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Generalized invariant for a charged particle interacting with a linearly polarized hydromagnetic plane wave

by

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FOREWORD

The note entitled "Generalized invariant for a charged particle interacting with a linearly polarized hydromagnetic plane wave" will be published in Bulletin de la Classe des Sciences de l'Académie Royale de Belgique.

AVANT-PROPOS

L'article intitulé "Generalized invariant for a charged particle interacting with a linearly polarized hydromagnetic plane wave" sera publié dans le Bulletin de la Classe des Sciences de l'Académie Royale de Belgique.

VOORWOORD

Dit werk, "Generalized invariant for a charged parficle interacting with a linearly polarized hydromagnetic plane wave" zal verschijnen in Bulletin de la Classe des Sciences de l'Académie Royale de Belgique.

VORWORT

Die Arbeit "Generalized invariant for a charged particle interacting with a linearly polarized hydromagnetic plane wave" wird in Bulletin de la Classe des Sciences de l'Académie Royale de Belgique herausgegeben werden.

GENERALIZED INVARIANT FOR A CHARGED PARTICLE INTERACTING WITH A LINEARLY POLARIZED HYDROMAGNETIC PLANE WAVE

by

M. ROTH

Abstract

Gyroresonant interactions between charged particles and hydromagnetic waves are important processes in magnetospheric and solar wind physics.

In this paper we formulate analytically a generalized invariant for a charged particle motion in the electromagnetic field of a linearly polarized hydromagnetic plane wave. The propagation is in the direction of a uniform magnetic field and it is assumed that the magnetic field of the wave is weak. Using a canonical transformation, the invariant J is developed to the first order in the modulation amplitude. It is shown that the curves J = constant represent satisfactorily the orbits in phase space.

Résumé

Les interactions gyrorésonnantes entre particules chargées et ondes hydromagnétiques jouent un rôle important dans de nombreux problèmes rencontrés dans l'étude de la magnétosphère et du vent solaire.

Dans cet article, on formule analytiquement un invariant généralisé pour le mouvement d'une particule chargée dans le champ électromagnétique d'une onde hydromagnétique polarisée linéairement. Cette onde plane se propage dans la direction d'un champ magnétique uniforme et on suppose que l'amplitude de la composante magnétique de l'onde est faible. A l'aide d'une transformation canonique simplificatrice, l'invariant J est développé jusqu'au premier ordre en l'amplitude de la modulation. Dans l'espace des phases, on montre que les courbes J = constante reproduisent de manière satisfaisante les trajectoires de phase calculées numériquement à partir des équations de mouvement.

Samenvatting

De gyroresonante interactie tussen geladen deeltjes en hydromagnetische golven speelt een belangrijke rol in talrijke problemen in verband met de magnetosfeer en de zonnewind.

In dit artikel formuleren wij analytisch een veralgemeende invariant van de beweging van een geladen deeltje in het electromagnetisch veld van een lineair gepolariseerde hydromagnetische vlakke golf. De voortplanting gebeurt in de richting van een uniform magnetisch veld, en er wordt verondersteld dat het magnetisch veld van de golf zwak is. Met behulp van een vereenvoudigende canonische transformatie wordt de invariant J ontwikkeld tot de eerste orde in de amplitudemodulatie. Verder wordt aangetoond dat, in de faseruimte, de krommen J = constante, goed de banen weergeven die berekend werden uit de bewegingsvergetijkingen.

Zusammenfassung

Die Resonanzinteraktionen zwischen geladete Teilchen und Hydromagnetische Wellen spielen eine wichtige Rolle in der Magnetosphere der Erde und im Sonnen Wind.

In dieser Arbeit geben wir ein analytische Formel für einen adiabatischen Invarianten im Falle wenn ein Teilchen mit eine Hydromagnetische Welle interagiert. Mit Hilfe einer kanonischen Transformation ist dieses Invariant J zur ersten Ordnung in das Wellenamplitude entwickelt worden. In dem Phasenraum sind die Teilchenbahnen gut durch die Lienen J = Konstante angenähert.

I. INTRODUCTION

The motion of a charged particle in a magnetic field often can be described quite accurately by a superposition of a gyration and a drift motion (Northrop, 1963). If both the Larmor radius and drift velocity change slowly during a Larmor period, the behaviour of the particle can be characterized by quantities which are approximate constants of motion. These *adiabatic invariants* approach a constant value in the limit of infinitely weak variations of the magnetic and electric fields. In particular, the magnetic moment is an approximate constant of motion. It is known as the first adiabatic invariant and was introduced by Alfvèn (1950).

The importance of a generalized invariant for a charged particle motion in an electromagnetic field arises whenever the conditions of adiabatic invariance are not satisfied.

In the magnetosphere as well as in the interplanetary space, the applicability of the three classical adiabatic invariants is quite restrictive. Violation of the invariants can be caused by non-adiabatic time variations of the magnetic and electric fields, but also by interactions with electromagnetic or hydromagnetic waves (gyroresonant interactions) or by collisions in the ambient medium (atmosphere, ionosphere).

It is generally accepted that the gyroresonant interactions occuring in the magnetosphere between charged particles and hydromagnetic waves (in particular : whistler and ion cyclotron waves) are responsible for a large number of magnetospheric processes : e.g. the limit on stably trapped particle flux in the radiation belts (Dragt, 1961 ; Kennel and Petschek, 1966), energetic particles precipitation, formation of aurorae, turbulent loss of ring current protons and SAR arc formation (Cornwall *et al.*, 1970; 1971).

In the interplanetary space, collisionless particles interact with hydromagnetic waves or with random magnetic fields. These irregularities can destroy the invariance of the magnetic moment and lead to a non-adiabatic change of the pitch angle, scattering the particles in the collisionless solar wind region.

In this paper we consider the effect of a linearly polarized hydromagnetic wave, on the motion of a charged particle. The wave propagation is in the direction of a uniform magnetic field $\vec{B}_o$. It is shown that, when the wave amplitude is weak, it is possible to find a more general invariant than the magnetic moment. In the second section, we describe our assumptions and notations and give the equations of motion by using a canonical transformation with zero order Larmor radius and phase as variables. In Sec. 3, we determine a generalized invariant of motion. A perturbation theory is used when h, the relative amplitude of the wave, is a small parameter. In a frame of reference moving with a speed equal to the phase velocity of the wave, the field is purely magnetostatic and the kinetic energy of the particle is conserved. Therefore, any constant of motion $J = J_o + hJ_1 + h^2 J_2 + \ldots$, is solution of the equation $(J, H) = O$ where the left hand side is the Poisson bracket of J with the Hamiltonian H. The method used is similar to that employed by Dunnett *et al.* (1968) and Dunnett and Jones (1972) for square wave and sine wave magnetic field modulations. Using the canonical transformation introduced in section 1, it is shown that all the differential equations determining J_i have the same form and can be integrated immediately.

Numerical results are discussed in Sec. 4 to test the validity of the first order invariant : $J_o + hJ_1$.

