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On the choice of boundary conditions for the matching

of kinetic to hydrodynamic polar wind models

by

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FOREWORD

The article "On the Choice of Boundary Conditions for the Matching of Kinetic to Hydrodynamic Polar Wind Models" will be published in the "Bulletin de la Classe des Sciences de l'Académie Royale de Belgique".

AVANT-PROPOS

Le travail intitué "On the Choice of Boundary Conditions for the Matching of Kinetic to Hydrodynamic Polar Wind Models" sera publié dans le Bulletin de la Classe des Sciences de l'Académie Royale de Belgique.

VOORWOORD

De tekst getiteld "On the Choice of Boundary Conditions for the Matching of Kinetic to Hydrodynamic Polar Wind Models" zal gepubliceerd worden in het "Bulletin de la Classe des Sciences de l'Académie Royale de Belgique".

ON THE CHOICE OF BOUNDARY CONDITIONS FOR THE MATCHING

OF KINETIC TO HYDRODYNAMIC POLAR WIND MODELS

by

M. SCHERER

Abstract

Kinetic polar wind models based on an asymmetric anisotropic Maxwellian at the exobase depend on four parameters which in principle can be determined by imposing fixed values for the number density, the flow speed, the temperature and the temperature anisotropy at the exobase. It is shown however that not any arbitrary set of values for these quantities can be chosen but that some forbidden regions do exist.

Résumé

Un modèle cinétique du vent polaire, basé sur une distribution de vitesse maxwellienne asymétrique et anisotropique, dépend de quatre paramètres qui, en principe, peuvent être déterminés en fixant à l'exobase les valeurs de la densité, de la vitesse d'écoulement, de la température et de l'anisotropie. Il est démontré cependant que ces quatre valeurs ne peuvent pas être choisies arbitrairement, mais qu'il existe certains domaines défendus.

Samenvatting

Kinetische poolwindmodellen steunend op een asymmetrische aniso-trope Maxwellverdeling aan de exobasis, bevatten vier parameters die in principe bepaald kunnen worden door het toekennen van een welbe-paalde waarde aan de deeltjesdichtheid, de stroomsnelheid, de tempera-tuur en de temperatuursanisotropie aan de exobasis. Er wordt aange-toond dat een willekeurige keuze van waarden voor deze grootheden niet kan gemaakt worden, maar dat er verboden gebieden bestaan.

1. INTRODUCTION

It is well known that in the polar regions of the Earth the light ions (H^+ and He^+) escape along the open geomagnetic field lines. This phenomena called the Polar Wind (Axford, 1968) has been described originally by means of a hydrodynamic approach (Banks and Holzer, 1968). Later on a kinetic approach was developed (Lemaire and Scherer, 1970) and it was shown that both methods are complementary (Donahue, 1971; Lemaire and Scherer, 1973) : the hydrodynamic equations are valid in the collision dominated region called barosphere, whereas the kinetic approach is appropriate in the collisionfree exosphere. These two regions are separated by a small transition region in which the number of collisions rapidly decreases with height. This intermediate region which has a thickness of only a few scale heights, is generally neglected by assuming that the barosphere and the exosphere are separated by a sharply defined surface called the exobase or baro-pause.

Lemaire and Scherer (1975) calculated a continuous polar wind model by matching at the exobase the solutions of the hydrodynamic Navier-Stokes equations for the hydrogen ions, to a simple kinetic model. For this purpose it was assumed that the velocity distribution of the hydrogen ions was given by a truncated Maxwellian in which only pitch angles in the upward loss cone were taken into account. Although the calculated number density, escape flux, temperature, temperature anisotropy and energy flux were continuous functions of altitude, a major shortcoming of the model was that due to the assumption concerning the velocity distribution at the exobase, the hydrogen temperature anisotropy always equals the physical unrealistic constant value $1 - \frac{2}{\pi}$.

