

On the entropy balance of the earth–atmosphere system

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(Received 17 July 1979; revised 12 February 1980)

SUMMARY

The entropy balance associated with a Budyko–Sellers climatic model is developed. It is shown that different regimes, associated with decreasing, as well as increasing values of entropy production (which measures the rate of dissipation in the system) in the course of time are possible. This immediately poses the problem of stability of steady states of the climatic system. An explicit criterion of climatic stability is thus derived, which is expressed in terms of thermodynamic quantities related to excess entropy production. The results are illustrated on simple cases involving diffusive energy transport. A comparison with Paltridge's minimum entropy exchange principle is also attempted.

1. INTRODUCTION

The complexity of the dynamical processes determining long-term climatic trends is well known. Nevertheless, the need of an approach involving only a few global variables is nowadays widely recognized. Suffice it to quote the energy-balance models of the Budyko–Sellers type (Budyko 1969; Sellers 1969), which have been developed further by such investigators as North (1975, a, b), Ghil (1976) and Coakley (1979), and which led to a qualitative understanding of a great many features of climate and its evolution.

Reduction of state space by suitably averaging the initial dynamical variables is a well-known procedure in many areas of physics. The most characteristic example is certainly Statistical Mechanics, where different averages have led, successively, from the Liouville equation to Boltzmann-like equations, to Markov chains, or to macroscopic balance equations like those of hydrodynamics and chemical kinetics.

A second line of approach to the study of complex systems, which is also suggested by the statistical mechanics 'prototype', is the development of a *thermodynamic* description. The primary objective is to cast some basic features of the system in the properties of state functionals – like entropy (or more generally, a Lyapounov functional; see Prigogine *et al.* 1977) or entropy production – which are largely independent of the details of the individual degrees of freedom. Typical examples of such properties are the Clausius inequality, the theorem of minimum entropy production in the linear range of irreversible processes (Prigogine 1947), or the stability criterion of steady states far from equilibrium (Glansdorff and Prigogine 1971). Surprisingly, this second line of approach is much less common in climate modelling. It is only recently that Paltridge (1975, 1978), Golitsyn and Mokhov (1978) and North *et al.* (1979) examined the possible existence of a variational principle governing climate. Paltridge's approach is specially significant for our discussion. Assuming certain relationships between atmospheric and oceanic dissipation rates, he showed that a maximization of the steady-state overall dissipation rate of the earth–atmosphere system yields uniquely defined spatial distributions of surface temperature, cloud cover and meridional energy fluxes, which closely resemble the observed zonally averaged mean-annual values.

The purpose of this paper is to develop the entropy-balance equation associated with an energy-balance equation of the Budyko–Sellers type. From this equation we identify, in section 2, the appropriate expressions for the entropy flux and entropy production which

are valid for stationary as well as for time-dependent states. In section 3 we evaluate the time-derivative of the entropy production and show that, in general, it has no definite sign. As a result the steady state solution of the system does not correspond to a minimum of entropy production, even if linear relations between energy flux and temperature gradient are considered. This provides an extension of Prigogine's minimum entropy production theorem. It also shows that entropy production can no longer serve as a Lyapounov functional, whose variational properties guarantee the stability of the reference state. This raises, therefore, the question of stability of the climatic system. In the remainder of section 3 we show how this question can be tackled by the methods of thermodynamics of irreversible processes.

Section 4 is devoted to the explicit evaluation of the time course of entropy production for a simple climatic model involving, successively, an ice-free earth (section 4a) and a climate close to the present-day one (section 4b). In both cases we show that entropy production may decrease or increase in time, depending on the initial state. This corroborates the general results of the analysis of section 3, according to which the steady state climate does not appear to satisfy an obvious variational principle, at least at the level of a Budyko-Sellers type of model. Nevertheless, some general trends appear to recur continuously. For instance, entropy production tends to increase whenever the equator-pole temperature difference becomes more pronounced.

Section 5 is devoted to the solution of the energy balance equation using Paltridge's maximum entropy production conjecture. This yields a meridional energy flux which is in fair agreement with present-day data, but a somewhat less satisfactory temperature distribution.

In the final section 6 we draw the main conclusions of the analysis. We point out the intrinsic variability of the climatic system, as illustrated from the different behaviour obtained for the entropy production by different assumptions on the energy flux (such as a diffusive energy transport or a maximum entropy production). It appears, therefore, that the basic problem one is faced with is to delimit the principal factors responsible for the selection of a particular steady state climatic regime.

2. THE MODEL. ENTROPY-BALANCE EQUATION

A one-dimensional model involving meridional energy transport will be adopted (North 1975a, b; Coakley 1979) as described in Fig. 1.

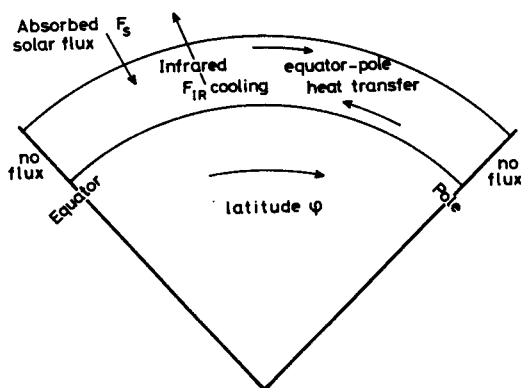


Figure 1.