II. EQUATIONS OF MOTION AND CANONICAL TRANSFORMATION

We consider a uniform magnetic field of intensity $\vec{B}_o$ in a direction parallel to the Z-axis and a monochromatic transversal hydromagnetic wave linearly polarized along the X-axis propagating with a phase velocity $\vec{U}$ parallel to $\vec{B}_o$. If Ω and $\vec{k}$ are, respectively, the angular frequency and the wave vector, the magnetic and electric fields of the wave are connected by Maxwell's equation :

$$\delta\vec{B} = \frac{\vec{k} \wedge \delta\vec{E}}{\Omega} \qquad (1)$$

The electromagnetic field is completely described by the equations :

$$\vec{B} = \vec{B}_o + \delta\vec{B} = B_o \vec{e}_z + h\, B_o \cos(kZ - \Omega t)\, \vec{e}_y \tag{2}$$

$$\vec{E} = \delta\vec{E} = h\, \frac{\Omega}{k}\, B_o \cos(kZ - \Omega t)\, \vec{e}_x \tag{3}$$

where h is the relative amplitude of the magnetic modulation and t the time variable.

In the frame of reference moving with a speed equal to the phase velocity of the wave, the electromagnetic field has an additional component resulting from the Lorentz transformation. As the ratio $\frac{U}{c}$ is much smaller than unity, it follows that :

$$t' = t \tag{4}$$

$$Z' = Z - Ut \tag{5}$$

$$\vec{E}' = \vec{E} + \vec{U} \wedge \vec{B} = O \tag{6}$$

$$\vec{B}' = \vec{B} = \vec{B}_o + hB_o \cos\left(\frac{\Omega Z'}{U}\right) \vec{e}_y \tag{7}$$

In these equations the prime indicates that the variable is relative to the frame of reference of the wave. Since henceforth we only consider this frame we will omit the primes.

In the cartesian coordinates system (X, Y, Z), the Hamiltonian of a charged particle of mass m and charge q in a magnetostatic field is :

$$\mathcal{H} = \frac{1}{2}\, m\, (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) \tag{8}$$

The Lagrangian $\mathcal{L}$ is given by :

$$\mathcal{L} = \frac{1}{2}\, m\, (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + q\vec{A}.\vec{V} \tag{9}$$

$\vec{A}$ is the potential vector from which the magnetic field $\vec{B}$ is derived. Its components are :

$$A_x = -\frac{1}{2} B_o Y + h B_o \cdot \frac{U}{\Omega} \cdot \sin \frac{\Omega}{U} Z \tag{10}$$

$$A_y = \frac{1}{2} B_o X \tag{11}$$

$$A_z = O \tag{12}$$

The generalized momentum (P_x, P_y, P_z,) are defined by :

$$\begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial \dot{X}} \\ \frac{\partial}{\partial \dot{Y}} \\ \frac{\partial}{\partial \dot{Z}} \end{Bmatrix} \mathcal{L}(X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}, t) \qquad (13)\ (14)\ (15)$$

In order to simplify the notations it is convenient to introduce the dimensionless quantities x, y, z, p_x, p_y, p_z, τ, v, H and L defined by :

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \lambda \tag{16}$$

$$\frac{P_x}{p_x} = \frac{P_y}{p_y} = \frac{P_z}{p_z} = \frac{1}{2} m \omega \lambda \tag{17}$$

$$\frac{t}{\tau} = \frac{2}{\omega} \tag{18}$$

$$\frac{V}{v} = \frac{1}{2} \omega \lambda \tag{19}$$

$$\frac{\mathcal{H}}{H} = \frac{\mathcal{L}}{L} = \frac{1}{4} m\,\omega^2\,\lambda^2 \tag{20}$$

ω is the angular frequency of gyration in the field B_o

$$\omega = \frac{qB_o}{m} \tag{21}$$

and λ is the wavelength of the modulation

$$\lambda = \frac{2\pi}{k} = \frac{2\pi U}{\Omega} \tag{22}$$

Then, the Hamiltonian H becomes :

$$H = H_o + h\,H_1 + h^2\,H_2 \tag{23}$$

with

$$H_o = \frac{1}{2}\,[\,(p_x + y)^2 + (p_y - x)^2 + p_z{}^2\,] \tag{24}$$

$$H_1 = -\frac{1}{\pi}\,(y + p_x)\sin 2\pi z \tag{25}$$

$$H_2 = \frac{1}{2\pi^2}\,\sin^2 2\pi z \tag{26}$$

From equations (13), (14) and (15), one deduces the generalized momentum :

$$p_x = \frac{dx}{d\tau} - y + \frac{h}{\pi}\,\sin 2\pi z \tag{27}$$

$$p_y = \frac{dy}{d\tau} + x \tag{28}$$

$$p_z = \frac{dz}{d\tau} \tag{29}$$

The equations of motion deduced from Hamilton's equations are given by

$$\frac{1}{2}\frac{d^2x}{d\tau^2} - \frac{dy}{d\tau} + h\frac{dz}{d\tau}\cos 2\pi z = 0 \tag{30}$$

$$\frac{1}{2}\frac{d^2y}{d\tau^2} + \frac{dx}{d\tau} = 0 \tag{31}$$

$$\frac{1}{2}\frac{d^2z}{d\tau^2} - h.\frac{dx}{d\tau}.\cos 2\pi z = 0 \tag{32}$$

By integration of equations (30) and (31), it follows .

$$-\frac{1}{2}\frac{dx}{d\tau} + y - \frac{h}{2\pi}\sin 2\pi z = \frac{1}{2}(y - p_x) = Q_1 = C^t \tag{33}$$

$$\frac{1}{2}\frac{dy}{d\tau} + x = \frac{1}{2}(x + p_y) = P_1 = C^t \tag{34}$$

where Q_1 and P_1 are two constants of motion.

In a zero order approximation, P_1 and Q_1 determine the cartesian coordinates of the guiding center C of the particle. Indeed from fig. 1, it can be seen that :

$$\vec{OC} = \vec{OP} + \vec{r}^{(o)} \tag{35}$$

where the dimensionless Larmor radius $\vec{r}^{(o)}$ is defined by

$$\vec{r}^{(o)} = \frac{\vec{v}_{\perp}^{(o)} \wedge \vec{B}^{(o)}}{2B^{(o)}} \tag{36}$$

The superscripts correspond to zero order values evaluated in the limit $h \to o$. The components of the Larmor radius, deduced from equations (27) and (28), are respectively :

$$r_x^{(o)} = \frac{1}{2}\ (p_y - x) \text{ and } r_y^{(o)} = -\ \frac{1}{2}\ (p_x + y) \tag{37}$$