Later on Scherer (1978) developped a more sophisticated kinetic polar wind model, based on a truncated asymmetric anisotropic Maxwellian

$$f[r_o, \vec{v}(r_o)] = \pi^{-3/2} N \beta_{\parallel}^{1/2} \beta_{\perp} \exp[-\beta_{\parallel}(v_{\parallel} - u)^2 - \beta_{\perp} v_{\perp}^2] \quad (1)$$

with $\quad \beta_{\parallel} = m/2kT_{\parallel} \; ; \; \beta_{\perp} = m/2kT_{\perp} \quad\quad (2)$

where r_o is the radial distance of the exobase, $\vec{v}(r_o)$ is the velocity vector at the exobase of a particle with mass m, $v_{\parallel}$ and $v_{\perp}$ are the components of this vector respectively parallel and perpendicular to the geomagnetic field, k is the Boltzmann constant.

N, u, $T_{\parallel}$ and $T_{\perp}$ are four parameters related to the number density $n(s_o)$, the flow speed $w(s_o)$, the temperature $T(s_o)$, and the temperature anisotropy $a(s_o)$ at the exobase, through the relations (Scherer, 1978) :

$$n(s_o) = \frac{1}{2} N \exp(-U^2) \, erfex(-U) \quad\quad (3)$$

$$w(s_o) = \frac{1}{2} c_o [1 + \sqrt{\pi} U \, erfex(-U)]/erfex(-U) \quad\quad (4)$$

$$T(s_o) = \frac{1}{3} T_{\perp} \left[(2+t) \, erfex(-U) - \frac{2 t}{\pi \, erfex(-U)} - \frac{2}{\sqrt{\pi}} tU \right]/erfex(-U) \quad\quad (5)$$

$$a(s_o) = t \left\{ 1 - \frac{2 U}{\sqrt{\pi} \, erfex(-U)} - \frac{2}{\pi[erfex(-U)]^2} \right\} \quad\quad (6)$$

with $\quad U = \beta_{\parallel}^{1/2} u \; ; \quad c_o = (8kT_{\parallel}/\pi m)^{1/2} \; ; \quad t = T_{\parallel}/T_{\perp} \; ;$

$$(7)$$

$$erfex(z) = \frac{2}{\sqrt{\pi}} \exp(z^2) \int_z^{\infty} \exp(-y^2) \, dy$$

Assuming that at the exobase the number density, the flow speed, the temperature and the temperature anisotropy are known quantities the system of equations (3) to (6) can be used to determine the para- meters N, U, $T_{\parallel}$ and $T_{\perp}$. Once that these values are fixed any moment of the velocity distribution in the exosphere is assigned (Scherer, 1978). In order to have physical acceptable solutions however, the values of N, $T_{\parallel}$ and $T_{\perp}$ must be positive and U must be real.

In Sec. 2 it will be shown that as a consequence of these limitations not any set of values $n(s_o)$, $w(s_o)$, $T(s_o)$ and $a(s_o)$ can be chosen as boundary conditions at the exobase, but that there exist forbidden values which do not give physical acceptable solutions. Finally, in Sec. 3 to 5 the choice of the boundary conditions at the exobase, for kinetic polar wind models based respectively on an asymmetric Maxwellian ($T_{\parallel} = T_{\perp}$, $U \neq 0$), a bi-Maxwellian ($T_{\parallel} \neq T_{\perp}$, $U = 0$), and a Maxwellian ($T_{\parallel} = T_{\perp}$, $U = 0$) will also be considered.

2. DISCUSSION OF THE SYSTEM OF EQUATIONS (3) to (6)

Straightforward calculations show that the system of equations (3) to (6) is equivalent to

$$N = 2n(s_o) \exp (U^2)/\mathrm{erfex}(-U) \tag{8}$$

$$T_{\parallel} = \frac{\pi m}{2k} \cdot \frac{w^2(s_o)}{\ell^2(U)} \tag{9}$$

$$T_{\perp} = \frac{3T(s_o)}{2 + a(s_o)} \tag{10}$$

$$T_{\parallel} = \frac{3T(s_o)}{g(U)} \cdot \frac{a(s_o)}{2 + a(s_o)} \tag{11}$$

where the following shorthand notations were used :

$$\ell(U) = \sqrt{\pi} \, U + \frac{1}{\mathrm{erfex}(-U)} \tag{12}$$

$$g(U) = 1 - \frac{2}{\sqrt{\pi}} \frac{U}{\mathrm{erfex}(-U)} - \frac{2}{\pi} \left[\frac{1}{\mathrm{erfex}(-U)} \right]^2 \tag{13}$$