$$\begin{aligned} \frac{1}{2} \frac{dS}{dt} &= \int_0^1 dx \frac{ds}{dt} = \int_0^1 dx \left[\frac{f(T,x)}{T} - \frac{1}{T} \frac{\partial}{\partial x} (1-x^2)^{\frac{1}{2}} J_x \right] \\ &= \int_0^1 dx \left[\frac{f(T,x)}{T} - \frac{\partial}{\partial x} \frac{1}{T} (1-x^2)^{\frac{1}{2}} J_x \right] + \\ &\quad + \int_0^1 dx J_x (1-x^2)^{\frac{1}{2}} \frac{\partial T^{-1}}{\partial x} \end{aligned}$$

Performing the x -integration in the first term and using the boundary condition

$$J_x = 0 \quad \text{at} \quad x = 0, x = \pm 1 \quad . \quad . \quad . \quad (5)$$

as well as Eq. (1), we arrive at the expression

$$\frac{1}{2} \frac{dS}{dt} = \int_0^1 dx \frac{f(T,x)}{T} + \int_0^1 dx J_x \nabla_x T^{-1} \quad . \quad . \quad . \quad (6)$$

Hence, we identify the entropy flux

$$\frac{d_e S}{dt} = 2 \int_0^1 dx \frac{f(T,x)}{T} \quad . \quad . \quad . \quad (7)$$

and the entropy production

$$\mathcal{P} \equiv \frac{d_i S}{dt} = 2 \int_0^1 dx J_x \nabla_x T^{-1} \quad . \quad . \quad . \quad (8)$$

Note that this separation implies that the function $f(T,x)$, that is the absorbed and outgoing radiations F_S and F_{IR} is entirely associated to non-dissipative processes. In this view therefore, the main role of the radiative flux is to create a *lateral temperature gradient* (that is, a non-equilibrium state), whose maintenance is associated with the entropy production $d_i S/dt$, Eq. (8).

We are now in position to identify the variables to be related by the constitutive equations, namely J_x and $\nabla_x T^{-1}$. Let us discuss a few representative situations (see also Glansdorff and Prigogine 1971):

(a) We first assume that the system operates in the linear range of irreversible processes. This will be reflected by the linear relation

$$J_x = L \nabla_x T^{-1} \quad . \quad . \quad . \quad (9a)$$

where the phenomenological coefficient L ($L \geq 0$) is constant. In this relation, $\nabla_x T^{-1}$ is to be viewed as a *generalized thermodynamic force* conjugate to the energy flux J_x .

(b) The phenomenological coefficient L is not constant. Rather, when Eq. (9a) is written in the Fickian or Fourier form:

$$J_x = -\lambda \nabla_x T \equiv -\frac{L}{T^2} \nabla_x T \quad . \quad . \quad . \quad (9b)$$

the transport coefficient $\lambda \equiv L/T^2$ is constant.

In both cases (a) and (b) we have a phenomenological law reminiscent of a diffusive mechanism of energy transport. Naturally, this does *not* mean that molecular diffusion and heat conduction are the dominant transport mechanism. Rather, these laws must be viewed as a phenomenological way of expressing turbulent transfer of latent heat and sensible heat in a medium of variable temperature. Several authors have discussed the properties of the phenomenological transfer coefficient (Stone 1973; Newell 1974; Lin 1977), and in particular its possible dependence on both local temperature *and* temperature gradients. This leads us to discuss a third type of situation:

(c) The system operates in the nonlinear range of irreversible processes, in the sense that the flux-force relationship is nonlinear. One way to express this is to take the coefficients L or λ in Eq. (9a) or (9b) to depend on both T and $\nabla_x T$:

$$J_x = L(T, |\nabla_x T|) \nabla_x T^{-1}$$

or

$$J_x = -\lambda(T, |\nabla_x T|) \nabla_x T \quad (9c)$$

Contrary to the case of certain physico-chemical systems (Glansdorff and Prigogine 1971) it does not seem possible to specify the particular form of nonlinearity involved in Eq. (9c). Hence, one can envisage a large number of different situations corresponding to different choices of constitutive relations. All these choices may well be compatible with present-day climatic data, if the coefficients involved in L or λ are adequately fitted. Already in the case of Eq. (9b), North (1975b) was able to reproduce a reasonable present-day meridional temperature distribution and flux by fitting a single parameter λ . It is therefore important to be able to remove somehow this high degeneracy in the choice of J_x . One way to achieve this is Paltridge's maximum entropy production Ansatz. We will have a detailed look at this possibility in section 5. In the following section we adopt a different approach. We intend to see how far one can go in the analysis of the climatic system as represented by the energy-balance equation, on the basis of the properties of thermodynamic state functions like entropy and entropy production.

3. VARIATIONAL PROPERTIES OF ENTROPY PRODUCTION, LYAPOUNOV FUNCTIONALS

One of the most important results of the thermodynamic theory of irreversible processes is Prigogine's theorem of minimum entropy production (Prigogine 1947). It asserts that in purely dissipative systems in which the fluxes and forces are related by linear laws of the form (9a), entropy production at the steady state settles to a minimum value compatible with the constraints acting on the system. It follows that these steady states are stable toward all possible disturbances, provided that thermodynamic equilibrium itself is stable. In other words, entropy production acts like a Lyapounov functional (see e.g. Cesari 1962) ensuring global stability.

Let us now see whether this result can be extended to our climatic model, Eq. (2). To this end we examine the behaviour of entropy production as a function of time. To remain as long as possible within the hypotheses of Prigogine's theorem we first consider the linear flux-force relation (9a), where the phenomenological coefficient L is constant.