Since the components of $\vec{OP}$ are x and y, it follows from (35), that the components of $\vec{OC}$ are

$$(\vec{OC})_x = \frac{1}{2}\ (x + p_y) = P_1 \tag{38}$$

$$(\vec{OC})_y = \frac{1}{2}\ (y - p_x) = Q_1 \tag{39}$$

If $\phi^{(o)}$ is the zero order Larmor phase angle, the projection of the particle in the Oxy plane is also given by

$$x = P_1 + r^{(o)} \sin \phi^{(o)} \tag{40}$$

$$y = Q_1 + r^{(o)} \cos \phi^{(o)} \tag{41}$$

Substituing (40) and (41) in (38) and (39) we obtain

$$p_x = - Q_1 + r^{(o)} \cos \phi^{(o)} \tag{42}$$

$$p_y = P_1 - r^{(o)} \sin \phi^{(o)} \tag{43}$$

It is convenient to introduce a transformation $(x, y, z; p_x, p_y, p_z) \rightarrow (Q_1, Q_2, Q_3; P_1, P_2, P_3)$ defined by

$$\left\{\begin{array}{ll} x = P_1 + P_2^{1/2} \sin Q_2 & (44) \\ y = Q_1 + P_2^{1/2} \cos Q_2 & (45) \\ z = Q_3 & (46) \\ p_x = -Q_1 + P_2^{1/2} \cos Q_2 & (47) \\ p_y = P_1 - P_2^{1/2} \sin Q_2 & (48) \\ p_z = P_3 & (49) \end{array}\right.$$

If Q_2 is identified as the zero order Larmor phase angle ($\phi^{(o)}$ in Fig. 1) and P_2 as the square of the zero order Larmor radius (36), it can be seen that (44), (45) correspond to (40), (41) and (47), (48) correspond to (42), (43).

In fact the relations (44) to (49) define a canonical transformation since, in the formalism (Q_i, P_i), the equations of Hamilton can be reduced to the former equations of motion (30), (31) and (32). Indeed, from the expression of the new Hamiltonian

$$H = H_o + h H_1 + h^2 H_2 \tag{23}$$

with

$$H_o = 2P_2 + \frac{1}{2} P_3^2 \tag{50}$$

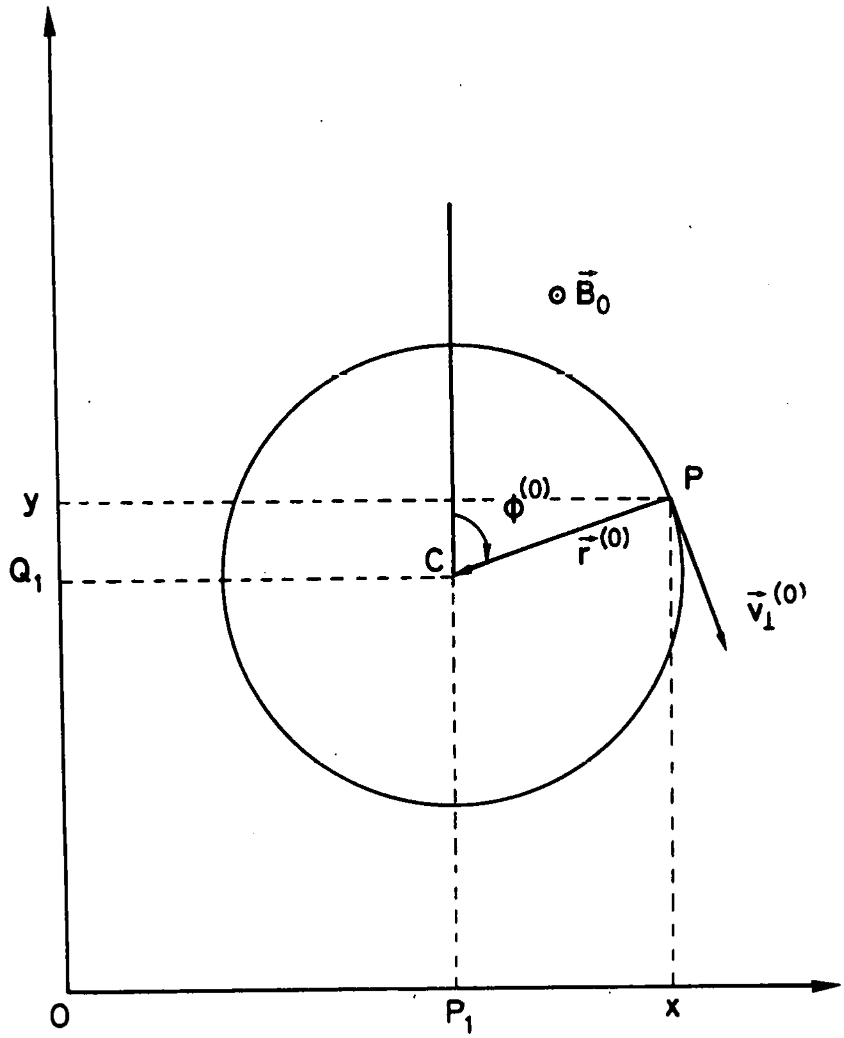

Fig. 1.- Zero order representation of the particle gyromotion.

$$H_1 = -\frac{2}{\pi} \quad P_2^{\ 1/2} \cos Q_2 \ . \sin 2\pi Q_3 \tag{51}$$

$$H_2 = \frac{1}{2\pi^2} \quad \sin^2 2\pi Q_3 \tag{52}$$

the equations of Hamilton become

$$\left\{ \begin{aligned} \frac{dP_1}{d\tau} &= o && (53) \\ \frac{dP_2}{d\tau} &= -\left(\frac{2h}{\pi} \quad P_2^{\ 1/2} \sin Q_2 \sin 2\pi Q_3 \right) && (54) \\ \frac{dP_3}{d\tau} &= -\left(-4h\, P_2^{\ 1/2} \cos Q_2 \cos 2\pi Q_3 + \frac{h^2}{\pi} \quad \sin 4\pi Q_3 \right. && (55) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{dQ_1}{d\tau} &= o && (56) \\ \frac{dQ_2}{d\tau} &= 2 - \frac{h}{\pi} \quad P_2^{\ -1/2} \cos Q_2 \sin 2\pi Q_3 && (57) \\ \frac{dQ_3}{d\tau} &= P_3 && (58) \end{aligned} \right.$$

The equations (53), (55), (56) correspond to the equations of motion (31), (32), (30). Equation (58) is identical with the equation (29) defining p_z. The equations (54) and (57) describe the evolution of the zero order approximations of the Larmor square radius and phase angle. The conservation of the kinetic energy

$$\frac{d}{d\tau}\left[\left(\frac{dx}{d\tau}\right)^2+\left(\frac{dy}{d\tau}\right)^2+\left(\frac{dz}{d\tau}\right)^2\right]=0 \tag{59}$$

results from these same equations.

It can therefore be concluded that the transformation $(x, y, z; p_x, p_y, p_z) \rightarrow (Q_1, Q_2, Q_3; P_1, P_2, P_3)$ is canonical.

