that the value of C is equal to unity.

For temperature values near the present-day climate, $a(\bar{x})$ is usually taken to be a roughly linear function of its argument (Cess, 1976; Nicolis, 1980). On the other hand, for very low $\bar{x}$, a must tend to the albedo of ice, a_{ice} whereas for high $\bar{x}$, a should also saturate to some value, a_{hot} descriptive of an ice-free earth. The simplest representation taking these features into account is the zero-dimensional piecewise linear model proposed by Crafoord and Källén (1978) and summarized in Fig. 1. Analytically, we write:

$$1 - a(\bar{x}) = 1 - a_{ice} = \gamma_1, \qquad \bar{x} < T_1$$
$$1 - a(\bar{x}) = 1 - a + \beta\bar{x} = \gamma_0 + \beta\bar{x},$$
$$T_1 < \bar{x} < T_2 \qquad (3.2)$$
$$1 - a(\bar{x}) = 1 - a_{hot} = \gamma_2, \qquad \bar{x} > T_2$$

In actual fact the albedo will always be a smooth function of temperature in the vicinity of the transition values T_1 and T_2—a property which is also a mathematical prerequisite for the derivation of a Fokker-Planck equation. Nevertheless because of the detailed balance condition, eq. (2.7), the steady-state probability $P_s(x)$ can be evaluated using the piecewise differentiable model (3.2).

Using the explicit dependence of the albedo on T as given by eqs. (3.2) in eq. (3.1), we see that for appropriate values of the parameters γ_0, γ_1, γ_2 and β the system may admit *three* steady state
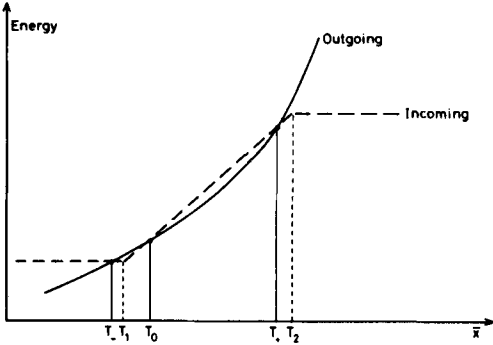
Fig. 1. Incoming and outgoing radiative energy curves as functions of $\bar{x}$ (global average temperature). Their intersections T_+, T_- and T_0 are the three steady states.

solutions. One of them, denoted hereafter by T_+, corresponds to the present-day climate and is asymptotically stable, provided the parameters γ_0 and β are chosen in such a way that the planetary albedo is 0.30 and the emissivity is $\varepsilon = 0.61$. The second solution, denoted by T_-, corresponds to a deep-freeze climate and is also asymptotically stable. A third solution T_0 lies between T_+ and T_- and is unstable.

Let us now turn to the stochastic aspects. We first limit ourselves to the steady-state, postponing until Section 4 the analysis of the time-dependent behavior. According to Section 2, the principal quantity determining the behavior of fluctuations is the potential $U(x)$. Such a potential was introduced for a spatially one-dimensional climate model by Ghil (1976) and further analyzed by North et al. (1979). As the deterministic equation has three steady-state solutions, the potential will have two minima at $T_\pm$ separated by a maximum at T_0. We call this a bistable potential. From definition (2.9) and from eqs. (3.1) and (3.2) we obtain the following explicit expression of U, after setting arbitrarily $U(0) = 0$:

$$-U(x) = Q\gamma_1 x - \frac{\varepsilon\sigma}{5} x^5 \qquad x < T_1$$

$$-U(x) = Q\left[\gamma_1 T_1 + \gamma_0(x - T_1) + \frac{\beta}{2}(x^2 - T_1^2) \right]$$

$$- \frac{\varepsilon\sigma}{5} x^5 \qquad T_1 < x < T_2$$

$$-U(x) = Q\left[\gamma_1 T_1 + \gamma_0(T_2 - T_1) + \frac{\beta}{2}(T_2^2 - T_1^2) \right.$$

$$\left. + \gamma_2(x - T_2) \right] - \frac{\varepsilon\sigma}{5} x^5 \qquad x > T_2 \qquad (3.3)$$