The balance equation (2) takes the form

$$C \frac{\partial T}{\partial t} = f(T, x) - \frac{\partial}{\partial x} (1 - x^2) L \frac{\partial T^{-1}}{\partial x} \quad (10)$$

and \mathcal{P} , Eq. (8) becomes:

$$\mathcal{P} = 2 \int_0^1 dx (1 - x^2) L \left(\frac{\partial T^{-1}}{\partial x} \right)^2 \geq 0 \quad (11)$$

Taking the time derivative and remembering that L is constant, we obtain:

$$\frac{d\mathcal{P}}{dt} = -4L \int_0^1 dx (1 - x^2) \frac{\partial T^{-1}}{\partial x} \frac{\partial}{\partial x} \frac{1}{T^2} \frac{\partial T}{\partial t}$$

Substituting $\partial T / \partial t$ from Eq. (10), performing a partial integration and taking into account the boundary conditions (6) we obtain:

$$\begin{aligned} \frac{d\mathcal{P}}{dt} &= -\frac{4L^2}{C} \int_0^1 dx \frac{1}{T^2} \left[\frac{\partial}{\partial x} (1-x^2) \frac{\partial T^{-1}}{\partial x} \right]^2 + \frac{4L}{C} \int_0^1 dx \frac{1}{T^2} f(T, x) \left[\frac{\partial}{\partial x} (1-x^2) \frac{\partial T^{-1}}{\partial x} \right] \\ &\equiv \frac{d_i \mathcal{P}}{dt} + \frac{d_e \mathcal{P}}{dt} \end{aligned} \quad (12)$$

The first term of this relation, $d_i \mathcal{P}/dt$ describes the time variation of \mathcal{P} arising solely from internal dissipative processes. If only this term were present relation (12) would be equivalent to the theorem of minimum entropy production, $d\mathcal{P}/dt \leq 0$. In Eq. (12) however we have a second term $d_e \mathcal{P}/dt$ which depends on the radiative flux $f(T, x)$ and which has no definite sign. Its presence offers new possibilities like, for instance, the inversion of the sign of $d\mathcal{P}/dt$ under certain conditions.

The reason for this lack of universality, as opposed to the universality of Prigogine's theorem, is in the *boundary conditions*. In Prigogine's theorem, the latter (fixed or zero-flux conditions) rule out all spatial configurations that could lead to a value of \mathcal{P} smaller than the steady-state one. In the present case however the lateral (zero-flux boundary conditions), Eq. (5), are not sufficiently stringent to eliminate such possibilities. As a matter of fact, the only exchanges between system and surroundings are along the vertical direction which has been lumped, owing to the one-dimensional character of the model described by Fig. 1 and Eq. (2). As a result, the radiative flux $f(T, x)$ which is at the origin of the term $d_e \mathcal{P}/dt$ in Eq. (12), acts like a constraint of a new type as it is *incorporated* into the structure of the energy balance equation. Interestingly, this constraint does not act directly as the driving force for a dissipative flux. Rather (see also comments following Eq. (8)), it is associated with a process of energy storage. In this respect the behaviour of $d\mathcal{P}/dt$ as deduced from Eq. (12) is somewhat reminiscent of that of electrical circuits comprising resistors and inductances. As pointed out by Landauer (1975), in such systems involving inertial elements in addition to dissipative ones, the entropy production may indeed increase in time, even if the circuit characteristics are completely linear.

So far we discussed strictly linear phenomenological laws, Eq. (9a). The results can, however, be extended straightforwardly to case (9b) of a Fourier type of law. The balance equation (2) and the entropy production \mathcal{P} , Eq. (8) take the form:

$$C \frac{\partial T}{\partial t} = f(T, x) + \lambda \frac{\partial}{\partial x} (1-x^2) \frac{\partial T}{\partial x} \quad (13)$$

$$\mathcal{P} = 2 \int_0^1 dx (1-x^2) \frac{1}{T^2} \lambda \left(\frac{\partial T}{\partial x} \right)^2 \geq 0 \quad (14)$$

Note the presence of the weighting factor $1/T^2$ in the integrand. The computation of $d\mathcal{P}/dt$ follows the same lines as before. The final result is:

$$\begin{aligned} \frac{d\mathcal{P}}{dt} &= -\frac{4\lambda^2}{C} \int_0^1 dx \frac{1}{T^2} \left[\frac{\partial}{\partial x} (1-x^2) \frac{\partial T}{\partial x} \right]^2 - \frac{4\lambda}{C} \int_0^1 dx \frac{f(T, x)}{T} \frac{\partial}{\partial x} (1-x^2) \frac{\partial \ln T}{\partial x} \\ &\quad + \frac{4\lambda^2}{C} \int_0^1 dx (1-x^2) \frac{1}{T^3} \left(\frac{\partial T}{\partial x} \right)^2 \frac{\partial}{\partial x} (1-x^2) \frac{\partial T}{\partial x} \end{aligned} \quad (15)$$

Without the last two terms, $d\mathcal{P}/dt$ would again be negative definite. As in the previous subsection, the presence of the radiative-flux term $f(T, x)$ offers some new possibilities. So does the last term, which, however, is of a higher order in $(\partial T/\partial x)$ than the other two terms. For this reason it is expected to give a negligible contribution. This will be verified in the explicit calculations reported in section 4.

