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Derivation of Zero-Dimensional from One-Dimensional Climate Models

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A one-dimensional energy-balance equation involving diffusive energy transport and taking into account the ice-albedo feedback is considered. A systematic elimination of the spatial degrees of freedom is performed. This gives rise to a zero-dimensional climate model displaying the explicit dependence of planetary albedo on planetary temperature and on some model parameters. In the general case, the zero-dimensional model involves memory effects as well as two characteristic relaxation rates.

1. INTRODUCTION

One of the main tasks in the mathematical modeling of climate in terms of simple energy balance equations is to incorporate the most important feedback mechanisms present in the earth-atmosphere system. Previous studies (e.g. Budyko, 1969; Sellers, 1969) have shown the importance of a positive feedback due to surface albedo, in the framework of a one-dimensional (1-D) latitudinal model. In view of the role of this mechanism in determining climate sensitivity a number of authors developed rather sophisticated albedo representations (Lian and Cess, 1977; Oerlemans and Van Den Dool, 1978) in terms of such factors as temperature, cloudiness and solar zenith angle. A common element in most of these representations is the occurrence of some discontinuity, related to the existence of an ice edge.

On the other hand, it has been shown recently that many of the qualitative effects predicted by one-dimensional models, such as the occurrence of transitions between different climatic regimes, are also reproduced by globally averaged, zero-dimensional (0-D) models (Fraedrich, 1978; Crafoord and Källén, 1978). In the latter, some very simple continuous (linear or piecewise linear) dependences of albedo on planetary temperature have been postulated. The question therefore arises whether such relationships can be justified from the experimental data, which all refer to zonally averaged latitude-dependent values. This is not merely an academic problem, since certain linear albedo-temperature feedbacks have been shown to be unphysical when used in the framework of a 1-D model (Schneider and Gal-Chen, 1973).

The purpose of this note is to express, in a self-consistent way, the dependence of the planetary albedo on planetary temperature, starting from a 1-D energy balance model. The main point we make is that the 0-D model can be viewed as an exact consequence of a 1-D model when the spatial degrees of freedom are systematically eliminated. This will enable us to deduce an explicit form of the albedo-temperature feedback which is: (i) continuous in a certain temperature range and (ii) dependent explicitly on such parameters as the eddy diffusivities and the infrared cooling coefficients.

In Section 2 we describe the 1-D model used. In Section 3 we perform an elimination of the space variables in the 1-D model, based on the wide separation of the time scales occurring in the problem. The procedure may be summarized as follows. Let X and Y be two groups of variables whose evolution is governed by a set of coupled first-order, autonomous differential equations. We assume that in the equation for Y there is a large parameter λ describing a fast relaxation process. Under certain conditions one may divide both members of the equation by λ and switch to suitable dimensionless variables. One then obtains:

$$dX/dt = f(X, Y, \varepsilon), \tag{Fa}$$

$$\varepsilon dY/dt = g(X, Y, \varepsilon) \tag{Fb}$$

where $\varepsilon = \lambda^{-1} \ll 1$ and f, g are smooth functions of ε in the vicinity of $\varepsilon = 0$. We are interested in the behavior of the above system as $\varepsilon \to 0$. According to a theorem due to Tikhonov (Wasow, 1965, sec. 39), under certain conditions, as $\varepsilon \to 0$, the solutions of the full system (F) tend to solutions of the reduced system:

$$dX/dt = f(X, Y, 0), \tag{Ra}$$

$$g(X, Y, 0) = 0. \tag{Rb}$$

From relation (Rb) we may obtain Y as a function of X:

$$Y = W(X),$$

in which case the equation for X takes the closed form:

$$dX/dt = f(X, W(X), 0).$$

In Section 3 the above procedure is applied to identify an effective planetary albedo; the latter is studied both numerically and analytically for small deviations from the present-day regime. Section 4 is devoted to the time-dependent problem. We show that the time dependences of the spatial degrees of freedom are reflected by memory effects at the level of the 0-D reduced balance equation for the planetary temperature. A brief discussion of the results is given in Section 5.