The properties of the stationary distribution $P_s(x)$ depend crucially on the normalization factor Z, eq. (2.8). For the function $U(x)$ defined by eqs. (3.3) the integral over x appearing in the expression of Z cannot be evaluated exactly for an arbitrary value of q. However in the limit of small fluctuations, the maxima of $\exp[-(2/q^2) U(x)]$ become very sharp and as a result the integral over x can be evaluated by steepest descent methods (see e.g. Matthews and Walker, 1965). Remembering that these maxima are at the deterministic stable states T_+ and T_-, and that there is also a minimum at the unstable state T_0, we obtain:

$$Z = \int_0^{T_0} \exp\left[-\frac{2}{q^2} U(x) \right] dx$$

$$+ \int_{T_0}^{\infty} \exp\left[-\frac{2}{q^2} U(x) \right] dx \qquad (3.4)$$

A steepest descent calculation amounts to expanding $U(x)$ in the two integrals around T_- and T_+, respectively, and to retaining only the quadratic terms in the expansion. We thus obtain:

$$Z \cong q\left[\left(\frac{\pi}{U''(T_-)} \right)^{1/2} \exp\left(-\frac{2}{q^2} U(T_-) \right) \right.$$

$$\left. + \left(\frac{\pi}{U''(T_+)} \right)^{1/2} \exp\left(-\frac{2}{q^2} U(T_+) \right) \right]$$

hence

$$P_s(x) \cong (q\pi^{1/2})^{-1}\left[(4\varepsilon\sigma T_-^3)^{-1/2} \exp\left(-\frac{2}{q^2} U(T_-) \right) \right.$$

$$\left. + (4\varepsilon\sigma T_+^3 - Q\beta)^{-1/2} \exp\left(-\frac{2}{q^2} U(T_+) \right) \right]^{-1}$$

$$\times \exp\left(-\frac{2}{q^2} U(x) \right) \qquad (3.5)$$

The behavior of this function is conditioned by the deepest of the two minima $U(T_-)$ and $U(T_+)$. Suppose first that $U(T_+) < U(T_-)$. Because of the inverse of a small factor $1/q^2$ in the exponent, the difference between these two quantities will be amplified enormously and the term containing $U(T_-)$ will give a vanishingly small contribution

in eq. (3.5). Moreover, P_s itself will be nonvanishing only in a small vicinity around T_+, and will therefore reduce to a Gaussian centered on this state:

$$P_s(x) \cong (\pi^{1/2} q)^{-1} (4\varepsilon\sigma T_+^3 - Q\beta)^{1/2}$$

$$\times \exp\left[-\frac{2}{q^2}(U(x) - U(T_+))\right]$$

$$\cong (\pi^{1/2} q)^{-1} (4\varepsilon\sigma T_+^3 - Q\beta)^{1/2}$$

$$\times \exp\left[\frac{1}{q^2}(-4\varepsilon\sigma T_+^3 + Q\beta)(x - T_+)^2\right]$$

$$U(T_+) < U(T_-) \qquad\qquad (3.6)$$

If on the other hand it turned out that $U(T_-) < U(T_+)$, an expression similar to (3.6) would obtain provided one retains the contributions around T_- as the dominant terms:

$$P_s(x) \cong (\pi^{1/2} q)^{-1} (4\varepsilon\sigma T_-^3)^{1/2}$$

$$\times \exp\left[\frac{2}{q^2}(U(x) - U(T_-))\right]$$

$$\cong (\pi^{1/2} q)^{-1} (4\varepsilon\sigma T_-^3)^{1/2}$$

$$\times \exp\left[-\frac{4}{q^2}(\varepsilon\sigma T_-^3 (x - T_-)^2)\right]$$

$$U(T_-) < U(T_+) \qquad\qquad (3.7)$$

At the borderline between these two cases, $U(T_+) \cong U(T_-)$, one should keep the contribution from both T_+ and T_- in eq. (3.5). The result is a two-hump distribution with equal height maxima, which is very well approximated by two Gaussians peaked sharply around T_+ and T_-, and joined smoothly in a shallow minimum around T_0. Setting

$$U(T_-) = U(T_+) = U_m$$

we obtain:

$$P_s(x) \cong (\pi^{1/2} q)^{-1} [(4\varepsilon\sigma T_-^3)^{-1/2}$$

$$+ (4\varepsilon\sigma T_+^3 - Q\beta)^{-1/2}]^{-1} \exp\left[-\frac{2}{q^2}(U(x) - U_m)\right]$$