Let us briefly summarize the situation. We have shown that, because of energy storage processes, the time variation of entropy production is not necessarily negative, as in curve (a) of Fig. 2 characterizing usual physico-chemical systems obeying Prigogine's theorem. Different possibilities can be envisaged, like for instance curve (b) of Fig. 2. We discuss

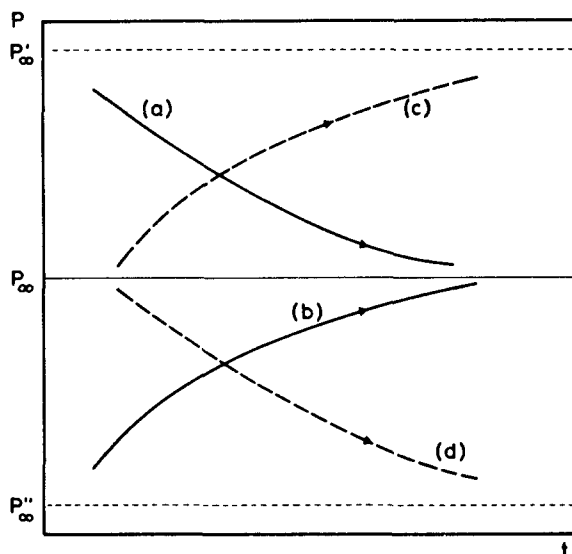


Figure 2. Possible time evolutions of entropy production compatible with Eqs. (12) and (15). Curve (a): same behaviour as in Prigogine's minimum entropy production theorem. (b): Entropy production is increasing, but reference state remains stable. (c), (d): Reference state is unstable, and system evolves to new steady states.

their climatic significance in section 4. For the purposes of our present qualitative discussion however, both situations (a) and (b) are indicative of the *stability* of the stationary climatic regime. Of more interest are therefore situations corresponding to curves (c) or (d), which are perfectly compatible with Eq. (12) or (15) and which indicate, nevertheless, that the system may evolve away from a certain reference state and tend to a new climatic configuration.

We would now like to obtain a criterion which would show when such instabilities are possible. In irreversible thermodynamics, it turns out that one cannot derive such a criterion using the variational properties of entropy production. Following Glansdorff and Prigogine (1971) we introduce therefore a new functional related to the *excess entropy* around the reference state. Let us first outline the formulation in the general case where no particular constitutive relation is postulated.

Let \hat{T} be a reference temperature, for instance that corresponding to the present-day climate. We consider a slight perturbation, δT , from this state, and set

$$T = \hat{T} + \delta T, \quad \left| \frac{\delta T}{\hat{T}} \right| \ll 1 \quad (16)$$

Using Eq. (4) one can easily construct the excess entropy function

$$\frac{1}{2} \delta^2 s = -\delta T^{-1} \delta e = -\frac{1}{2} \frac{C}{\hat{T}^2} (\delta T)^2 \quad (17)$$

Note that

$$\delta^2 S = \int_{-1}^1 dx \delta^2 s \leq 0 \quad (18)$$

Because of this, we regard $\delta^2 S$ as a Lyapounov functional (Cesari 1962) and we evaluate its time derivative along the motion described by the balance equation (2). Keeping in mind that \hat{T} is time-independent, we obtain:

$$\frac{d}{dt} \frac{1}{2} (\delta^2 S) \equiv -2C \int_0^1 dx \frac{1}{\hat{T}^2} \delta T \frac{\partial \delta T}{\partial t} \quad (19a)$$

with (cf. Eq. (2))

$$C \frac{\partial \delta T}{\partial t} = \left(\frac{\partial f}{\partial T} \right)_{\hat{T}} \delta T - \frac{\partial}{\partial x} (1-x^2)^{\frac{1}{2}} \delta J_x \quad (19b)$$

More explicitly:

$$\frac{d}{dt} \frac{1}{2} (\delta^2 S) = -2 \int_0^1 dx \frac{1}{\hat{T}^2} \left(\frac{\partial f}{\partial T} \right)_{\hat{T}} (\delta T)^2 + 2 \int_0^1 dx \delta J_x \cdot \delta \nabla_x T^{-1} \quad (20)$$

The first term of this expression reflects the effect of radiative flux. The second term has the same structure as the entropy production, Eq. (9), except that we now deal with the *excess flux* δJ_x , and the *excess force* $\delta \nabla_x T^{-1}$. We shall refer to this combination as the *excess entropy production* (Glansdorff and Prigogine 1971).

By Lyapounov's stability theorem (Cesari 1962), we conclude that \hat{T} will be asymptotically stable as long as $d/dt \frac{1}{2} (\delta^2 S)$ has a sign opposite to $\delta^2 S$, or

$$\int_0^1 dx \left[-\frac{1}{\hat{T}^2} \left(\frac{\partial f}{\partial T} \right)_{\hat{T}} (\delta T)^2 + \delta J_x \delta \nabla_x T^{-1} \right] \geq 0 \quad (21)$$

along all solutions of Eq. (19b).

To give a more explicit form to this expression, we must specify how J_x is related to $\nabla_x T^{-1}$. Choosing as an example, the non-linear relation (Stone 1973):

$$J_x = - \left[\lambda_0(x) + \lambda_1 \left(\frac{\partial T}{\partial x}, x \right) \right] \nabla_x T \quad (22)$$

we obtain the explicit form of the stability condition:

$$\int_0^1 dx \left[\frac{\lambda_0 + \lambda_1}{\hat{T}^2} (1-x^2) \left(\frac{\partial \delta T}{\partial x} \right)^2 - \frac{1}{\hat{T}^2} \left(\frac{\partial f}{\partial T} \right)_{\hat{T}} (\delta T)^2 + (1-x^2) \frac{\lambda'_1}{\hat{T}^2} (\nabla_x \hat{T}) \left(\frac{\partial \delta T}{\partial x} \right)^2 \right] \geq 0 \quad (23)$$

where λ'_1 denotes the derivative of λ_1 with respect to its argument $\partial T / \partial x$.

