Associated Homogeneous Distributions in Clifford Analysis

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Abstract

We introduce Ultrahyperbolic Clifford Analysis (UCA) as a motivation for studying Associated Homogeneous Distributions (AHDs). UCA can be regarded as a higher-dimensional function theory that generalizes the theory of complex holomorphic functions. In UCA, the algebra of complex numbers is replaced with a Clifford algebra $Cl_{p,q}$ and the classical complex Cauchy-Riemann equation is replaced with a Clifford algebra-valued equation, having physical relevance.

The convolution kernel in Cauchy's integral formula from complex analysis, $\frac{1}{2\pi i}z^{-1}$, becomes in UCA a (non-trivial) AHD. In the theoretical development of UCA and also for its practical application, it is necessary that we can convolve and multiply AHDs. The aim of this talk is to show that UCA can be founded on classical distribution theory, so that it is not necessary to use a more general generalized function algebra for this purpose. This is achieved by using a new convolution and isomorphic multiplication algebra of (one-dimensional) AHDs developed earlier by the author, entirely within the setting of Schwartz' distributions.

1 Introduction

Ultrahyperbolic Clifford Analysis (UCA) is a particular generalization of complex analysis to hypercomplex analysis. Let $p, q \in \mathbb{N}$, $n \triangleq p+q$, P the canonical quadratic form of signature (p, q), $\mathbf{R}^{p,q} \triangleq (R^n, P)$ the inner product space with inner product induced by P and $Cl_{p,q}$ the Clifford algebra generated by $\mathbf{R}^{p,q}$. Then, UCA can be regarded as the study of a particular subset of functions from $R^n \to Cl_{p,q}$. A physical interpretation of UCA is that of a theory of functions defined on a generalized Lorentzian space with an arbitrary number of time (p) and space dimensions (q). UCA generalizes Hyperbolic Clifford Analysis (HCA), corresponding to $p = 1$ or $q = 1$, and Elliptic Clifford Analysis (ECA), corresponding to $p = 0$ or $q = 0$. ECA is about 30 years old and now a mature part of analysis, [2], [4]. HCA and UCA are still under development.

The set of Associated Homogeneous Distributions (AHDs) with support in R, denoted by $\mathcal{H}'(R)$, is the distributional analogue of the set of power-log functions with domain in R, [18], [25], [11]. $\mathcal{H}'(R)$ contains the majority of the (one-dimensional) distributions one typically encounters in physics applications, such as δ, $\eta \triangleq \frac{1}{\pi}x^{-1}$ (a normalized Cauchy's principal value $\mathrm{Pv}\,\frac{1}{x}$), the Heaviside step distributions $1_{\pm}$, pseudo-functions generated by taking Hadamard's finite part of certain divergent integrals, associated Riesz kernels, generalized Heisenberg distributions, all their generalized derivatives and primitives, etc.

There is a close relationship between UCA and AHDs. First, the development of UCA requires us to study AHDs since the latter appear as cornerstone objects in the formulation of UCA. In addition, one needs their properties, e.g. for solving Boundary Value Problems (BVPs) and Riemann-Hilbert Problems (RHPs).

In particular, HCA with $p = 1$ and $q = 3$ appears to be a very suitable mathematical tool for solving physics applications, e.g. in Electromagnetism (EM) and Quantum Physics (QP). The latter physical relevance explains why AHDs appear so often in applications.

In earlier work, I constructed a convolution algebra and an isomorphic multiplication algebra of AHDs on R within Schwartz' distribution theory, [11]–[17]. We will see that higher dimensional versions of these algebras on R^n, obtained as pullbacks along the quadratic form P, play a key role in UCA. Consequently, UCA can be founded on Schwartz' distribution theory and it is thus not necessary to use a more general generalized function algebra for its construction.

2 Ultrahyperbolic Clifford Analysis

For an in depth overview of Clifford analysis, see [2], [4], [3], [9], [10].