$$U(T_+) \sim U(T_-) \qquad\qquad (3.8)$$

The interpretation of eq. (3.6) to (3.8) is fairly obvious: If $U(T_+) < U(T_-)$ the influence of the deep-freeze state disappears in the limit of long times. T_+ is therefore dominant and attracts all

initial conditions. If on the contrary $U(T_-) < U(T_+)$, state T_- is dominant, and the present-day climate will sooner or later be subjected to a runaway effect leading to the deep-freeze state. Note however that, as we will show in Section 4, the characteristic times associated with this transition may be exceedingly long. Finally, if $U(T_+) \cong U(T_-)$ states T_+ and T_- are equally dominant and will survive with equal probability in the long time limit. This means that, while the system will jump back and forth between them, the average residence times on both states will be equal (and, presumably, exceedingly long as pointed out earlier). Figure 2 gives the shape of the steady-state probability in the above three cases.

We see that the situation is reminiscent of the diffusive motion of a material point in a bistable well, or of the passage from vapor to liquid phase in the region of coexisting phases as described by the Van der Waals free energy (see e.g. Landau and Lifshitz, 1959). In order to impress on the reader these analogies, we coin the term *climatic potential* for the quantity $U(x)$. When the amplitude of fluctuations q is no longer small the probability distribution becomes broader and the distinction between dominant states is not as sharp as before, nevertheless the qualitative picture drawn above remains correct in its essential aspects.

What are the elements which decide the dominance of a particular climate? From eqs. (3.3) we see that the relation between $U(T_+)$ and $U(T_-)$ depends on the system's parameters. As $P_s(x)$ is vanishingly small for $x > T_2$, it is likely that the particular choice of the value of a_{hot} is not crucial. Fixing a_{hot} to 0.25, as well as the present-day
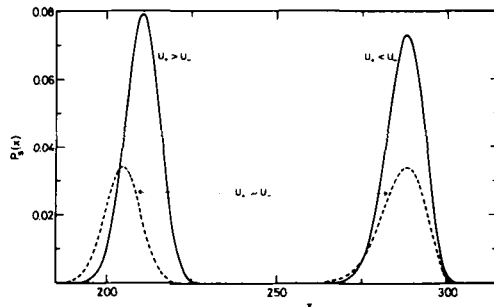

Fig. 2. Steady-state probability distribution $P_s(x)$ for three representative cases: $U_+ > U_-$, $U_+ \sim U_-$ and $U_+ < U_-$ and for $q = 7$ yr$^{-1/2}$ K. Note that, in the second case, $P_s(x)$ is a two-hump distribution.

planetary albedo to 0.30, we are left with two free parameters, γ_1 (i.e., a_{ice}) and β, the albedo-temperature slope. The condition that $U(T_+) \sim U(T_-)$, in other words that both stable states T_+ and T_- are equally dominant, corresponds to a given relationship between γ_1 and β. Figure 3 depicts this *climatic coexistence curve* for the model discussed in the present Section. A different picture of the situation is provided by Fig. 4, which gives, through various *ice-isoalbedo curves*, the way the difference $U(T_+) - U(T_-)$ varies with the albedo-temperature feedback slope β. We see that high values of β favor, for reasonable choices of a_{ice}, the deep-freeze state. Conversely, for moderate values of β, the present-day climate tends to dominate, in the sense that it constitutes the most probable state of the stationary probability distribution.
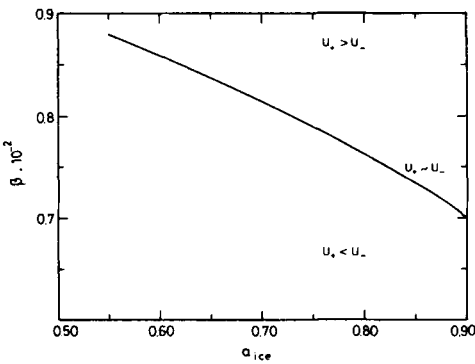

Fig. 3. Climatic coexistence curve: Values of temperature feedback parameter β versus albedo of ice, a_{ice} for which $U_+ \sim U_-$.
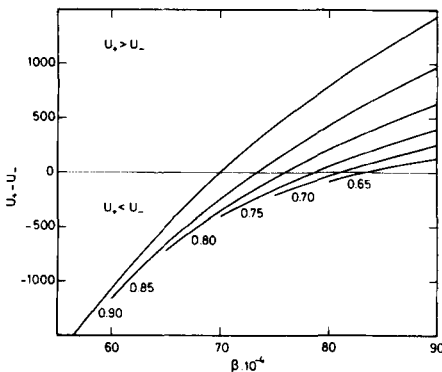

Fig. 4. Dependence of $(U_+ - U_-)$ on parameter β for different values of a_{ice}.