III. DETERMINATION OF A GENERALIZED INVARIANT OF MOTION

Since the Hamiltonian H does not depend explicitly on time, any function J, for which $(J, H) = O$, is a constant. So if J is developped following the powers of h

$$J = \sum_{n=o}^{\infty} h^n J_n \tag{60}$$

the expansion of the Poisson bracket leads to a set of recurrence equations

$$(H_o, J_o) = O \tag{61}$$

$$(H_o, J_1) + (H_1, J_o) = O \tag{62}$$

$$(H_o, J_2) + (H_1, J_1) + (H_2, J_o) = O \tag{63}$$

$$(H_o, J_3) + (H_1, J_2) + (H_2, J_1) = O \tag{64}$$

......

$$\sum_{k=o}^{n} (H_k, J_{n-k}) = O, H_k = O \text{ for } k > 2 \tag{65}$$

Any absolute invariant of motion is necessarily a combination $J(Q_1, P_1, H)$ of the constants Q_1, P_1 and H. Two obvious solutions of this type are $J = H$ or $J_o = H_o, J_1 = H_1$, $J_2 = H_2, J_{k>2} = O$ and $J = \sum_{i=o}^{\infty} h^i J_i$ where all the J_i are some arbitrary functions $J_i(Q_1, P_1)$of Q_1 and P_1. Of course, all the invariants of this type are consequences of the conservation of the kinetic energy.

Besides these absolute invariants we will determine generalized adiabatic invariants which are only slightly varying quantities along the orbit of the particle. For instance, in the case of a nearly uniform magnetic field, the magnetic moment, $\frac{1}{2} mV_\perp^2/B$, is a well-known adiabatic invariant. It can be considered as a zero order invariant and identified with J_o. When the characteristic length of the field inhomogeneity is comparable to the distance the particle travels in a Larmor period (i.e. near resonance), this zero order invariant is poorly conserved and higher order terms hJ_1, $h^2 J_2$,... should be considered in the series expansion (60) defining J.

The problem is then to determine a quantity J (or J_o, J_1, J_2, ...) which generalizes the magnetic moment near or at the resonance condition. This condition is given by

$$V_{/\!/} = \frac{dZ}{dt} = \frac{\omega\lambda}{2\pi} \quad \text{or} \quad v_{/\!/} = \frac{dz}{d\tau} = \frac{1}{\pi} \tag{66}$$

In the following paragraphs solutions for J_o and J_1 will be determined and the general form of the equation governing J_k will be given.

a) Zero order adiabatic invariant : J_O

J_o is a solution of the equation

$$\sum_{i=1}^{3} \left(\frac{\partial H_o}{\partial Q_i} \cdot \frac{\partial J_o}{\partial P_i} - \frac{\partial H_o}{\partial P_i} \cdot \frac{\partial J_o}{\partial Q_i} \right) = O \tag{61}$$

which explicitly becomes

$$2 \frac{\partial J_o}{\partial Q_2} + P_3 \frac{\partial J_o}{\partial Q_3} = O \qquad (67)$$

The general solution of this equation is any arbitrary function of Q_1, P_1, P_2, P_3 and $P_3 Q_2 - 2Q_3$. Since in the zero order approximation, the parallel and perpendicular energies are conserved, P_2 (i.e. $r^{(o)2}$ and P_3 (i.e. v_z) are constant. In the same approximation $P_3 Q_2 - 2Q_3$ (i.e. $v_z \phi^{(o)} - 2z$) vanishes. Therefore we can consider J_o only as a function of P_3.

b) First order adiabatic invariant : $J_O + hJ_1$

The first order term J_1 is a solution of the equation (62). When J_o depends only on P_3, this equation becomes

$$2 \frac{\partial J_1}{\partial Q_2} + P_3 \frac{\partial J_1}{\partial Q_3} = - 4 P_2{}^{1/2} \cos Q_2 \cdot \cos 2\pi Q_3 \cdot \frac{\partial J_o}{\partial P_3} \qquad (68)$$

With the transformation $(Q_2, Q_3, P_3) \to (s, q, p)$

$$Q_2 = \frac{2(s + q)}{p} \quad , \quad Q_3 = q \quad , \quad P_3 = p \qquad (69)$$

equation (68) becomes

$$p \frac{\partial J_1}{\partial q} = - 4 P_2{}^{1/2} \cos 2(\frac{s + q}{p}) \cdot \cos 2\pi q \cdot \frac{\partial J_o}{\partial p} \qquad (70)$$

Since J_o is a known function of p (or P_3), an analytical expression of J_1 can be obtained by an integration over q. When P_3 (or p) $\neq \pm 1/\pi$, the solution of (70) is

$$J_1 = P_2{}^{1/2} \frac{\partial J_o}{\partial P_3} \frac{1}{(1 - \pi^2 P_3^2)} (2\pi P_3 \sin 2\pi Q_3 \cos Q_2 - 2 \cos 2\pi Q_3 \sin Q_2) \qquad (71)$$

For $P_3 = \pm 1/\pi$, J_1 is a discontinuous function. Near these resonances J_1 diverges unless $\frac{\partial J_o}{\partial P_3}$ goes to zero at least as $(1 - \pi^2 P_3^2)$ when $P_3 \to \pm 1/\pi$. This requirement limits the choice of the function $J_o(P_3)$. A suitable choice for J_o is

$$J_o = \left(1 - \frac{b_o^2}{\beta^2}\right) + \left(1 - \frac{\beta^2}{2}\right) \arcsin\left(-\frac{b}{\beta}\right) - \frac{1}{2} b(\beta^2 - b^2)^{1/2}$$

$$- \left(1 - \frac{\beta^2}{2}\right) \arcsin\left(-\frac{b_o}{\beta}\right) + \frac{1}{2} b_o(\beta^2 - b_o^2)^{1/2} \tag{72}$$

where

$$b = \pi P_3 \tag{73}$$

$$\beta = \pi v \tag{74}$$

b_o being the initial value of b. When h = O, $b \equiv b_o$ and J_o is identical to the dimensionless magnetic moment

$$\xi = 1 - \frac{b^2}{\beta^2} = 1 - \frac{P_3^2}{v^2} \tag{75}$$

It follows from (72) that

$$\frac{\partial J_o}{\partial P_3} = -\frac{1 - \pi^2 P_3^2}{v\, \xi^{1/2}} \tag{76}$$

vanishes as $1 - \pi^2 P_3^2$ at the resonance points $P_3 = \pm 1/\pi$.