2. THE MODEL

The 1-D model of North (1975) will be used. The time-dependent energy balance equation in this model is of the form

$$C\frac{\partial T}{\partial t} = QS(x)a(x, x_s) - I(x) + \frac{\partial}{\partial x}\left[(1-x^2)D\frac{\partial}{\partial x}T(x)\right], \tag{1}$$

where

Q is the solar constant divided by 4, taken equal to $340\,\mathrm{W\,m^{-2}}$,
x is the sine of latitude and x_s corresponds to the ice boundary,
$I(x)$ is the outgoing infrared radiation,
$a(x, x_s)$ is the absorption function [$=1-$albedo],
D is the eddy diffusion coefficient,
C is the thermal inertia coefficient, taken equal to $3.138\times10^8\ \mathrm{J\,m^{-2}}$,
T is the temperature,
t is time,
and $S(x)$ is the normalized mean annual meridional distribution of solar radiation determined from astronomical calculations.

The following approximation will be used (Coakley, 1979):

$$S(x) = 1.0 - 0.477\,P_2(x), \tag{2}$$

where P_2 is the 2nd Legendre polynomial. The parameterization used for $I(x)$ is the one developed by Cess (1976) for the Northern hemisphere. Assuming a constant 50% cloud cover:

$$I(x) = A + BT(x),$$

with (3)

$$A = 211.5\,\mathrm{W\,m^{-2}} \text{ and } B = 1.575\,\mathrm{W\,m^{-2}}.$$

The absorption function used, taking symmetric hemispheres, is

$$a(x,x_s)\begin{cases} =b_0, & x>x_s, \\ =a_0+a_2P_2(x), & x<x_s, \end{cases} \tag{4}$$

where $b_0=0.38$ is the absorption coefficient over ice or snow when 50% covered with clouds (Budyko, 1969), $a_0=0.697$ and $a_2=-0.0779$ are the absorption coefficients over ice free areas obtained after analyzing the albedo distribution by Fourier–Legendre series.

The ice boundary is determined using the prescription of Budyko and Sellers:

$T>-10$°C, no ice present,

$T<-10$°C, ice present.

Finally Eq. (1) is subject to the boundary conditions

a) no heat transport at the pole, nor across the equator,

b) the temperature and its gradient must be continuous at the ice edge.

To solve the balance equation Eq. (1), we expand $T(x)$ in a series of Legendre polynomials:

$$T=\sum_{n=0}^{\infty} T_nP_n(x), \tag{5}$$

where T_0 is the planetary temperature. We then deduce from Eq. (1):

$$C\partial T_0/\partial t=QH_0(x_s)-(A+BT_0), \tag{6a}$$

$$C\partial T_n/\partial t=QH_n(x_s)-[n(n+1)D+B]T_n, \quad n\geqq 2, \tag{6b}$$

$$\sum_n T_nP_n(x_s)=-10, \tag{6c}$$

with

$$H_n(x_s)=(2n+1)\int_0^1 S(x)a(x,x_s)P_n(x)dx \quad (n=0,2,\ldots). \tag{6d}$$

Eqs. (6) are coupled solely through the value of x_s. On the other hand the higher the Legendre mode, the faster its relaxation to the steady-state value will be, owing to the factor $n(n+1)$ multiplying T_n. We are, therefore, within the domain of validity of the Tikhonov theorem referred to in the introduction. Hence, the first non-trivial approximation to Eqs. (6) taking spatial effects into account, amounts to setting:

$$\partial T_n/\partial t \simeq 0, \quad n \geqslant 4.$$

It follows that

$$T_n \simeq QH_n(x_s)/[n(n+1)D+B], \qquad n \geqslant 4.$$

Thus T_n is a decreasing function of n both because the denominator is quadratic in n and because H_n decreases with n. Hence, we may expect that the ice-edge position, Eq. (6c), will not be substantially affected by these higher modes. From now on therefore we illustrate the main idea on a two-mode truncation involving T_0 and T_2 alone, although the results could in principle be extended to higher T_n's.