Equation (10) immediately yields the parameter value $T_{\perp}$.
Elimination of $T_{\parallel}$ between the equations (9) and (11), produces the transcendental equation

$$\frac{g(U)}{\ell^2(U)} = \frac{6k}{\pi m} \cdot \frac{a(s_o)}{2 + a(s_o)} \cdot \frac{T(s_o)}{w^2(s_o)} \qquad (14)$$

from which U can be determined. Once the solution U known, the values of N and $T_\parallel$ are calculated by means of respectively the equations (8) and (9) or (11).

It can be shown that for real values of U the left-hand side of equation (14) is a monotonic decreasing function, and that

$$\lim_{U \to -\infty} \frac{g(U)}{\ell^2(U)} = \frac{2}{\pi} \; ; \qquad \frac{g(0)}{\ell^2(0)} = 1 - \frac{2}{\pi}$$

$$\qquad (15)$$

$$\lim_{U \to +\infty} \frac{g(U)}{\ell^2(U)} = 0 \; .$$

This yields the condition

$$0 < \frac{a(s_o)}{2 + a(s_o)} \cdot \frac{T(s_o)}{w^2(s_o)} < \frac{m}{3k} \qquad (16)$$

which shows that at the exobase the flow speed, the temperature and the temperature anisotropy can not be chosen arbitrarly.

For a fixed value of $w(s_o)$, the inequality (16) expresses that in the $[T(s_o), a(s_o)]$- plane only the points in the region below the orthogonal hyperbola $a(s_o) T(s_o) = \frac{m}{3k} w^2(s_o)[2+a(s_o)]$ can be used as boundary conditions at the exobase. This is illustrated in fig. 1 in the case of hydrogen ions, for $w(s_o) = 2; 2.5; 3; \ldots, 10 \text{ km s}^{-1}$.