In J_1, one needs only a zero order relation (in h) between P_2 and ξ which is (see eq. 37)

$$P_2 = \frac{1}{4} \quad \xi v^2 \tag{77}$$

as in the case of a uniform magnetic field. According to eq. (71)

$$J_1 = \cos 2\pi Q_3 \sin Q_2 - \pi P_3 \sin 2\pi Q_3 \cos Q_2 \tag{78}$$

Therefore, for this particular choice of $J_o(P_3)$, the first order approximation of the adiabatic invariant $J = J_o + hJ_1$ remains finite even at the resonances.

c) Higher order approximations of the adiabatic invariant : $J = \sum_{k=o}^{n} h^k J_k, \; n > 1$

By means of the transformation (69), the differential equation (65) governing J_n has the form

$$p \frac{\partial J_n}{\partial q} = \frac{1}{\pi} \; P_2^{-1/2} \cos 2\left(\frac{s+q}{p}\right) \;.\; \sin 2\pi q \;.\; \left(\frac{\partial J_{n-1}}{\partial Q_2}\right)_{Q_2 \,=\, 2\left(\frac{s+q}{p}\right)}$$

$$+ \frac{2}{\pi} \; P_2^{\,1/2} \sin 2\left(\frac{s+q}{p}\right) \;.\; \sin 2\pi q \;.\; \frac{\partial J_{n-1}}{\partial P_2} - 4\, P_2^{\,1/2} \cos 2\left(\frac{s+q}{p}\right) \;.$$

$$\cos 2\pi q \;.\; \frac{\partial J_{n-1}}{\partial p} + \frac{1}{\pi} \sin 4\pi q \;.\; \frac{\partial J_{n-2}}{\partial p} \tag{79}$$

The successive J_n (s, q, p, P_2) can, in principle, be determined by a simple integration over q. The problem is similar to that analysed by Dunnett and Jones (1972) for a sinusoidally modulated magnetic field with axial symmetry. In the next section, we limit the development to the first order approximation $J_o + hJ_1$.

IV. NUMERICAL RESULTS

In order to check if the choice (72) of J_o (P_3) leads to an appropriate invariant of motion, we compare in this section, the value of ξ along the orbit of the particle with the value of ξ determined from the algebraic equation $J = C$ where the constant C is determined by the initial condition $C = J[(Q_3)_o, (Q_2)_o, \xi_o]$.

First, consider a particle injected at the origin (z = o) with a velocity $V = \omega\lambda/2\pi$ (i.e. $v = 1/\pi$) parallel to the magnetic field direction O_z. The amplitude of the magnetic field modulation is h = 0.025. The orbit of this particle, obtained by integrating Eqs. (32) to (34), is illustrated in Figs. 2 and 3. Since the initial velocity along Oz satisfies the resonance condition (66), the particle acquires an appreciable perpendicular velocity $v_\perp$ (Fig. 3) and its Larmor radius (Fig. 2) increases during the first 7 Larmor periods. After about the 13th Larmor period the initial conditions are more or less recovered and a new increase of the transversal kinetic energy is again observed during the 7 following periods. Obviously in this case the magnetic moment is not conserved.

The trajectories of the particle for different initial conditions can also be represented in a two dimensional phase space by evaluating the quantity ξ at the point P_n corresponding to successive periods of the magnetic field modulation, i.e. for $Z = n\lambda$ (or $Q_3 = z = n$), n = 0, 1, 2, ... For these points the magnetic field B has always the same value, and ξ is therefore proportional to the magnetic moment. The particle motion can then be described in a plane by the two parameters Q_2 (phase angle) and ξ (magnetic moment).

Figure 4 shows for h = 0.025 and $v = 1/\pi$, the values of ξ and Q_2 at the successive points P_n. The points $(Q_2, \xi)_n$ in the phase plane (Q_2, ξ) are located on different curves, each of them corresponding to a definite set of initial conditions $[(Q_2)_o, \xi_o]$. These curves demonstrate the existence of a functional relationship between ξ and Q_2 and indicate that an invariant exists even when the usual magnetic moment is not conserved as it is the case for the low values of ξ where the closed curves show the resonance phenomenon. For large values of ξ, i.e. for large pitch angles, the curves approximate to curves of constant ξ. In this case, the magnetic moment is an adiabatic invariant. This could be expected since for large

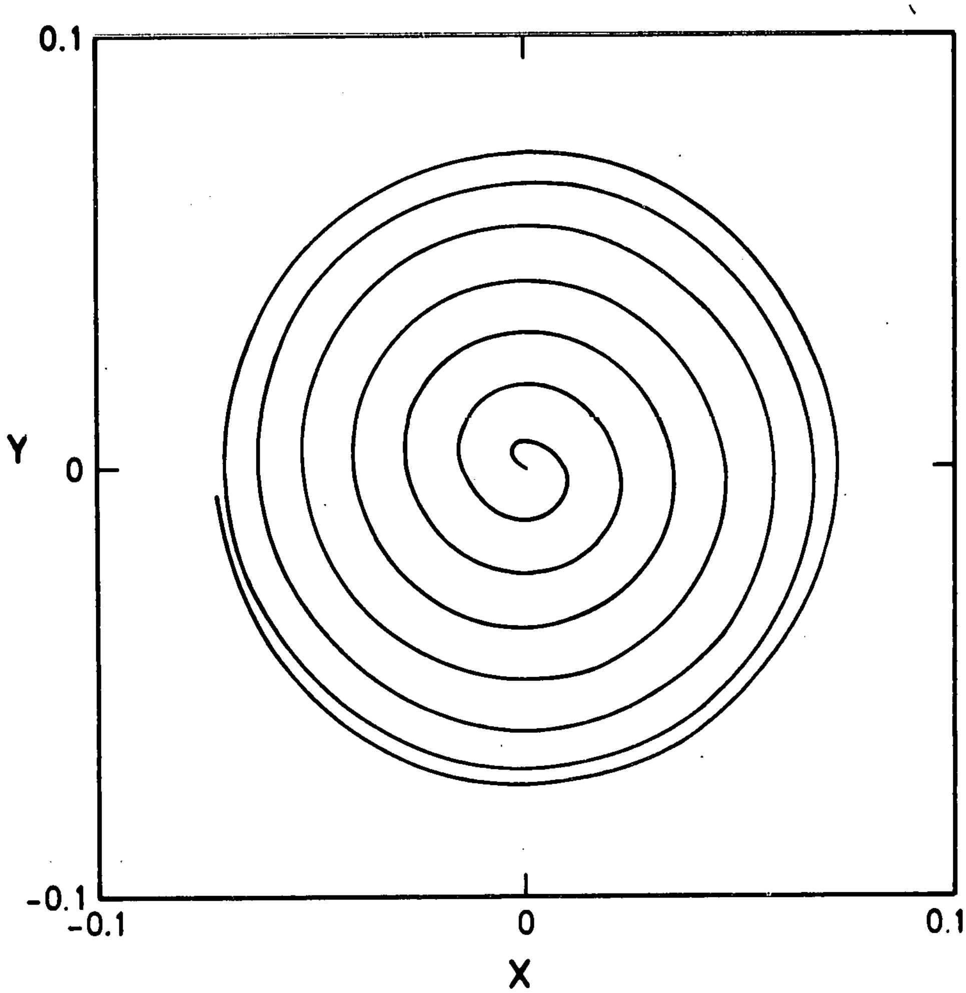

Fig. 2.- Particle orbit in a plane perpendicular to the magnetic field. The particle initially at the origin is injected along the magnetic field z-axis with a velocity $v = 1/\pi$.