At the level of the two-mode approximation, we will require that the model reproduces as closely as possible the present-day steady state values of T_0 and T_2 ($T_0 = 14.9°C$, $T_2 \simeq -28°C$, cf. Coakley, 1979). From this it follows that the ice edge is at $x_s \simeq 0.96$. To insure that, we adjusted the eddy diffusion coefficient D and the infrared cooling coefficient A. The values which fitted the model best are $A = 214.2\,\mathrm{W\,m^{-2}}$ and $D = 0.591\,\mathrm{Wm^{-2}}$.

3. QUASI-STATIC ELIMINATION OF THE SPATIAL DEGREES OF FREEDOM

We want to see now whether the 1-D model summarized in Eqs. (6) may induce a closed equation for the planetary temperature; such an equation would constitute a 0-D model. Clearly this requires the elimination of all degrees of freedom but T_0. In the two-mode approximation the variables to be eliminated are therefore T_2 and x_s.

A necessary assumption to be made at this stage concerns again time scales. Comparing Eq. (6a) with Eq. (6b) for $n = 2$, we estimate the relaxation time $C/B \simeq 6.4$ yr for T_0 whereas $C/6D + B \simeq 2$ yr for T_2. We may regard therefore the time of evolution of T_0 as being the rate determining step. Alternatively, we may set Eq. (6b) for T_2 at a quasi-steady state as suggested by the Tikhonov theorem:

$$\partial T_2/\partial t \simeq 0. \tag{7}$$

By combining expressions (6b) and (6c) we get an algebraic equation of ninth degree in x_s:

$$QH_2(x_s) + (6D+B)(10+T_0)/P_2(x_s) = 0. \tag{8}$$

As x_s becomes now a function of T_0, Eq. (6a) takes a closed form. Setting

$$H_0(x_s)=H_0(x_s(T_0))=1-\alpha_p(T_0), \tag{9}$$

we can thus identify an explicit dependence of the "effective" planetary albedo α_p in terms of the planetary temperature T_0; the derivative with respect to T_0 of this dependence is given by

$$\frac{d\alpha_p}{dT_0}=\frac{\partial\alpha_p}{\partial x_s}\frac{dx_s}{dT_0}. \tag{10}$$

It should be realized that this refers to fixed cloud characteristics. Hence it is only a part of the total variation of albedo in terms of the planetary temperature. In actual fact albedo depends on T_0 also through other factors such as cloud amount, water vapor and so forth. Schematically, denoting these latter factors by Y, we have

$$\alpha_p=\alpha_p(x_s(T_0),\ Y(T_0)) \tag{11}$$

and thus

$$\frac{d\alpha_p}{dT_0}=\frac{\partial\alpha_p}{\partial x_s}\frac{dx_s}{dT_0}+\frac{\partial\alpha_p}{\partial Y}\frac{dY}{dT_0}. \tag{12}$$

In what follows we shall focus on the first term of the right-hand side only. Our purpose is to display the dependence of α_p on T_0 in a completely self-consistent way without using data other than the basic premises of the 1-D model. So far, it seems impossible to carry out a similar program for the second term.

We first study Eq. (8) and Eq. (9) numerically for different values of T_0. The results are given in Figure 1 for $-10<T_0<16$. We see that the dependence of the effective planetary albedo is nearly linear in that temperature range. Moreover, the derivative of the albedo in the vicinity of the present day regime is:

$$d\alpha_p/dT=-0.0032. \tag{13}$$

This is less than the values deduced by Cess (1976), using satellite data of mid latitudes. However, as pointed out by Cess, such values are probably overestimations. In addition, explicit consideration of atmospheric feedback mechanisms, which appear only implicitly in our calculation, is likely to further influence results.