On the other hand the inequality (16) can also be studied in the $[a(s_o), w(s_o)]$-plane for a fixed value of $T(s_o)$. This is illustrated in fig. 2 in the case of hydrogen ions, for $T(s_o) = 1000, 2500, 5000, 7500$ and 10000 K. Only the points above the curves correspond to acceptable
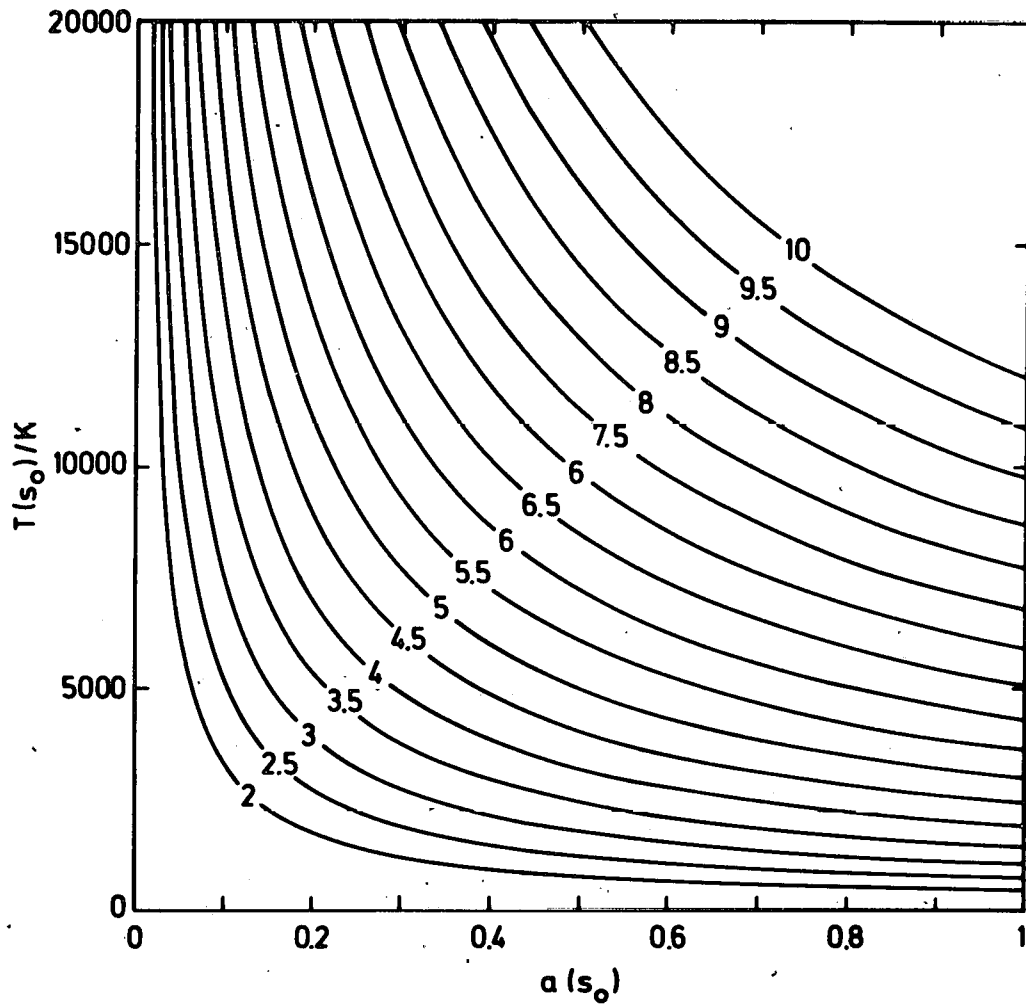
Fig. 1.- Only the points $[a(s_o), T(s_o)]$ below the orthogonal hyperbolae corresponding to the values $w(s_o)$ = 2; 2.5; 3; ...; 10 km s^{-1}, are suitable boundary conditions at the exobase, for kinetic hydrogen polar wind models based on an anisotropic asymmetric Maxwellian.
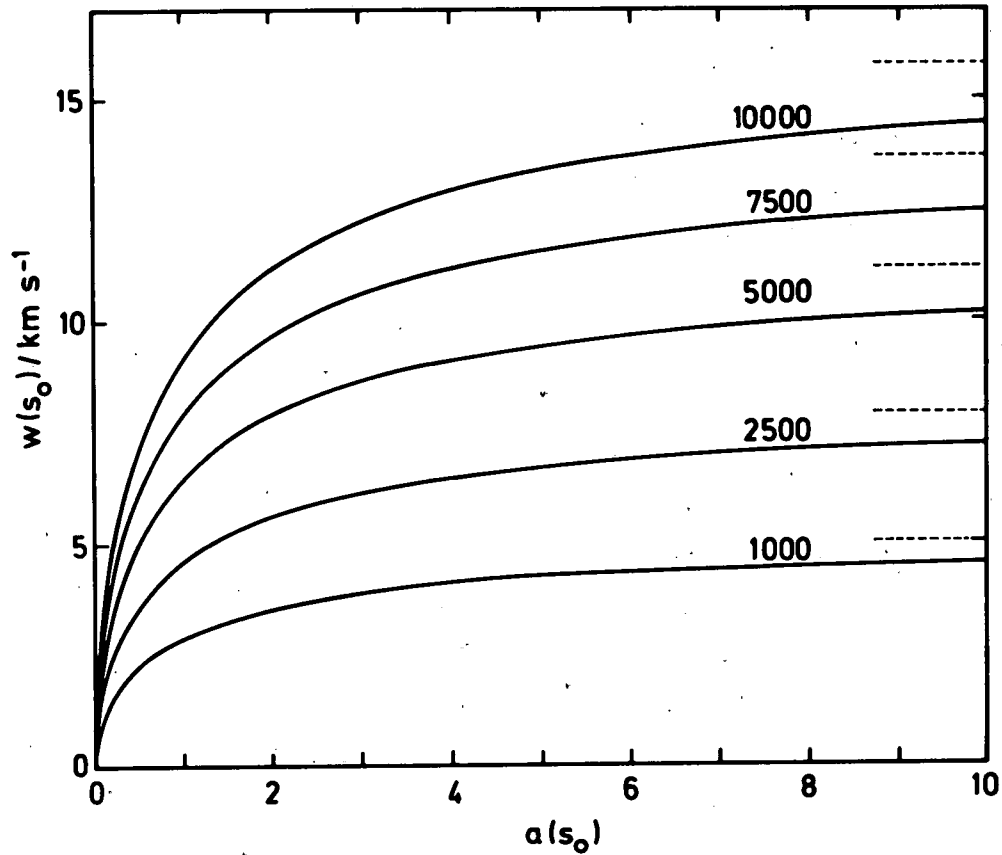