In this inequality none of the terms, except the first, have a definite sign. Hence, under certain conditions their sign can become negative and their absolute value can exceed that of the first term. In this case one would have $d/dt (\delta^2 S) < 0$, and since $(\delta^2 S)$ remains always negative, by Lyapounov's theorem \hat{T} would be unstable. We may refer to this situation as a *climatic catastrophe*. We see that it is reflected by a clearcut change in the thermodynamic properties of the system. In a sense, *climatic change becomes a problem of thermodynamic stability*. Note that the terms threatening stability in Eq. (23) are related either to the storage term f , or to the non-linearity in the $J_x - \nabla_x T$ relationship. This is in agreement with the fact that the source of non-linearity making bifurcations possible in the energy-balance equation (cf. Eq. (2) or (19b)) is, precisely, in these two terms.

Finally, it is easy to verify that the left-hand side of relation (21) or (23) is closely related to the second variation of the functional recently proposed by North *et al.* (1979) in their variational formulation of Budyko-Sellers climate models.

4. ILLUSTRATIONS

In this section we illustrate the structure of the general expressions derived so far on simple examples.

(a) *An ice-free earth*

We first consider Eq. (2) in the case of an ice-free earth. It is believed (Budyko 1977) that this was indeed the case in the mesozoic and early cenozoic eras up to the beginning of the quaternary glaciations.

We adopt relation (9b) for the energy flux, and the following expressions for the radiative flux terms $f(T, x)$:

$$\begin{aligned} J_x &= -\lambda \nabla_x T \\ f(T, x) &= Q(1-\alpha)S(x) - (A + BT) \end{aligned} \quad (24)$$

where the albedo α is taken to be constant (a rather legitimate approximation for an ice-free earth). Q is the solar constant, A and B are the infrared cooling coefficients, and the insolation $S(x)$ is approximated by (Coakley 1979):

$$S(x) = 1 + S_2 P_2 \quad (S_2 < 0) \quad (25)$$

P_2 being the second Legendre polynomial. Equation (2) takes thus the explicit form

$$C \frac{\partial T}{\partial t} = Q(1-\alpha)(1 + S_2 P_2) - (A + BT) + \lambda \frac{\partial}{\partial x} (1-x^2) \frac{\partial T}{\partial x} \quad (26)$$

The solution is easily found to be

$$T(x) = T_0 + T_2 P_2(x) \quad (27a)$$

where the planetary temperature T_0 obeys

$$C \frac{dT_0}{dt} = Q(1-\alpha) - (A + BT_0) \quad (27b)$$

and the amplitude T_2 to

$$C \frac{dT_2}{dt} = Q(1-\alpha)S_2 - (B + 6\lambda)T_2 \quad (27c)$$

Both T_0 , T_2 are to be expressed in degrees centigrade. The entropy production, Eq. (14), becomes:

$$\mathcal{P} = 2\lambda T_2^2 \int_0^1 dx (1-x^2) \frac{1}{[(273 + T_0) + T_2 P_2(x)]^2} \left(\frac{dP_2}{dx} \right)^2$$

One can easily see (Nicolis 1979) using the appropriate numerical values for Q , α , S_2 , A , B , λ that $273 + T_0 \gg T_2$. Thus, the above expression can be approximated by

$$\mathcal{P} \simeq \frac{12\lambda}{5} \frac{T_2^2(t)}{(273 + T_0(t))^2} \quad (28)$$

We proceed to the evaluation of $d\mathcal{P}/dt$. To simplify the picture as much as possible we consider only those evolutions that keep the planetary temperature T_0 invariant. This is legitimate, since the equations for T_0 and T_2 are uncoupled. The time dependence of T_2 is easily found to be

$$T_2(t) = T_{2\infty} + e^{-\mu t}(T_{20} - T_{2\infty}) \quad (29)$$

with

$$T_{2\infty} = \frac{Q(1-\alpha)S_2}{B+6\lambda} < 0$$

$$T_{20} = \text{initial condition}$$

$$\mu = \frac{1}{C}(B+6\lambda) > 0$$

It follows that

$$\frac{d\mathcal{P}}{dt} = -\mu \frac{24}{5(273+T_0)^2} T_2(T_{20} - T_{2\infty}) \quad (30)$$

Figure 3 depicts the evolution of T_2 and \mathcal{P} . We see that \mathcal{P} can decrease or increase in time, according as the initial value T_{20} is smaller or larger than the steady-state level $T_{2\infty}$. As $T_{2\infty}$ is negative, in actual fact this implies that for fixed values of the coefficients, \mathcal{P} decreases if the initial thermal gradient, measured by $|T_2|$, is large and it increases if the initial $|T_2|$ is small. This is quite reasonable, since the steeper the gradient, the larger the rate of dissipation will have to be.

We see in an explicit way the possibility of having maximum entropy production at the steady state for all families of initial conditions (or equivalently, for all 'virtual displacements') with $T_{20} > T_{2\infty}$. The same system however can give rise to a decrease in \mathcal{P} , for different types of initial conditions. Note however that all these new possibilities do not compromise the stability of the steady-state regime, $T_{2\infty}$. As a matter of fact, $(\mathcal{P} - \mathcal{P}_\infty)$ turns out to be a Lyapounov function ensuring stability both in the case of increasing and of decreasing \mathcal{P} 's (see also Fig. 2, curves (a) and (b)).