As mentioned earlier, certain parameters have been adjusted to fit the

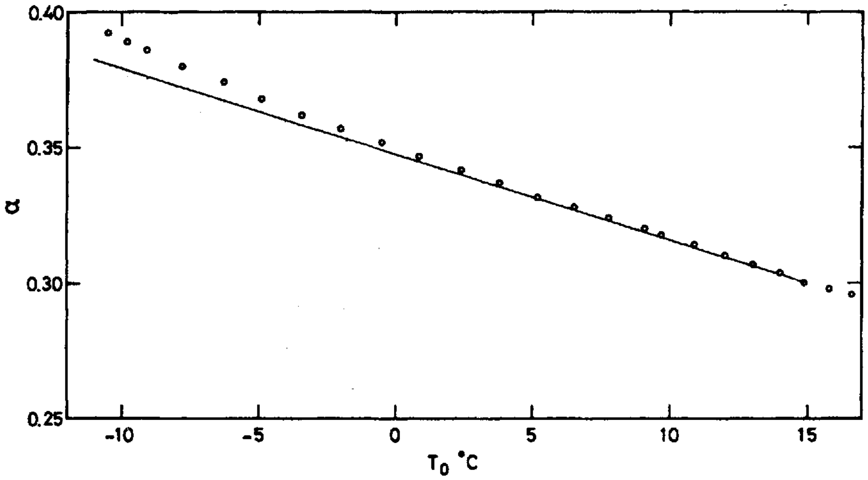

FIGURE 1 Dependence of planetary albedo α_p on planetary temperature T_0 as determined by numerical evaluation of Eqs. (8) and (9). Numerical values of the parameters are given in the text.

present-day regime. The results obtained are therefore significant for values of x_s and T_0 not too remote from present conditions. As long as one is restricted to small deviations around some given values of variables, one can also perform an analytic study of the albedo-temperature relationship. To this end we set:

$$x_s = x_s^* + \eta, \tag{14}$$

where x_s^* is the location of the present-day ice boundary ($x_s^* = 0.96$) and η is a small perturbation. Inserting Eq. (14) in both Eq. (8) and Eq. (9), expanding around x_s^* and keeping only linear terms we get a closed equation for T_0:

$$C\frac{\partial T_0}{\partial t} = Q\left[1 - \alpha_p^* - \left(\frac{\partial \alpha_p}{\partial x_s}\right)_{x_s^*}\eta\right] - (A + BT_0), \tag{15}$$

$$\eta = -\frac{QH_2(x_s^*) + (6D+B)(10+T_0)/P_2(x_s^*)}{Q[\partial H_2(x_s)/\partial x_s]_{x_s^*} - (6D+B)(10+T_0)3x_s^*/P_2(x_s^*)^2}, \tag{16}$$

where

$$1 - \alpha_p^* = H_0(x_s^*)$$

and

$$\left(\frac{\partial \alpha_p}{\partial x_s}\right)_{x_s^*} = -\left(\frac{\partial H_0(x_s)}{\partial x_s}\right)_{x_s^*}$$

Eq. (16) displays the dependence of the effective albedo on the planetary temperature and on the transport and cooling coefficients D and B. The latter coefficients appear explicitly, as well as implicitly, through the values of x_s^*. Differentiating this relation with respect to T_0^*, multiplying by $\partial\alpha_p/\partial x_s$ and setting $T = T_0 = 14.9$ in the result, we find a value close to the numerical one, Eq. (13).

It should be mentioned that Lian and Cess (1977) have actually expressed the temperature derivative of the planetary albedo in terms of zonal values integrated over all latitudes. They then evaluated their expression using experimentally determined values of the dependence of zonal albedo on temperature. Our work differs in that we remain in the framework of the 1-D energy balance model and we try to deduce all quantities of interest in a self consistent way without any further use of experimental data.

4. TIME-DEPENDENT PROBLEM—MEMORY EFFECTS

So far we have studied the dependence of the planetary albedo on parameters characterizing suitable averages of the 1-D problem. To this end we have assumed that the components T_n, $n \geqslant 2$, expressing the spatial dependence of the temperature, are in a quasi-steady-state. As mentioned in the previous section, this hypothesis is rather reasonable, in view of the difference in the relaxation times of T_0 and of T_n, $n \geqslant 2$. In this section we will nevertheless analyze the more general case of the time-dependent problem. For the sake of simplicity we shall limit ourselves again to the two-mode approximation, and furthermore to a linear analysis in the vicinity of the present value of the planetary temperature, or alternatively of the ice edge.