Fig. 2.- Only the points $[a(s_o), w(s_o)]$ above the curves corresponding to the values $T(s_o) = 1000$, 2500, 5000, 7500 and 10000 K are suitable boundary conditions at the exobase, for kinetic hydrogen polar wind models based on an anisotropic asymmetric Maxwellian. The dotted lines correspond to the horizontal asymptotes of the curves.
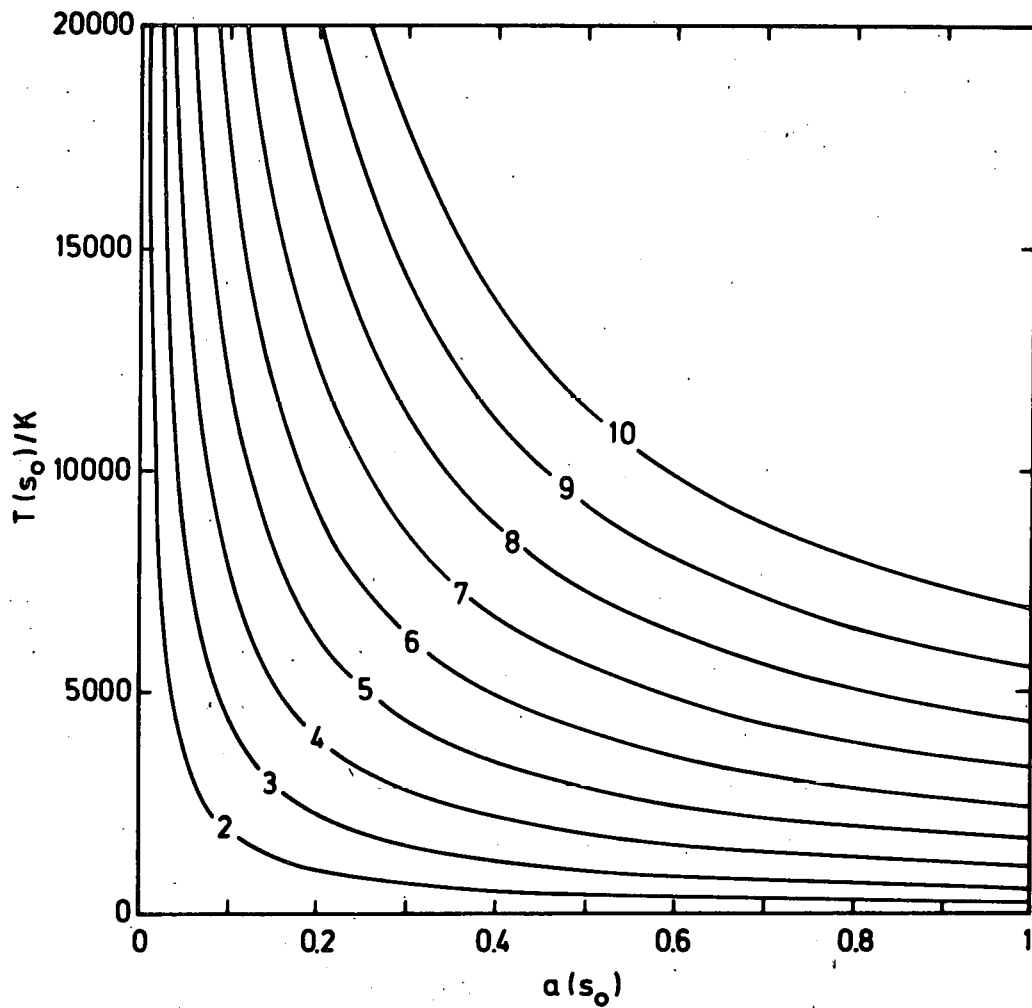
Fig. 3.- The points $[a(s_o), T(s_o)]$ below the orthogonal hyperbolae corresponding to the values $w(s_o) = 2$; 3; ...; 10 km s^{-1}, yield positive values for the asymmetry parameter in the velocity distribution at the exobase.