In the context of climatic history of the last 250 myr or so, one might question the assumption adopted implicitly so far that the coefficients A , B , λ do not evolve in time. Recently one of the authors (Nicolis 1979) developed plausible scenarios of evolution of these coefficients and analysed the consequences of such variations on the values of T_0 and T_2 , using the constraint (suggested by paleoclimatic data) that the equatorial temperature, T_{eq} remained practically invariant ($T_{eq} \approx 25^\circ\text{C}$). We have verified that this simultaneous evolution of both T and the parameters does not affect the qualitative trend shown in Fig. 3. Namely, a more pronounced pole-equator thermal gradient leads to a relaxation accompanied by a decreasing entropy production whereas the opposite is true if the pole-equator initial gradient is smaller than the steady state value.

(b) The influence of ice caps

We now extend the model of the previous subsection to account for the existence of ice caps characterizing the present climate. Equation (26) keeps then the same general form, except that the albedo α is replaced by an expression taking into account the existence of an ice edge. Specifically (North 1975a), denoting by x_s the position of the latter and assuming symmetric hemispheres:

$$1 - \alpha \equiv a(x, x_s) \begin{cases} = b_0, & x > x_s, \\ = a_0 + a_2 P_2(x), & x < x_s, \end{cases} \quad (31)$$

where b_0 is the absorption coefficient over ice or snow 50% covered with clouds, and a_0 , a_2

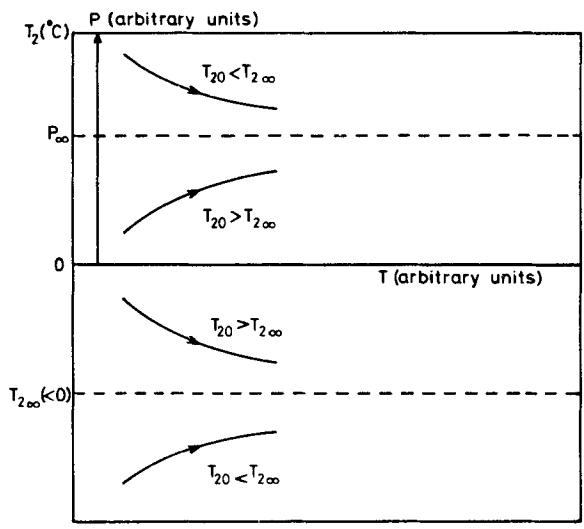


Figure 3. Time evolution of the amplitude T_2 of the second Legendre mode (lower part) and of entropy production (upper part) associated with model Eqs. (26).

are the absorption coefficients over ice-free areas obtained after analysing the albedo distribution by Legendre series. As usual, it is assumed that at x_s the temperature is -10°C .

An appropriate solution of Eqs. (26) and (31) can be found by expanding T in series of Legendre polynomials. Truncation to the second mode gives (North 1975a):

$$\begin{aligned} C\frac{dT_0}{dt} &= QH_0(x_s) - (A + BT_0) \\ C\frac{dT_2}{dt} &= QH_2(x_s) - (B + 6\lambda)T_2 \\ T_0 + T_2P_2(x_s) &= -10 \end{aligned} \tag{32}$$

where T_0 and T_2 are again expressed in degrees centigrade, and

$$\begin{aligned} H_m(x_s) &= (2n + 1) \int_0^1 dx S(x) a(x, x_s) P_m(x) \\ m &= 0, 2 \end{aligned} \tag{33}$$

Note that, contrary to the preceding subsection, these equations are coupled through the variable x_s .

The entropy production, Eq. (14), takes the same form as Eq. (28):

$$\mathcal{P}(t) \simeq \frac{12\lambda}{5} \frac{T_2^2(t)}{(273 + T_0(t))^2} \tag{34}$$

provided we again adopt the assumption $273 + T_0 \gg |T_2|$. As it turns out, the solution of Eq. (32) completely justifies this assumption.

In order to analyse the time dependence of $\mathcal{P}(t)$ we solved numerically the initial value problem for Eqs. (32) using the Hammin method. We first explored (Fig. 4) the vicinity of the

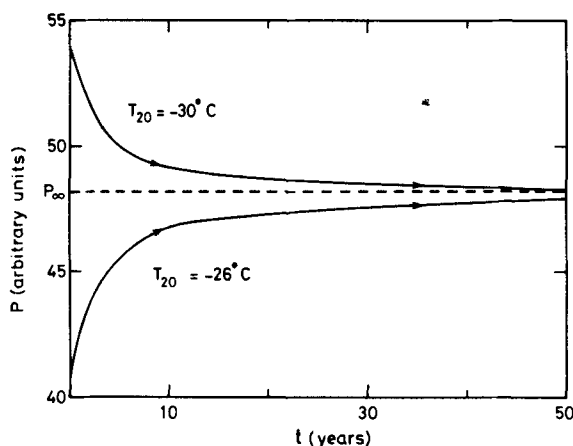


Figure 4. Time evolution of entropy production associated with model Eq. (32), using two different initial conditions for T_2 and $T_0 = 14.9^\circ\text{C}$. In both cases the present climatic regime is recovered asymptotically. Numerical values used: $Q = \frac{1}{4}$ (1360) W m^{-2} , $b_0 = 0.38$, $a_0 = 0.697$, $a_2 = -0.0779$, $C = 3.138 \times 10^8 \text{ J m}^{-2}$, $S_2 = -0.477$, $A = 214.2 \text{ W m}^{-2}$, $B = 1.575 \text{ W m}^{-2}$, $\lambda = 0.591 \text{ W m}^{-2}$.

steady-state solution of these equations corresponding to the present-day climate. For the numerical values given in the caption of Fig. 4, this state corresponds to $T_0 = 14.9^\circ\text{C}$, $T_2 = -28.2^\circ\text{C}$, $x_s = 0.96$ and is asymptotically stable. Figure 4 depicts the evolution of $\mathcal{P}(t)$ induced by a perturbed pole-equator gradient, keeping the planetary temperature invariant. We see that if $T_{20} = -30^\circ\text{C}$ \mathcal{P} decreases in time, whereas for $T_{20} = -26^\circ\text{C}$ \mathcal{P} increases until the present-day climate is recovered. We arrive therefore at the same qualitative behaviour as in the preceding subsection. We thus feel that there is no support for the claim (Golitsyn and Mokhov 1978) that the stability properties of the climate should be linked to the extremal properties of entropy production.