Starting from the full time-dependent equations (6), we differentiate Eq. (6c) with respect to time. Combining the resulting equation with Eq. (6b) and Eq. (6c) and using Eq. (6a) we obtain

$$C\frac{\partial x_s}{\partial t} = \frac{P_2(x_s)}{3x_s}\left\{\frac{1}{(10+T_0)}[QH_0(x_s) - (A+BT_0) + P_2(x_s)QH_2(x_s)] + (6D+B)\right\}. \tag{17}$$

Eq. (17) together with Eq. (6a) constitute a system of two highly nonlinear, first-order coupled differential equations for x_s and T_0. Within the framework of a linear analysis we let

$$x_s = x_s^* + \eta, \quad \eta \ll x_s^*,$$

and

$$T_0 = T_0^* + \theta, \quad \theta \ll T_0^*,$$

where η and θ, are respectively small deviations of the ice boundary and of the planetary temperature. Linearizing with respect to both variables, we obtain after some algebraic manipulations

$$\frac{\partial \eta}{\partial t} = +\hat{b}\theta(t) - \hat{a}\eta(t) \tag{18a}$$

and

$$\frac{\partial \theta}{\partial t} = -\frac{1}{C}\left[B\theta(t) + Q\left(\frac{\partial \alpha_p}{\partial x_s}\right)_{x_s^*}\eta(t)\right], \tag{18b}$$

with

$$\hat{a} = \frac{1}{C}\left\{\frac{1}{(10+T_0^*)}\frac{P_2^*}{3x_s^*}Q\left[\left(\frac{\partial \alpha_p}{\partial x_s}\right)_{x_s^*} - P_2^*\left(\frac{\partial H_2}{\partial x_s}\right)_{x_s^*}\right] + (6D+B)\right\},$$

$$\hat{b} = \frac{1}{C(10+T_0^*)}\frac{P_2^*}{3x_s^*}6D.$$

The solution of Eq. (18a) is

$$\eta(t) = e^{-\hat{a}t}\left[K + \hat{b}\int_0^t d\tau\,\theta(\tau)e^{\hat{a}\tau}\right], \tag{19}$$

where K is the constant of integration depending on the initial conditions. Assuming $K=0$, and substituting Eq. (19) into Eq. (18b), we obtain

$$\frac{\partial \theta}{\partial t} = -\frac{1}{C}\left[B\theta(t) + Q\left(\frac{\partial \alpha_p}{\partial x_s}\right)_{x_s^*}\hat{b}\int_0^t d\tau\, e^{-\hat{a}(t-\tau)}\theta(\tau)\right]. \tag{20}$$

We thus arrive again at a closed equation for the deviation θ of the planetary temperature from the reference value T_0^*. Contrary to the previous section, where we were dealing with ordinary differential equations, Eq. (20) is now an integro-differential equation displaying memory effects. The latter arise from the time-dependent elimination of spatial variables.

In order to solve Eq. (20) we use the Laplace transform (e.g. Matthews and Walker, 1965, sec. 4–3):

$$\tilde{\theta}(s)=\int_0^\infty e^{-st}\theta(t)dt. \tag{21}$$

Inserting Eq. (21) into Eq. (20) we obtain

$$\tilde{\theta}(s)=\theta(0)\Big/\left[s-\frac{\varepsilon}{\hat{a}+s}+\frac{B}{C}\right], \tag{22}$$

with

$$\varepsilon=-\frac{Q\hat{b}}{C}\left(\frac{\partial\alpha}{\partial x_s}\right)_{x_s^*},$$

and $\theta(0)$ the initial value of θ. Performing the inverse Laplace transform we have

$$\theta(t)=\frac{1}{2\pi i}\int_c \frac{ds\,e^{st}}{\left[s-\dfrac{\varepsilon}{\hat{a}+s}+\dfrac{B}{C}\right]}\theta(0). \tag{23}$$

Thus, the time dependence of $\theta(t)$ will be determined by the singularities of the integrand, that is by the zeros of the denominator. These are given by the equation

$$s^2+(\hat{a}+B/C)s-(\varepsilon-\hat{a}B/C)=0. \tag{24}$$

Its roots have the numerical values:

$$s_1=-0.038\,\text{yr}^{-1},\quad s_2=-0.334\,\text{yr}^{-1}, \tag{25}$$

for the parameters given above and for a thermal inertia coefficient† corresponding to a mixed layer of 75 m.