2.1 Clifford algebras

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denote an orthogonal basis for R^n. The universal (real) Clifford algebra $Cl_{p,q}$ over $\mathbf{R}^{p,q}$ is defined by

$$\begin{aligned} \mathbf{e}_1^2 &= \ldots = \mathbf{e}_p^2 = +1 \text{ and } \mathbf{e}_{p+1}^2 = \ldots = \mathbf{e}_n^2 = -1, && (1)\\ \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i &= 0, i \neq j, && (2) \end{aligned}$$

together with linearity over $\mathbb{R}$ and associativity. Clifford showed how to turn an n-dimensional linear space into an 2^n-dimensional algebra. Essential is that his algebra is not closed for vectors, but is closed for all anti-symmetric tensors which can be generated from the underlying linear space. These anti-symmetric tensors represent oriented subspaces of the original n-dimensional linear space. A (real) Clifford number (also called "multivector") x is therefore a hypercomplex number over $\mathbb{R}$ of the form

$$x = \underbrace{a1}_{1} + \underbrace{a^i\mathbf{e}_i}_{\binom{n}{1}} + \frac{1}{2!}\underbrace{a^{i_1 i_2}\left(\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2}\right)}_{\binom{n}{2}} + \ldots + \underbrace{a^{1,\ldots,n}\left(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n\right)}_{\binom{n}{n}}. \quad (3)$$

This can be regarded as a direct sum of $n+1$ grades $x = \oplus_{k=0}^{n} [x]_k$, making $Cl_{p,q}$ a graded linear space of dimension 2^n. We have the embeddings: $\mathbb{R} \hookrightarrow Cl_{p,q}$ by the grade 0 part and $\mathbf{R}^{p,q} \hookrightarrow Cl_{p,q}$ by the grade 1 part. A Clifford number of pure grade k has the geometrical interpretation of an oriented k-dimensional subspace. E.g., $x = [x]_1$ represents an oriented line segment (a vector), $x = [x]_2$ represents an oriented surface segment, etc. Clifford himself called his algebras geometrical algebras, because they are the natural choice when doing geometrical meaningful calculations with oriented subspaces of a given n-dimensional linear space.

The Clifford product of two numbers of pure grade, $x = [x]_k$ and $y = [y]_l$, is given by

$$xy = \sum_{i=|k-l|,2}^{k+l} [xy]_i \,. \tag{4}$$

In particular, the Clifford product of two vectors $\mathbf{v}$ and $\mathbf{w}$ decomposes into the sum of the inner and outer products,

$$\mathbf{vw} = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \wedge \mathbf{w}, \tag{5}$$

wherein the grade 0 part contains information about the angle between the vectors and the grade 2 part expresses that two vectors also span an oriented parallelogram.

Familiar Clifford algebras are: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (Hamilton's quaternions), P (Pauli's algebra), M (Majorana's algebra), and the time-space algebra $Cl_{1,3}$. Clifford algebras have been found to be very well-suited to formulate physical problems, [20], [21], [1], [24].

2.2 Generalized Cauchy-Riemann equation

Introduce the $Cl_{p,q}$-valued nabla operator $\partial \triangleq \sum_{i=1}^{n} \mathbf{e}_i \partial_i$, called Dirac operator, and let Ω be a domain in R^n .

Definition 1 *Ultrahyperbolic Clifford Analysis is the study of functions satisfying*

$$\partial F = -S, \tag{6}$$

with $F \in C^{\infty}(\Omega, Cl_{p,q})$ and for given $S \in C_c^{\infty}(R^n, Cl_{p,q})$, together with a boundary condition for F at infinity and possibly integrability conditions on S.

If $S = 0$, eq. (6) is a particular generalization of the Cauchy-Riemann equation from complex analysis and then defines functions F called (left) holomorphic.

2.3 Physical interpretation

Let us restrict eq. (6) to the Clifford algebra $Cl_{1,3}$, choose for S a smooth compact support multivector function having as only non-vanishing grades 1 and 3 (i.e., $S = [S]_1 + [S]_3$) and restrict F to be of pure grade 2 (i.e., $F = [F]_2$). Then eq. (6) reproduces the Maxwell-Heaviside equations for the EM field F, generated by a given electric monopole current density source $[S]_1$ and a given magnetic monopole current density source $[S]_3$, [19], [23], [26]. Hence, HCA of signature $(1,3)$ (and

CA	**EM**
Cauchy-Riemann eq.	Equation of EM
Clifford-valued functions	Generalized EM fields
Holomorphy	Holography
Singularities, Residues	Source fields
Cauchy/Integral theorems	Reciprocity theorems
Riemann-Hilbert problems	Scattering problems
etc.	etc.

Table 1: Correspondences between CA and EM.

with additional grade restrictions) is a mathematical function theory that models physical EM fields. This identification now leads to the correspondences summarized in Table 1.

The above physical interpretation can be readily generalized. Choose any Clifford algebra $Cl_{p,q}$, let F be a general $Cl_{p,q}$-valued function and S a given smooth compact support $Cl_{p,q}$-valued function. Then eq. (6) becomes a model for a generalized EM in a universe with p time dimensions and q space dimensions!