4. Time-dependent properties

In this Section we analyze the time-dependent behavior of the fluctuations for the climatic model discussed in Section 3. Essentially, we must solve the initial-value problem for the Fokker-Planck equation (2.5), in which the nonlinear friction coefficient $f(x, \lambda)$ has the structure described by eqs. (3.1) and (3.2). This type of problem was investigated recently by a number of authors in the context of bifurcations in nonlinear physico-chemical systems, for an f displaying a cubic nonlinearity (Suzuki, 1977; van Kampen, 1977; Caroli et al., 1979). At present it appears difficult to extend these calculations for the type of nonlinearity characterizing our model. We therefore resort, for the most of this Section, to numerical simulations.

The problem is highly nontrivial because of the stiffness properties of the Fokker-Planck equation. Numerical solutions using straightforward discretizations of the various derivatives of x can lead rapidly to negative solutions for the probability density $P(x, t)$, and to further inconsistencies. As it turns out, similar problems arise in plasma physics. We have therefore followed a proposal by Chang and Cooper (1970), according to which the discretization must be performed with a variable mesh size. The latter is chosen to reproduce as closely as possible, the exact steady-state probability distribution. The results are very satisfactory: in addition to positivity, normalization is secured provided that appropriate boundary conditions are imposed. The convergence is excellent and is maintained even if the time step is relatively large.

Throughout our simulations, the boundaries were chosen to be at $x = 0$ K and $x = 360$ K. Three different types of situation were considered (see also Fig. 5):

(i) Present climate at T_+ is dominant, that is to say $U(T_+) < U(T_-)$. For a value of the albedo of ice $a_{ice} = 0.80$ and a value of temperature feedback coefficient $\beta = 0.0065$, the minimum of the climatic potential U at T_+ is quite deep. In order to evolve to the deep-freeze state at T_- starting from T_+ (this will show up by the appearance of a second peak of increasing size at T_-) the system must diffuse through the *potential barrier* constituted by the maximum of U at the unstable state T_0.
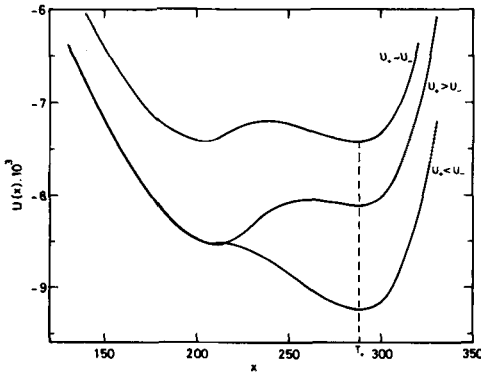
Fig. 5. Climatic potential $U(x)$ for three representative cases. The difference between the value $U(T_+)$ and the maximum value of U gives in each case, the magnitude of the barrier that the system must overcome before evolving to the low temperature state T_-, starting from T_+.

For the parameter values chosen, the jump of U between T_0 and T_+ turns out to be equal to $U_0 - U_+ \equiv \Delta U = 713 \text{ yr}^{-1} \text{ K}^2$.

(ii) T_+ and T_- are equally dominant. For a_{ice} also near 0.80 and for $\beta = 0.0075$, the potential barrier turns out to be $\Delta U = 213 \text{ yr}^{-1} \text{ K}^2$.

(iii) T_- is dominant. For $a_{ice} = 0.80$ and $\beta = 0.0085$ the barrier one has now to overcome to jump from T_+ to T_- is much lower, $\Delta U = 33 \text{ yr}^{-1} \text{ K}^2$.

In each of these three typical situations, the evolution of the probability distribution was followed for different values of the variance of the fluctuations, q^2. The initial condition was taken to be a Gaussian centered on T_+ (see eq. (3.6)), or a delta function also centered on T_+.