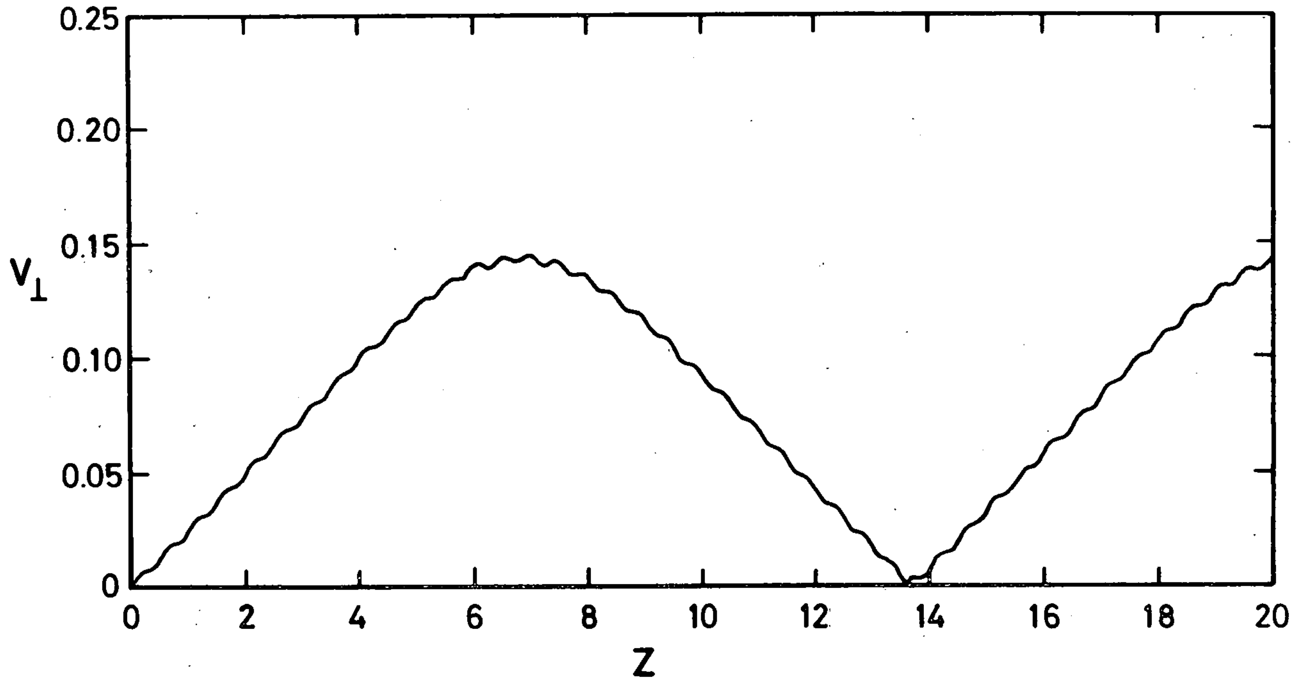

Fig. 3.- Variation of the perpendicular velocity $v_\perp$. The particle is initially injected along the z-axis with a velocity $v = 1/\pi$.

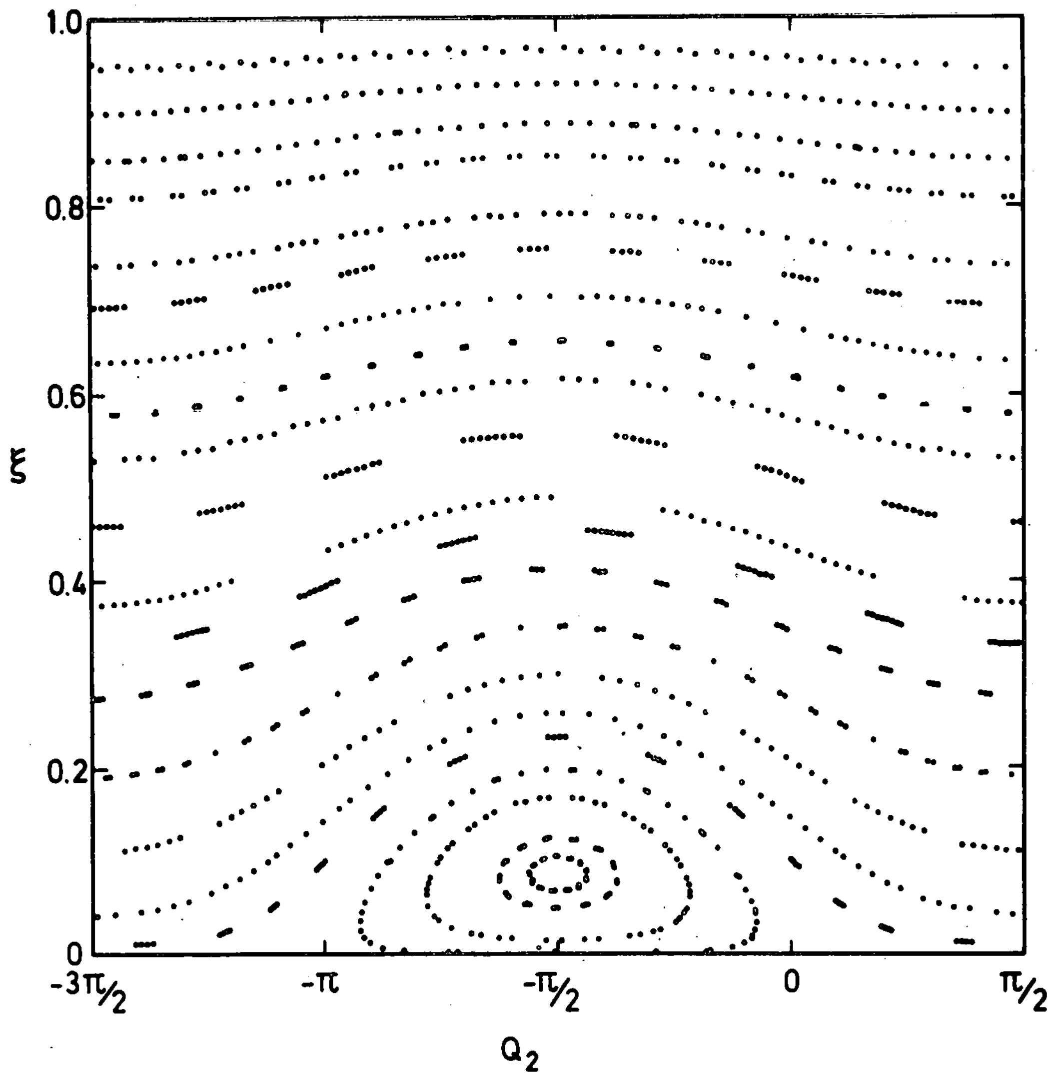

Fig. 4.- Integrated orbits in the phase plane (Q_2, ξ), $h = 0.025$, $v = 1/\pi$.

ξ. V_z is small and the zero order adiabatic criterium is $\dfrac{2\pi V_z}{\omega} \ll \lambda$.