In Fig. 5 we report the evolution of entropy production using the same parameter

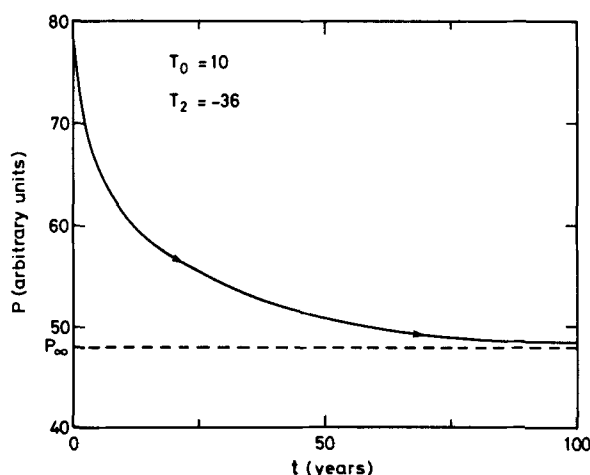


Figure 5. Time evolution of entropy production associated with model Eq. (32) using an initial condition ($x_s = 0.84$, $T_0 = 10^\circ\text{C}$, $T_2 = -36^\circ\text{C}$) simulating the last major glaciation. Numerical values used are as in Fig. 2.

values as in Fig. 4, but starting from initial conditions simulating the last major glaciation (18 000 years B.P.). We know that in this case the ice caps went as far down as 57° in the Northern Hemisphere, and that the planetary temperature was less by about 5°C . The ice boundary condition (third relation (32)) allows us then to compute T_2 . Taking $x_s = \sin 57^\circ = 0.84$, $T_0 = 10^\circ\text{C}$ we find $T_2 = -36^\circ\text{C}$. As seen from Fig. 5, the entropy production then decreases monotonically until the present-day climate is reached.

5. COMPARISON WITH PALTRIDGE'S IDEAS

The main focus of this paper was the *time-dependent* properties of entropy production in the vicinity of a steady-state climatic regime. As is usually done in the analysis of irreversible phenomena, both the equations of evolution of the state variables and the entropy function were evaluated by introducing suitable *constitutive relations* linking fluxes and forces. Thanks to these relations, the energy-balance equation became closed, and allowed for an explicit evaluation of the temperature profile across the system.

An altogether different approach was adopted in the work by Paltridge (1975, 1978). His main idea is to use an *unconstrained* energy balance equation, whereby the energy flux is not linked to the temperature gradient. At the steady state and in the framework of the one-dimensional model adopted in the present work, this yields:

$$\nabla^+ J_x = \frac{\partial}{\partial x}(1-x^2)^{\frac{1}{2}} J_x = Q(1-a(x))S(x) - (A + BT) \quad (35)$$

where $-\nabla^+$ is the adjoint gradient operator. As noted in section 2, the inverse of the radius of the earth has been absorbed into J_x . From this relation one may express T as a function of J_x :

$$T = \frac{Q(1-a(x))S(x) - A - \nabla^+ J_x}{B} \quad (36)$$

In this way the entropy production, Eq. (8), can be written entirely in terms of the flux J_x :

$$\mathcal{P} = -2B \int_0^1 dx \frac{\nabla^+ J_x}{273B + Q(1-a(x))S(x) - A - \nabla^+ J_x} \quad (37)$$

Following Paltridge, we may now seek for the function J_x^0 extremizing \mathcal{P} . We obtain in this way the following variational equation, $\delta\mathcal{P}/\delta J_x = 0$:

$$(1-x^2)^{\frac{1}{2}} J_x^0(x) = \int_0^x g(x) dx - \frac{1}{K} \int_0^x g^{\frac{1}{2}}(x) dx \quad (38)$$

where

$$g(x) = Q(1-a(x))S(x) - A + 273B \quad (39a)$$

and the constant K is adjusted to give zero flux at the poles:

$$K = \frac{\int_0^1 g^{\frac{1}{2}}(x) dx}{\int_0^1 g(x) dx} \quad (39b)$$

Figure 6 depicts the energy flux obtained by applying this procedure and by using the parameter values adopted earlier in the present work. The results are reasonable, both as far as the position of the maximum and the behaviour near the poles is concerned. On the

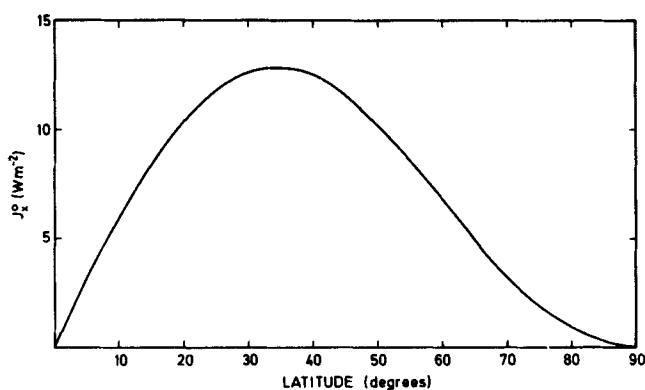


Figure 6. Latitude dependence of the energy flux divided by the earth's radius, obtained by the entropy production extremization (Eq. (38)). The parameter values used are the same as in Fig. 4.