A general property that emerges from all simulations is that the time evolution is exceedingly slow if the variance q^2 is small. For instance, taking $q = 0.5 \text{ yr}^{-1/2} \text{ K}$ (or $q^2 = 0.25 \text{ yr}^{-1} \text{ K}^2$, to be compared with the values of incoming and outgoing radiation, normalized by the thermal inertia coefficient, of the order of $200 \text{ yr}^{-1} \text{ K}$) and an initial probability distribution as in eq. (3.6), one finds that $P(x, t)$ hardly moves for times up to 10,000 yrs, both in the case where T_+ is dominant, and T_+, T_- are equally dominant. It is only in case (iii) mentioned above, where T_- dominates, that one finds a modest tendency to evolve slowly. What is happening here is that because of the relatively small height of the barrier ΔU, diffusion over it is possible. The states near

T_+ are therefore progressively depleted even if the initial condition favors T_+. A convenient way to express this is to introduce the probability diffusion flux at the barrier position T_0:

$$J_p(x = T_0, t) = -\frac{q^2}{2} \frac{\partial P(x, t)}{\partial x}\bigg|_{x=T_0} \quad (4.1)$$

At $t = 0$ and $t \to \infty$ this flux practically vanishes owing respectively, to the initial condition chosen and to detailed balance at the steady-state (see eq. (2.7)). For intermediate times and $q = 0.5 \text{ yr}^{-1/2} \text{ K}$ the system is unable to build up an appreciable flux as long as T_- is not dominant. But when this latter state becomes dominant one observes, after an initial overshoot, a plateau value which remains essentially constant up to 10,000 yrs. The value of the flux in this plateau is very different from the numerically determined steady-state one and reflects therefore the nonequilibrium behavior of the system. It is still, however, hardly detectable.

When q increases the diffusion over the barrier is in many cases accelerated. Take for instance $q = 3 \text{ yr}^{-1/2} \text{ K}$. When T_+ is dominant the probability flux still remains practically zero up to $t \sim 10,000$ yrs. However, when T_+ and T_- are equally dominant one observes a plateau value which is different from the steady-state one and subsists up to $t \sim 10,000$ yrs as shown in curve (a) of Fig. 6. On the other hand the time to reach the plateau is rather short, $t \sim 50$ yrs. For $q = 3 \text{ yr}^{-1/2} \text{ K}$ and T_- dominant the evolution becomes still faster. No plateau is reached, and there is a continuous decrease of J_p to the steady-state value of zero. After an initial transient, the regression of J_p is practically linear in time, see curve (b) of Fig. 6.

For larger values of q the evolution is further accelerated. Thus, for $q = 10 \text{ yr}^{-1/2} \text{ K}$ the steady-state is reached rapidly for all three characteristic cases (i) to (iii). Table 1 gives a recapitulative picture of the various forms of time-dependencies.

In addition to the probability flux, a useful index of the qualitative aspects of evolution is the variance of the fluctuations around the mean,

$$\int_D dx (x - \langle x \rangle)^2 P(x, t) \quad (4.2)$$

Fig. 7 represents the time evolution of the variance in two cases. One of them, curve (a), corresponds to the situation of curve (b) of Fig. 6. We see that during the approach to the steady-state the

variance increases almost linearly by one order of magnitude in a couple of thousands of years. This reflects the depletion of the states around T_+ and the progressive appearance of a second probability peak around T_-, giving rise to a broad two-hump

Fig. 6. Time evolution of the probability flux J_p (in yr^{-1}) at the minimum T_0 of P, for $q = 3$ yr$^{-1/2}$ K. The initial condition is a Gaussian distribution centered on T_+. Curve (a): Probability flux when, T_+ and T_- are equally dominant. Curve (b): Probability flux when T_- is dominant.

Fig. 7. Time evolution of the variance of the probability distribution. Curve (a): State T_- is dominant and $q = 3$ yr$^{-1/2}$ K. Curve (b): States T_+ and T_- are equally dominant and $q = 10$ yr$^{-1/2}$ K.