2.4 Cauchy kernels

Of central importance in the formulation of UCA are the Cauchy kernels. The Cauchy kernel C_{x_0} in UCA is a vector-valued distribution, which derives from a scalar distribution $g_{x_0} \in \mathcal{D}'$ as

$$C_{x_0} = \partial g_{x_0}. \tag{7}$$

The scalar distribution g_{x_0} is a fundamental solution of the *Ultrahyperbolic Equation* (UE) (i.e., the wave equation) in $\mathbf{R}^{p,q}$

$$\square_{p,q} g_{x_0} = \delta_{x_0}. \tag{8}$$

The point $x_0 \in R^n$ will eventually play the role of calculation point in the generalized Cauchy's integral theorem in UCA, but can here be thought of as parametrizing a family of distributions.

Introduce the shorthands, $P_{x_0} \triangleq P(x - x_0)$ and $A_{n-1} \triangleq 2\pi^{n/2}/\Gamma(n/2)$. A (real) fundamental solution of the UE for general (p, q) with $2 \leq n$ is, [5], [8], [6], [22],

(i) for $n > 2$

$$g_{x_0} = -\frac{1}{(n-2)\, A_{n-1}} \frac{1}{2} \left(e^{iq\frac{\pi}{2}} (P_{x_0} + i0)^{1-\frac{n}{2}} + e^{-iq\frac{\pi}{2}} (P_{x_0} - i0)^{1-\frac{n}{2}} \right), \tag{9}$$

(ii) for $n = 2$

$$g_{x_0} = \frac{1}{4\pi} \frac{1}{2} \left(e^{iq\frac{\pi}{2}} \ln(P_{x_0} + i0) + e^{-iq\frac{\pi}{2}} \ln(P_{x_0} - i0) \right), \tag{10}$$

$$= \frac{1}{4} \left(\cos(q\pi/2) \frac{1}{\pi} \ln |P_{x_0}| - \sin(q\pi/2) 1_{-}(P_{x_0}) \right). \tag{11}$$

The distributions g_{x_0} are readily seen to be pullbacks along P_{x_0} of one-dimensional AHDs. This is how AHDs enter in the formulation of UCA.

3 Associated Homogeneous Distributions on R

3.1 Definition

Definition 2 *HD. A distribution $f_0^z \in \mathcal{D}'$ is called a (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ iff it satisfies for any $r > 0$,*

$$\left\langle f_0^z, \varphi\left(x/r\right)\right\rangle = r^{z+1}\left\langle f_0^z, \varphi\left(x\right)\right\rangle, \forall\varphi \in \mathcal{D}. \tag{12}$$

Definition 3 *AHD. A distribution $f_m^z \in \mathcal{D}'$ is called an associated (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ and order of association $m \in \mathbb{Z}_+$, iff there exists a sequence of associated homogeneous distributions f_{m-l}^z of degree of homogeneity z and associated order $m-l$, $\forall l \in \mathbb{Z}_{[1,m]}$, not depending on r and with $f_0^z \neq 0$, satisfying,*

$$\left\langle f_m^z, \varphi\left(x/r\right)\right\rangle = r^{z+1}\left\langle f_m^z + \sum_{l=1}^{m}\frac{\left(\ln r\right)^l}{l!} f_{m-l}^z, \varphi\left(x\right)\right\rangle, \forall\varphi \in \mathcal{D}. \tag{13}$$

For a more detailed overview of AHDs, see [18], [25], [11].

3.2 Preliminaries

We will use hereafter the following terminology.

Definition 4 *A partial distribution is a linear and continuous functional that is only defined on a proper subset $\mathcal{D}_r \subset \mathcal{D}$.*

Definition 5 *A $f_m^z \in \mathcal{H}'\left(R\right)$ has a critical degree of homogeneity at $z = z_c$ iff $f_m^{z_c}$ exists as a partial distribution.*

Definition 6 *An extension f_ε from $\mathcal{D}_r$ to $\mathcal{D}$, of a partial distribution f, is a distribution $f_\varepsilon \in \mathcal{D}'$, defined $\forall\varphi \in \mathcal{D}$, such that $\langle f_\varepsilon, \psi\rangle = \langle f, \psi\rangle$, $\forall\psi \in \mathcal{D}_r \subset \mathcal{D}$.*

Definition 7 *A regularization of a partial distribution $f_m^{z_c} \in \mathcal{H}'\left(R\right)$ is any extension $\left(f_m^{z_c}\right)_e$ in $\mathcal{H}'\left(R\right)$ of $f_m^{z_c}$ from $\mathcal{D}_r$ to $\mathcal{D}$.*