The orbits represented in Fig. 5 are obtained for the same value of h and for a two time larger value of the velocity $v = 2/\pi$. It can be seen that for small values of ξ, i.e. for small pitch angles, the invariant curves are approximately horizontal straight lines. In this case, the magnetic moment and J_o can be considered as good adiabatic invariants. However, for larger values of ξ or of the pitch angles, the closed curves demonstrate the existence of a resonance. This resonance occurs for $\xi \sim 0.75$ (i.e., $v_z \sim 1/\pi$) and $Q_2 = -\pi/2$.

All the results illustrated in Figs. 4 and 5 have been obtained by numerical integration of the equations of motion (Eqs. 53 to 58) by a Runge-Kutta method.

A similar representation of the orbit in the (Q_2, ξ) plane can be obtained from the expression of the invariant $J = J_o + hJ_1$ where J_o and J_1 are given respectively by Eqs (72) and (78). In this representation J must be expressed in terms of the variables (Q_2, ξ). P_3 is the only dynamical variable included in J_o and J_1 and this is exactly $v(1 - \xi)^{1/2}$.

Curves $J_o + hJ_1 = C$ are shown in Figs. 6 and 7 for $v = 1/\pi$ and $v = 2/\pi$ respectively, with the same parameters ($h = 0.025$, $Q_3 = n$) as used in Figs. 4 and 5. From the comparaison of Figs. 4 and 6, or Figs. 5 and 7, it can be seen that the invariant curves calculated by a perturbation theory show satisfactory agreement with the exact results obtained from the integration of the equations of motion.

This shows that the expression of $J_o + hJ_1$ defines a satisfactory first order invariant which can then be used for predicting the variation of the pitch angle along the trajectory of the particle, without integrating differential equations.

CONCLUSIONS

When a charged particle interacts with an electromagnetic wave its magnetic moment is not an adiabatic invariant, especially when its velocity satisfies the resonance condition.

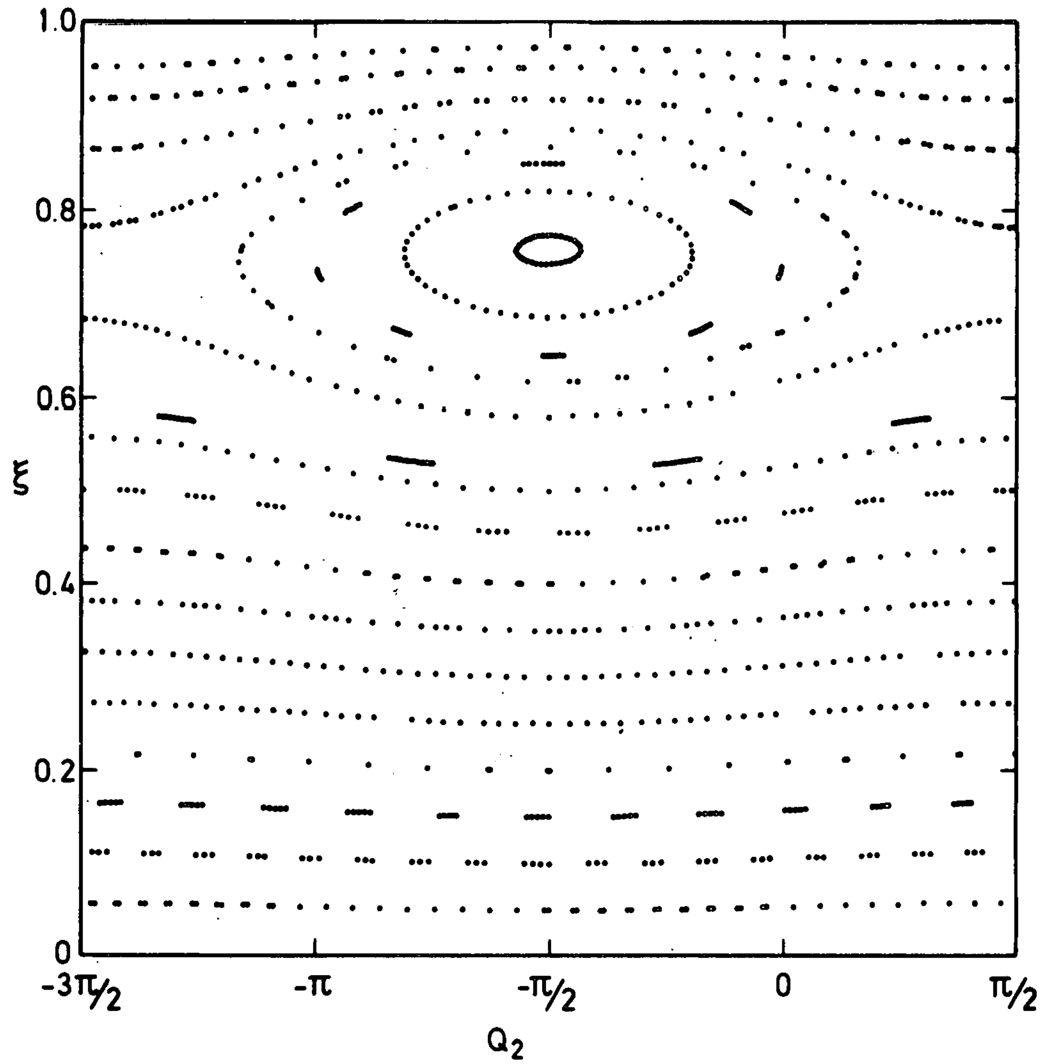

Fig. 5.- Integrated orbits in the phase plane (Q_2, ξ), $h = 0.025$, $v = 2/\pi$.