distribution. The second case, curve (b), refers to a fast evolution for large fluctuations, $q = 10$ yr$^{-1/2}$ K, under conditions of equal dominance of states T_+ and T_-. Again, as a result of the building of a second probability peak, the variance increases dramatically in a few thousands of years.
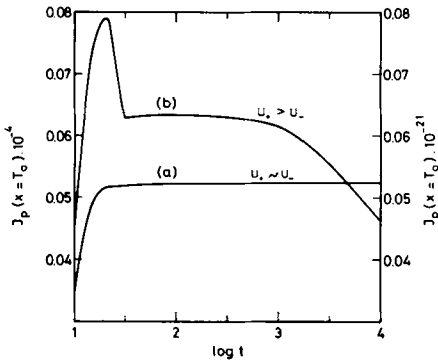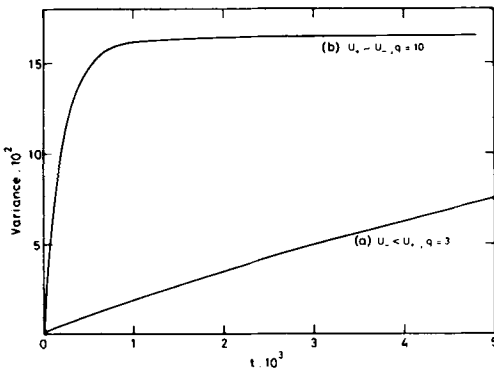
The results reported so far in this Section are reminiscent of the basic ideas underlying Kramers' phenomenological theory of chemical kinetics (see e.g. Wax, 1954): A chemical reaction is viewed as a diffusion problem over a potential barrier, corresponding to the activation energy that must be overcome before an initial chemical bond is broken and a new one is formed. After an initial lapse of time, and well before the reaction is completed, a weak diffusion flux over the barrier is postulated and assumed to remain practically constant for all x near the barrier position, and slowly depending on time provided q is small enough. Using the Fokker-Planck equation (2.5) one obtains, from eq. (2.6) and the definition of the potential U:

$$J_p(x, t) = -\left(\frac{\partial U}{\partial x} P(x) + \frac{q^2}{2} \frac{\partial P}{\partial x} \right)$$

$$= -\frac{q^2}{2} \exp\left(-\frac{2}{q^2} U \right) \frac{\partial}{\partial x} \left[P \exp\left(\frac{2}{q^2} U \right) \right]$$

(4.3)

For $J_p(x)$ independent of x one can integrate both sides over x to obtain, in the notation of our model:

$$J(t) = \frac{q^2}{2} \left(\int_{T_-}^{T_+} \exp\left[\frac{2}{q^2} U \right] dx \right)^{-1}$$

$$\times \left\{ P(T_+, t) \exp\left[\frac{2}{q^2} U(T_+) \right] \right.$$

$$\left. - P(T_-, t) \exp\left[\frac{2}{q^2} U(T_-) \right] \right\}$$

(4.4)

Table 1

	$q = 0.5$	$q = 3$	$q = 10$
T_+ dominant	no evolution visible up to $t \sim 10,000$ yrs	no evolution visible up to $t \sim 10,000$ yrs	steady-state rapidly reached $t \leqslant 100$ yrs
T_+, T_- equally dominant	no evolution visible up to $t \sim 10,000$ yrs	nonequilibrium plateau with $J_p \sim 0.5 \times 10^{-22}$ yr^{-1}	steady-state rapidly reached $t \leqslant 5000$ yrs
T_- dominant	nonequilibrium plateau with very small flux is reached	regression of J_p to steady-state value $J_{p,s} = 0$	steady-state rapidly reached $t \leqslant 1000$ yrs

To fix ideas, consider the case depicted on curve (b) of Fig. 6 of a climate dominated by T_-. Still, if we start from a sharply peaked distribution around T_+ the state T_- will remain unoccupied for a very long period of time as long as the variance of fluctuations q is small (this is entirely confirmed by the numerical simulations). Hence, to a good approximation eq. (4.3) will become;

$$|J(t)| \cong \frac{q}{2\pi^{1/2}} \, (-4\varepsilon\sigma T_0^3 + Q\beta)^{1/2} \exp\left(-\frac{2}{q^2}\,\Delta U\right)$$

$$\times \, P(T_+, t) \qquad (4.5)$$

where a steepest descent evaluation of the denominator was performed and, as before, $\Delta U = U(T_0) - U(T_+)$.