boundary conditions. For flow speeds at the exobase exceeding the asymptotic value $w(s_0) = [3kT(s_0)/m]^{1/2}$ (illustrated by the horizontal dotted lines in fig. 2), there is no limitation on the choice of the temperature anisotropy.

Up till now any real value of U was considered. If however, one is only interested in positive values of the parameter U, it follows from (14) and (15) that the following condition must be satisfied :

$$0 < \frac{a(s_0)}{2 + a(s_0)} \cdot \frac{T(s_0)}{w^2(s_0)} < \frac{m}{6k} (\pi - 2) \qquad (17)$$

In the case of hydrogen ions inequality (17) reduces to

$$T(s_0) < 22.88 \ w^2(s_0) \left[1 + \frac{2}{a(s_0)} \right] \qquad (18)$$

where $w(s_0)$ is expressed in km s^{-1} and $T(s_0)$ in K.

Only the regions below the orthogonal hyperbolae illustrated in Fig. 3 for $w(s_0) = 2; 3;..., 10$ km s^{-1}, yield boundary conditions $T(s_0)$ and $a(s_0)$ for which positive values of U can be found.

3. CASE OF AN ASYMMETRIC MAXWELLIAN

In the special case of an asymmetric Maxwellian, i.e. $T_\parallel = T_\perp = T$ and $U \neq 0$, only 3 parameters N, T and U have to be determined. Therefore the number density $n(s_0)$, the flow speed $w(s_0)$, the temperature $T(s_0)$, and the temperature anisotropy $a(s_0)$ can no longer be chosen as boundary conditions at the exobase, but only 3 of these four quantities can be used to fix the parameters.

a.   Assuming that $n(s_0)$, $w(s_0)$ and $T(s_0)$ are given, the system of equations (3) to (5) is equivalent to the system build up by (8), (9) and

$$T = \frac{3T(s_o)}{2 + g(U)} \tag{19}$$

Elimination of $T = T_{\parallel}$ between (9) and (19) yields a transcendental equation in U

$$\frac{2 + g(U)}{\ell^2(U)} = \frac{6k}{\pi m} \frac{T(s_o)}{w^2(s_o)} \tag{20}$$

The left-hand side of equation (20) is a monotonic decreasing function of U. Moreover it can be shown that

$$\lim_{U \to -\infty} \frac{2 + g(U)}{\ell^2(U)} = +\infty \quad ; \quad \frac{2 + g(0)}{\ell^2(0)} = 3 - \frac{2}{\pi}$$

$$\tag{21}$$

$$\lim_{U \to +\infty} \frac{2 + g(U)}{\ell^2(U)} = 0$$

As a consequence of this any value of the flow speed and the temperature at the exobase will give a real value for the parameter U. The values for N and T follow then immediately from the equations (8) and (19).