Definition 8 *(i) The convolution product of any two AHDs on R of degrees $a-1$ and $b-1$ is called a critical convolution product, iff the resulting degree $a+b-1 \triangleq k \in \mathbb{N}$.*

Definition 9 *(ii) The multiplication product of any two AHDs on R of degrees a and b is called a critical multiplication product, iff the resulting degree $a+b \triangleq -l \in \mathbb{Z}_-$.*

3.3 Definition of the products

3.3.1 Convolution

Let $\mathcal{D}'_R$ denote the distributions based on R with support bounded on the left and $\mathcal{D}'_L$ denote the distributions based on R with support bounded on the right. A structure theorem for AHDs states that any AHD on R is the sum of an AHD in $\mathcal{D}'_L$ and an AHD in $\mathcal{D}'_R$. To define a convolution product on $\mathcal{H}'(R)$ we must consider three cases.

Case 1. The factors have one-sided support, bounded at the *same side*.

In this case we can use for any degree of the factors the standard definition involving the *direct product* (the standard convolution integral). This case is an example of the method of retarded distributions.

Case 2. The factors have one-sided support, bounded at *different sides*, and the resulting degree of homogeneity is *not a natural number*.

In this case, the convolution $f * g$, with $f \in \mathcal{D}'_L$ and $g \in \mathcal{D}'_R$, can not straightforwardly be defined in terms of a direct product, because $\mathrm{supp}(f*g) \cap \mathrm{supp}(\varphi \in \mathcal{D}(R^2))$ is generally non-compact. This case is handled in two steps:

(i) First in $T \triangleq \{(a, b) \in \mathbb{C}^2 : 0 < \mathrm{Re}(a),\ 0 < \mathrm{Re}(b) \text{ and } \mathrm{Re}(a + b) < 1\}$ we use the standard convolution integral.

(ii) Then we extend by *analytic continuation* to $R \triangleq \{(a, b) \in \mathbb{C}^2 : a + b - 1 \notin \mathbb{N}\}$.

Case 3. The factors have one-sided support, bounded at *different sides*, and the resulting degree of homogeneity $a + b - 1$ is *a natural number* k (critical product). It was observed that:

(a) Any critical convolution product $f^{a-1} * f^{b-1}$ is in general a partial distribution only defined on $\mathcal{S}_{\{k\}}$. $S_{\{k\}}$ is the subspace of S whose members have zero k-th order moment.

(b) A particular extension of the partial distribution $f^{a-1} * f^{b-1}$ from $\mathcal{S}_{\{k\}}$ to $\mathcal{S}$ can be realized as *an* analytic finite part.

(c) This finite part, being a limit in $\mathbb{C}^2$, is in general non-unique.

(d) *Fortunately*, it turned out that this non-uniqueness only involves an arbitrary term of the form cx^k, $c \in \mathbb{C}$ arbitrary.

A critical convolution product, only existing as a partial distribution $f^{a-1} * f^{b-1}$, is then *defined as any extension* in $\mathcal{H}'(R)$ and so obtains meaning as a distribution.

3.3.2 Multiplication

Let $f^a, g^b \in \mathcal{H}'(R)$ of degree a and b. The multiplication of AHDs is defined in terms of the convolution product by

$$f^a \,.\, g^b \triangleq \mathcal{F}\left(\left(\mathcal{F}^{-1} f^a\right) * \left(\mathcal{F}^{-1} g^b\right)\right). \tag{14}$$

3.4 Properties of the products

The constructed algebras of AHDs on R have the following properties.

A. Non-commutativity

(i) Non-critical products are always commutative.

(ii) Critical products are generally non-commutative.

(iii) Deviation from commutative is by a term of the form cx^k (convolution) or $c\delta^{(l)}$ (multiplication), $k \in \mathbb{N}$, $l \in \mathbb{Z}_+$ and $c \in \mathbb{R}$ arbitrary.

B. Non-associativity

(i) Non-critical triple products are always associative.

(ii) Critical triple products are generally non-associative.

(iii) Deviation from commutative is by a term of the form cx^k (convolution) or $c\delta^{(l)}$ (multiplication), $k \in \mathbb{N}$, $l \in \mathbb{Z}_+$ and $c \in \mathbb{R}$ arbitrary.

4 Conclusion

The here presented connection between UCA and AHDs clearly reveals the importance of this rather small subset of Schwartz distributions. On the one hand, they appear as crucial building blocks in the construction of advanced higher dimensional analysis. On the other hand, and essentially because of their role in UCA, they appear ubiquitous in physics applications.

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