As regards the time dependence of J, we see from eq. (4.5) that it is identical to that of $P(T_+)$. In the Kramers regime, the latter will be simply given by

$$\frac{dP(T_+, t)}{dt} = -\frac{1}{\tau} P(T_+, t) \qquad (4.6a)$$

since the only process going on is the depletion of the states in the potential well around T_+. Thus:

$$P(T_+, t) = P(T_+, 0) \exp\left(-\frac{t}{\tau}\right) \qquad (4.6b)$$

Within the range of Kramer's theory the characteristic time turns out to be (Wax, 1954, Caroli et al., 1979)

$$\tau \sim \pi(-4\varepsilon\sigma T_0^3 + Q\beta)^{-1/2}(4\varepsilon\sigma T_+^3 - Q\beta)^{-1/2}$$

$$\times \exp\left(\frac{2}{q^2}\,\Delta U\right) \qquad (4.7)$$

For values of the variance q^2 significantly smaller than the height of the barrier ΔU, τ will be very long and, concomitantly, $J(t)$ will be very small. For instance, for $q = 3$ and T_- dominant we obtain a time scale of the order of glaciation onset, $\tau \sim 10^4$ yrs, and values of $J(t)$ in agreement with curve (b) of Fig. 6. On the other hand, for the same value of q the time scale is much longer, $\tau \sim 10^{23}$ yrs, if the states T_+ and T_- are equally dominant. This again agrees with curve (a) of Fig. 6.

The possibility of reproducing, for suitable values of q and ΔU, characteristic time scales reminiscent of glaciations is a significant feature of our work. At present, however, it remains difficult

to draw definite conclusions because of the uncertainties of values of the model parameters.

5. Discussion

In this paper we performed a stochastic analysis of a zero-dimensional climatic model showing bistable behavior, which is the simplest nontrivial form of climatic transition. We showed that both the static and the time-dependent properties of the fluctuations are monitored by two basic quantities: The climatic potential, U, and the variance of the noise, q^2. A sensitivity analysis of U with respect to the system's parameters—particularly the temperature feedback coefficient β—led us to distinguish between a regime where present climate dominates, and a regime where a deep-freeze climate dominates. We also determined conditions of "coexistence" of these two regimes in terms of the characteristic parameters.

At a more quantitative level, we found that for a small variance the stationary probability distribution is very sharply peaked around the dominant state, and that the time scale of evolution becomes exceedingly slow. Moreover, an increase of the temperature feedback coefficient tends to favor the deep-freeze climate and to accelerate the evolution toward it, by diminishing the height of the *potential barrier* between present climate, T_+, and the unstable state, T_0.

We believe that the evaluation of the probability of climatic fluctuations, initiated in the present paper, is a prerequisite in the understanding of climatic change. The earth is a noisy environment. A local imbalance between incoming and outgoing energy—provoking for instance a sudden cooling—can occur anywhere any time, with a certain probability. Depending on the time scale of evolution triggered by such a fluctuation, one will have a qualitatively new behavior or an effect which will be masked by other factors acting on the system. In the first class one has the rather fast evolution depicted in curve (b) of Fig. 6 whose characteristic scale is about 10^4 yrs, comparable to the onset time of a glaciation. In the second class one has the exceedingly slow evolution of curve (a) of Fig. 6, with a time scale larger than the age of the earth itself! As we saw in Section 4, a convenient criterion of evolution is the way the variance of the fluctuations behaves in the course of time. This is an interesting and workable

criterion since the variance is a measurable quantity.

Our results can also be interpreted in an alternative way: Namely, in a climatic system involving more than one simultaneously stable states, fluctuations provide a *mechanism of selection* between these states. This joins a proposal recently formulated by Paltridge (1979) in the context of his maximum entropy production conjecture.

The work we reported can be extended in at least two directions. First, relax the hypothesis of additive noise and analyse the effect of fluctuations of such parameters as Q or ε, which couple to the system in a multiplicative way. A preliminary study of this aspect was recently carried out (Nicolis and

Nicolis, 1979). And second, use more sophisticated energy-balance models like the one-dimensional model studied by North (1975). This latter extension is especially crucial, in view of the local character of fluctuations. Work in both directions is in progress.

6. Acknowledgements

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СТОХАСТИЧЕСКИЕ АСПЕКТЫ КЛИМАТИЧЕСКИХ ПЕРЕХОДОВ.
I. АДДИТИВНЫЕ ФЛУКТУАЦИИ

Анализируется уравнение Фоккера-Планка, соответствующее нульмерной климатической модели, которое указывает на существование двух устойчивых состояний. Вводится Функция климатического потенциала, вариационные свойства которой определяют наиболее вероятные состояния стационарного распределения вероятностей. Изучение свойств, зависящих от времени, дает возможность определить характерные временные масштабы эволюции системы.