It is worth-while to note that in this case, the temperature anisotropy at the exobase can not be chosen arbitrarly, but will be given by

$$a(s_o) = g(U) \tag{22}$$

Straightforward calculations show that $g(U)$ is a monotonic increasing function of U with

$$\lim_{U \to -\infty} g(U) = 0 \quad \text{and} \quad \lim_{U \to +\infty} g(U) = 1. \tag{23}$$

Therefore kinetic polar wind models based on an asymmetric Maxwellian will always have temperature anisotropies at the exobase smaller than 1.

From (20) and (21) follows also that in the $[T(s_o), w(s_o)]$-plane the parabole

$$T(s_o) = \frac{3\pi-2}{6k} \, m \, w^2(s_o) \qquad (24)$$

divides the plane in two regions : the domain below the parabola corresponds to positive U-values whereas the domain above corresponds to negative U-values. This is illustrated in fig. 4 for the case of hydrogen ions. Note also that since $g(0) = 1 - \frac{2}{\pi}$, the temperature anisotropy $a(s_o)$ satisfies the inequalities

$$1 - \frac{2}{\pi} < a(s_o) < 1 \qquad \text{for} \quad U > 0$$

$$\qquad (25)$$

$$0 < a(s_o) < 1 - \frac{2}{\pi} \qquad \text{for} \quad U < 0$$

b. Assuming that $n(s_o)$, $w(s_o)$ and $a(s_o)$ are given the system of equations (3), (4) and (6) is now equivalent to (8), (9) and (22). In order to obtain a real U-value the temperature anisotropy must be smaller than 1. Once U known, the parameters N and $T = T_{\parallel}$ follow from the equations (8) and (9). The temperature at the exobase can not be chosen arbitrary but will be given by (19).

c. Assuming that $n(s_o)$, $T(s_o)$ and $a(s_o)$ are given, the system of equations (3), (5) and (6) is equivalent to (8), (19) and (22). For values of $a(s_o)$ smaller than 1, the solution of (22) yields a real value U, whereafter N and T follow from the equations (8) and (19). The flow speed at the exobase can not be chosen arbitrary but will be given by (20).
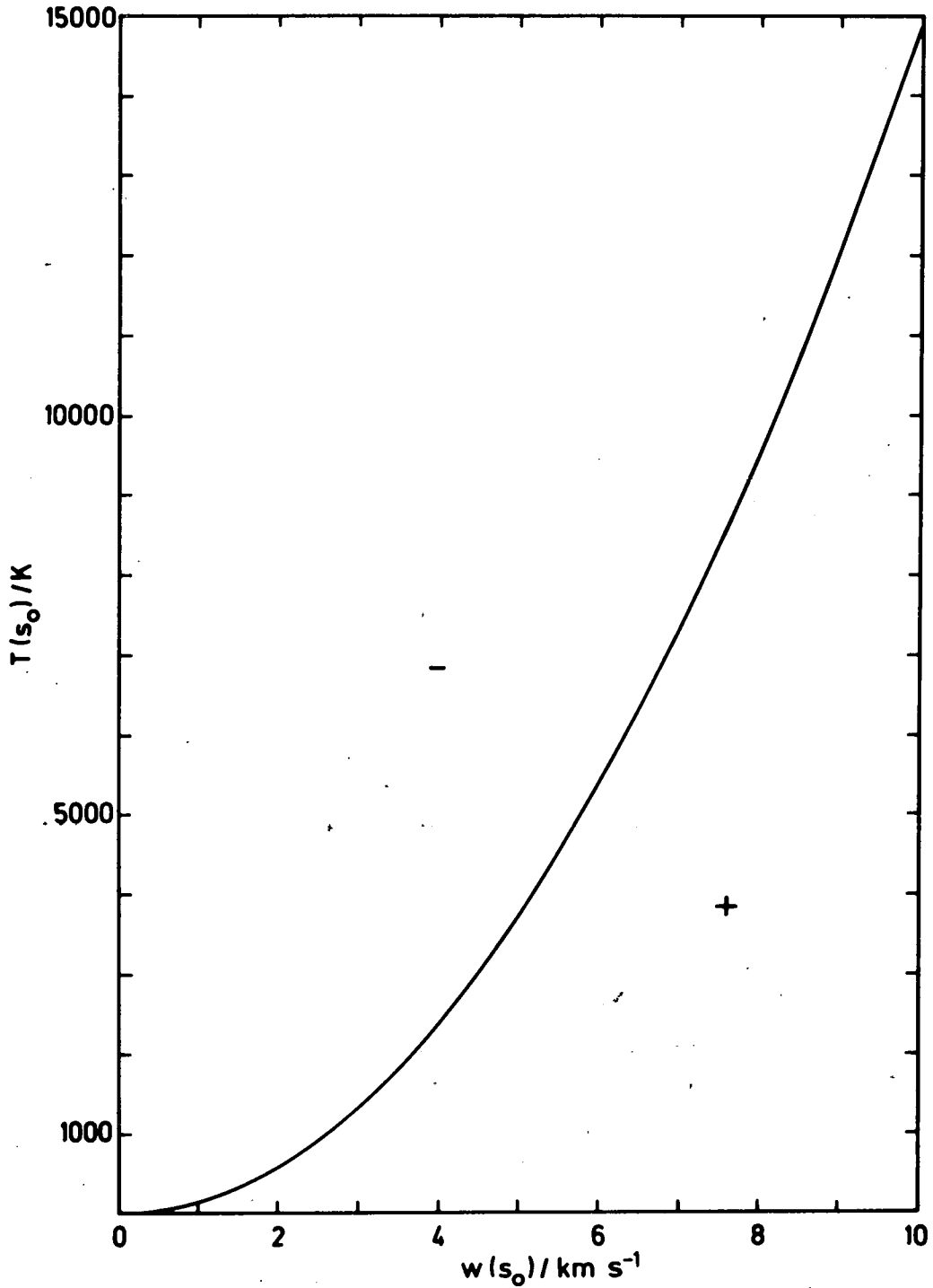
Fig. 4.- The points $[w(s_o), T(s_o)]$ below (above) the para-
bola correspond to positive (negative) values for
the asymmetry parameter in the case of kinetic
polar wind models based on an asymmetric
Maxwellian.