The values in Eq. (25) are to be compared to the unique relaxation rate characterizing the planetary model of the previous section (with memory effects neglected) which is

$$C^{-1}[Qd\alpha_p/dT_0+B]=0.048\,\text{yr}^{-1}.$$

†It should be pointed out that the evaluation of characteristic times carried out in this section depends crucially on the choice of the thermal inertia coefficient C. If instead of choosing a mixed layer of 75 m, we took a whole ocean depth, we would be led to much larger time scales. Alternatively, for smaller depths of the mixed layer, we would have shorter characteristic times of the order of a year. We believe that the choice of the appropriate value of C depends on the nature of the perturbations acting on the system.

We see that the effect of memory splits this unique characteristic rate into a slightly slower one s_1, and a much faster one s_2. Among these only the first one will give a significant contribution to the evolution of T_0, while the other will die out very quickly. In any case the system remains stable and overdamped: it cannot display (damped) oscillatory behavior.

As pointed out by Bhattacharya and Ghil (1978), oscillations may arise in the presence of time lags. Within the framework of our model such lags cannot arise, as the memory effect is fading away for all $0<\tau<t$. Nevertheless, we considered the consequences of replacing the kernel of Eq. (20) by a delta function

$$\int_0^t dt'\hat{a}e^{-\hat{a}(t-t')}\theta(t')\sim\theta(t-\tau). \tag{26}$$

Equation (20) becomes then:

$$\frac{\partial\theta(t)}{\partial t}=-\frac{1}{C}\left[B\theta(t)+Q\left(\frac{\partial\alpha_p}{\partial x_s}\right)_{x_s^*}\frac{\hat{b}}{\hat{a}}\theta(t-\tau)\right]. \tag{27}$$

Seeking for solutions of the form $\theta=\theta_0\exp(\omega t)$ and using the numerical values of the model coefficients, we find the characteristic equation for ω:

$$\omega=-(0.156-0.0971\,e^{-\omega\tau}). \tag{28}$$

It admits one real negative solution for all $\tau\geqslant 0$. Therefore instabilities and oscillations are ruled out. Additional conditions are necessary, such as those considered by Bhattacharya and Ghil (1978) in connection with genuinely nonlinear ice-sheet dynamics. (Källén *et al.*, 1978, 1979), in order to obtain such effects within the framework of a 1-D energy balance model.

5. DISCUSSION

We have seen that it is possible within the framework of energy-balance models to express a number of features of a system of low dimensionality starting from a model corresponding to a higher dimensionality. The technique used was the systematic elimination of spatial degrees of freedom, either by a quasi-static procedure (Section 3) or in a time-dependent way (Section 4). It led us to an explicit representation of the surface-albedo feedback in terms of small deviations of the planetary temperature from its present value.

The wide separation of characteristic times between the first two spatial modes (Section 4) shows that memory effects, although quantitatively important, do not introduce a qualitative change in the time evolution of temperature. This provides an *a posteriori* justification of the quasi-static assumption used in Section 3, as well as of the truncation to the second Legendre mode adopted throughout the present paper.

A crucial factor limiting the generality of our conclusions is that we did not consider explicitly feedback mechanisms other than the dependence of surface-albedo on temperature. Such mechanisms, denoted collectively by $Y(T_0)$ in Eq. (11), are likely to play a rather important role in the total value of the temperature derivative of the albedo (Lian and Cess, 1977). Unfortunately, it does not seem possible at this time to construct climate models taking these other effects into account in a self-consistent way.

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