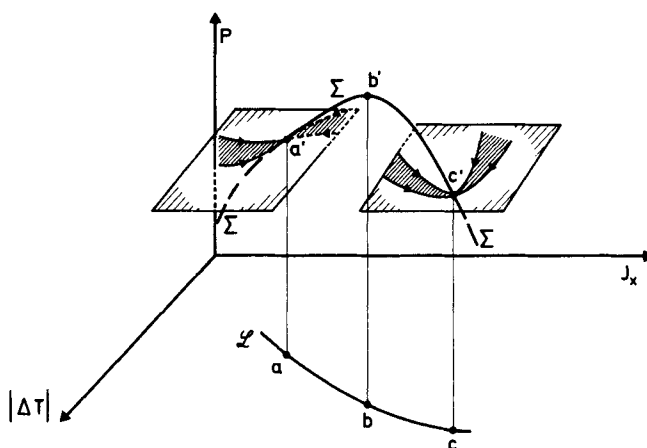


Figure 7. Entropy production surface, Σ as a function of the energy flux, J_x and of an average temperature gradient, $|\Delta T|$. \mathcal{L} : line of unconstrained steady states. An example of such states is (b), the state of maximum dissipation. (a), (c): steady states obtained after using a constitutive relation. In particular state (a) is taken to be the present-day climate as given by North's model discussed in section 4. Possible time-dependent behaviours of entropy production are described in the vicinity of points (a'), (c'), of the surface Σ . In particular, as shown in section 4 (b), (a') is a saddle point: for high initial $|\Delta T|$ P tends to decrease, whereas the opposite is true for small initial $|\Delta T|$'s. This behaviour is not to be confused with the fact that, among all possible *steady states*, (b) is the one with maximum dissipation. In other words, Paltridge's variational principle pertains to steady-state behaviour and not to the evolution in the vicinity of a steady state.

other hand, one can show that the corresponding temperature profile gives excessively high values at the equator and low values at the poles, as already pointed out by Golitsyn and Mokhov (1978).

Independently of these technical aspects, however, the main point to be retained is that entropy extremization does not require using a constitutive relation expressing J_x in terms of $\partial T/\partial x$ or fitting such coefficients as λ in order to obtain the steady state format of present climate. Thus, among all possible steady states that may be realized by the earth-atmosphere system under a given energy input, there is one (cf. Eq. (38) and Fig. 6) which extremizes the entropy production. Other steady states, such as those evaluated in section 4, are possible. They have, however, a smaller dissipation rate than the state J_x^0 , Eq. (38). The situation is described in Fig. 7.

6. CONCLUDING REMARKS

Our principal goal in this paper was to cast some basic features related to climate and its evolution into the properties of entropy and entropy production. We have found that the behaviour of these quantities is far from being simple and universal, just like climate itself is far from showing simple and universal trends. Rather, it appears that the direction of change of entropy production is conditioned by the initial strength of the equator–pole temperature gradient as compared to that of the final steady state. Now, a more pronounced thermal gradient is characteristic of glacial periods (Newell 1974). We may therefore summarize the results of section 4 by stating that *the evolution to a glaciation is accompanied by an enhanced rate of dissipation*, as measured by the entropy production. An additional illustration of this conclusion is provided by a direct comparison of Eqs. (28) and (34). Using paleoclimatic data from the mesozoic era (Nicolis 1979) we deduce that for an equatorial temperature of 25° and a polar one of 15°,

$$\mathcal{P}_{\text{Past}} \simeq \frac{12\lambda_{\text{Past}}}{5} \frac{(-7)^2}{(273+22)^2}$$

whereas for the present interglacial climate:

$$\mathcal{P}_{\text{Present}} \simeq \frac{12\lambda_{\text{Present}}}{5} \frac{(-28.2)^2}{(273+14.9)^2}$$

We see that the change in T_2 induces about a 16-fold increase of \mathcal{P} . Certainly, λ cannot have varied in the opposite direction by a comparable amount. Thus, the present interglacial climate appears to be more dissipative than the climate associated with an ice-free earth.

Although the results of sections 2 and 3 are quite general, the illustrations developed in section 4b are limited by several simplifications. Perhaps the most serious one – which is in fact a limitation of all diffusive models used so far in the literature – is the assumption that x_i adjusts instantaneously to the value of $T(x)$. This introduces an unrealistically fast time scale into the problem. Obviously, a natural boundary condition on the ice edge must be introduced in order to allow for the ice melting or advance in a self-consistent way (see also Pollard 1978). A second limitation is the two-mode truncation adopted. This prevents analysing the behaviour of dissipation under the effect of localized disturbances from some reference state. Such local disturbances are certainly more realistic. The time scale of evolution is also likely to be lengthened under these conditions.

The discussion of Fig. 7 in connection with the thermodynamic properties of steady states illustrates the considerable degeneracy associated with the modelling of the meridional flux. A basic problem which remains open at this time is therefore to come up with criteria determining the *selection mechanisms* of a particular steady state climatic regime. Paltridge (1979) suggests that the role of fluctuations is likely to be instrumental. He believes that fluctuations are capable of introducing a drift in state space, eventually driving the system to the state of maximum dissipation. A general answer to this major question is however still lacking. It may be expected that the systematic use of thermodynamics could prove useful in tackling this problem.

ACKNOWLEDGMENTS

We are grateful to G. W. Paltridge for enlightening discussions and to M. Ghil for pointing out to us the work by Golitsyn and Mokhov.

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