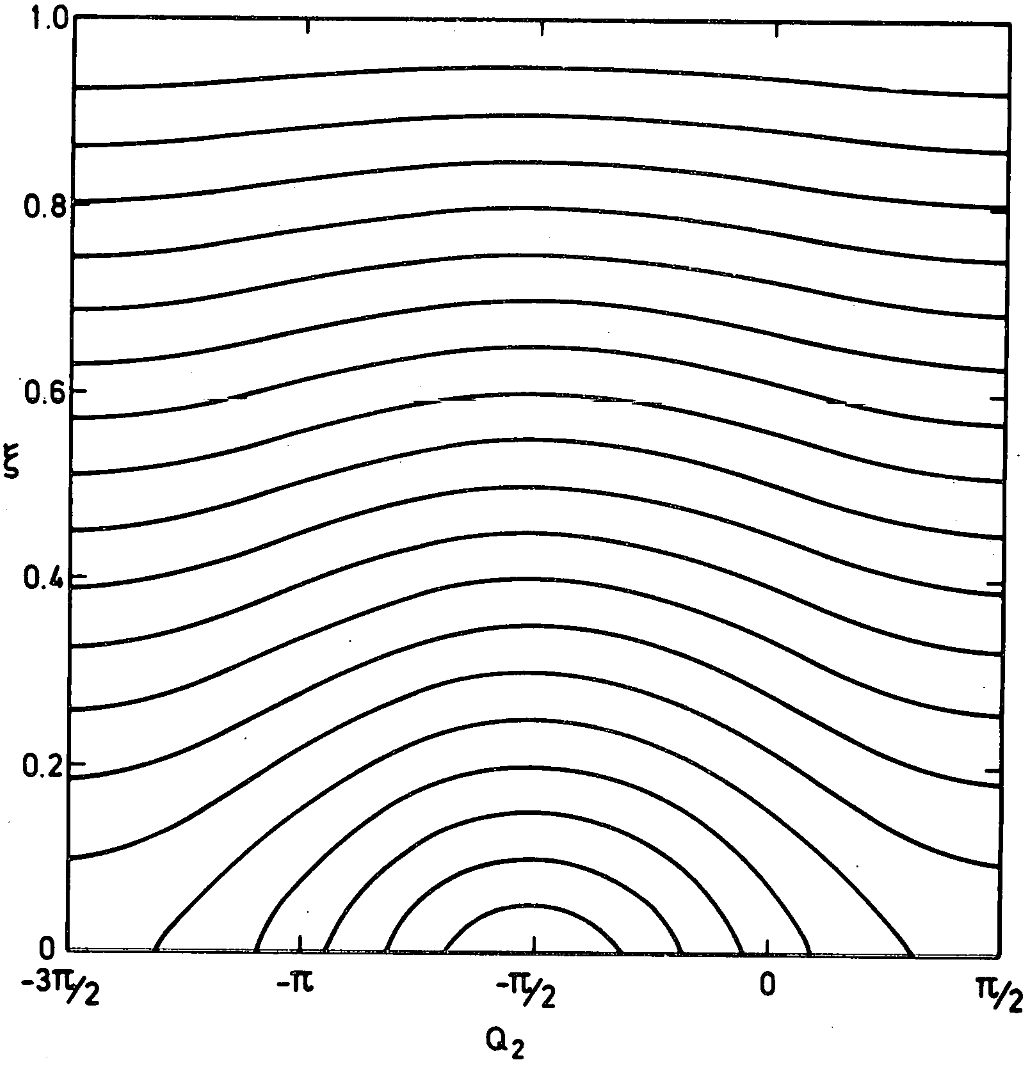

Fig. 6.- Invariant curves $J_o + hJ_1 = C$, $h = 0.025$, $v = 1/\pi$.

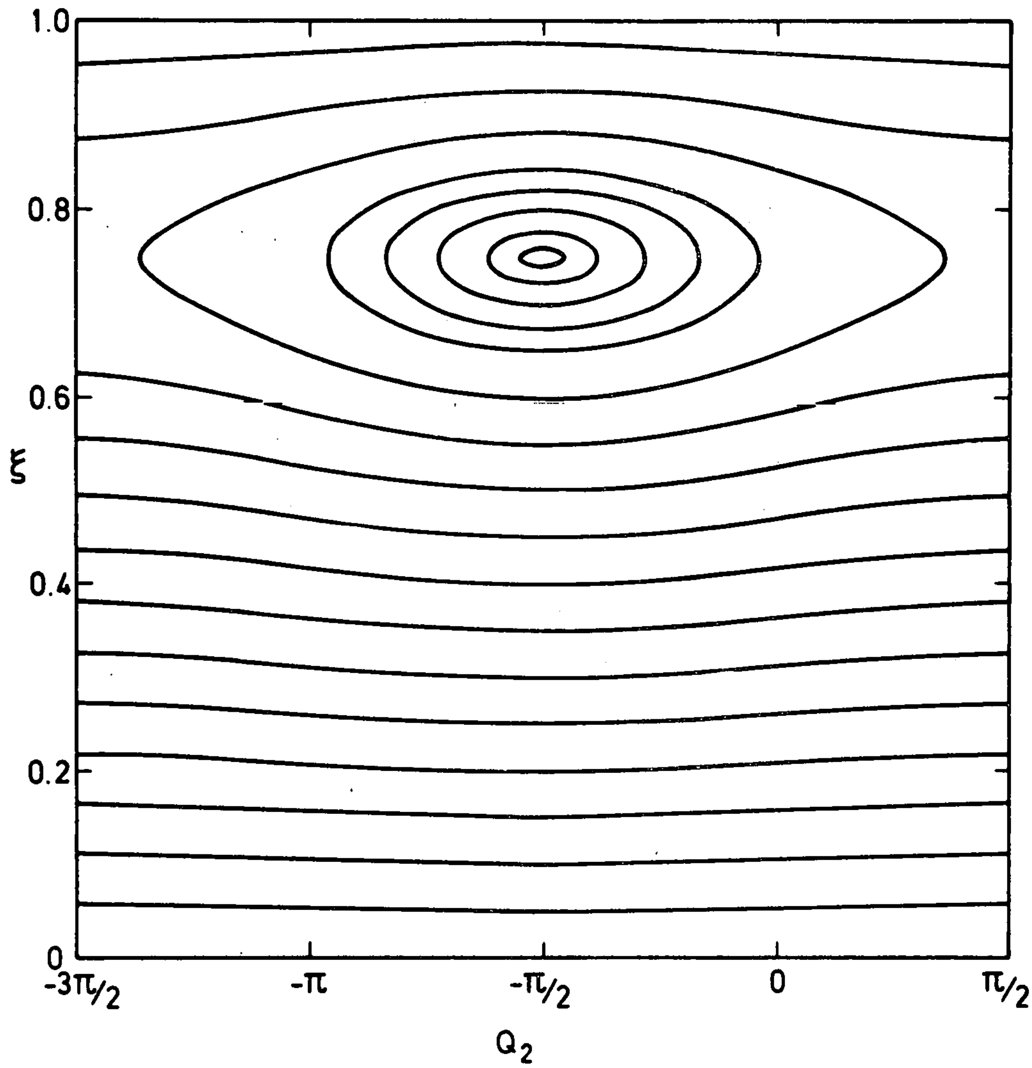

Fig. 7.- Invariant curves $J_o + hJ_1 = C$, $h = 0.025$, $v = 2/\pi$.

For a linearly polarized hydromagnetic plane wave propagating in the direction of a uniform magnetic field, we have obtained an analytical expression for a first order invariant which reduces to the magnetic moment when the modulation amplitude tends to zero. The comparaison of the invariant curves obtained by a perturbation method (Figs. 6 and 7) with the exact results calculated from the equations of motion (Figs. 4 and 5) proves the validity of the theory even when the resonance condition is satisfied.

The invariant is obtained as a series of powers of the modulation amplitude The coefficients can be deduced from a set of recurrence equations. For small values of the modulation amplitude this series can be limited to the first order term.

For larger modulation amplitudes higher order terms are needed. Using a suitable canonical transformation it is indicated how these higher order terms can be deduced.

This theory can be useful for the study of the magnetic interaction of a charged particle with Alfvèn waves. It can be used to describe the variation of the pitch angle along the particle trajectory without solving the equations of motion.

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