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4. CASE OF A BI-MAXWELLIAN

In the special case of a bi-Maxwellian (i.e. $T_{\parallel} \neq T_{\perp}$, $U = 0$) the velocity distribution (1) depends on 3 parameters, N, $T_{\parallel}$ and $T_{\perp}$ which can be determined by means of three of the four equations of the system (3) to (6).

a. Assuming that $n(s_0)$, $w(s_0)$ and $T(s_0)$ are given, the system (3) to (5) is equivalent to :

$$N = 2\, n(s_0) \tag{26}$$

$$T_{\parallel} = \frac{\pi m}{2k}\, w^2(s_0) \tag{27}$$

$$T_{\perp} = \frac{1}{2}\left[3T(s_0) - \frac{m}{2k}\,(\pi - 2)\, w^2(s_0) \right] \tag{28}$$

Since only positive values of $T_{\perp}$ can be considered, the values $T(s_0)$ and $w(s_0)$ in the region below the parabola $T(s_0) = \frac{m}{6k}\,(\pi - 2)\, w^2(s_0)$ can not be used as boundary conditions at the exobase. This is illustrated in fig. 5 for the case of hydrogen ions. The temperature anisotropy at the exobase can not be chosen but is given by

$$a(s_0) = \left(1 - \frac{2}{\pi}\right) \frac{T_{\parallel}}{T_{\perp}} \tag{29}$$

b. Assuming that $n(s_0)$, $w(s_0)$ and $a(s_0)$ are given the system of equations (3), (4) and (6) is equivalent to (26), (27) and (29). Elimination of T between (27) and (29) yields

$$T_{\perp} = \frac{m}{2k}\,(\pi - 2)\, \frac{w^2(s_0)}{a(s_0)} \tag{30}$$

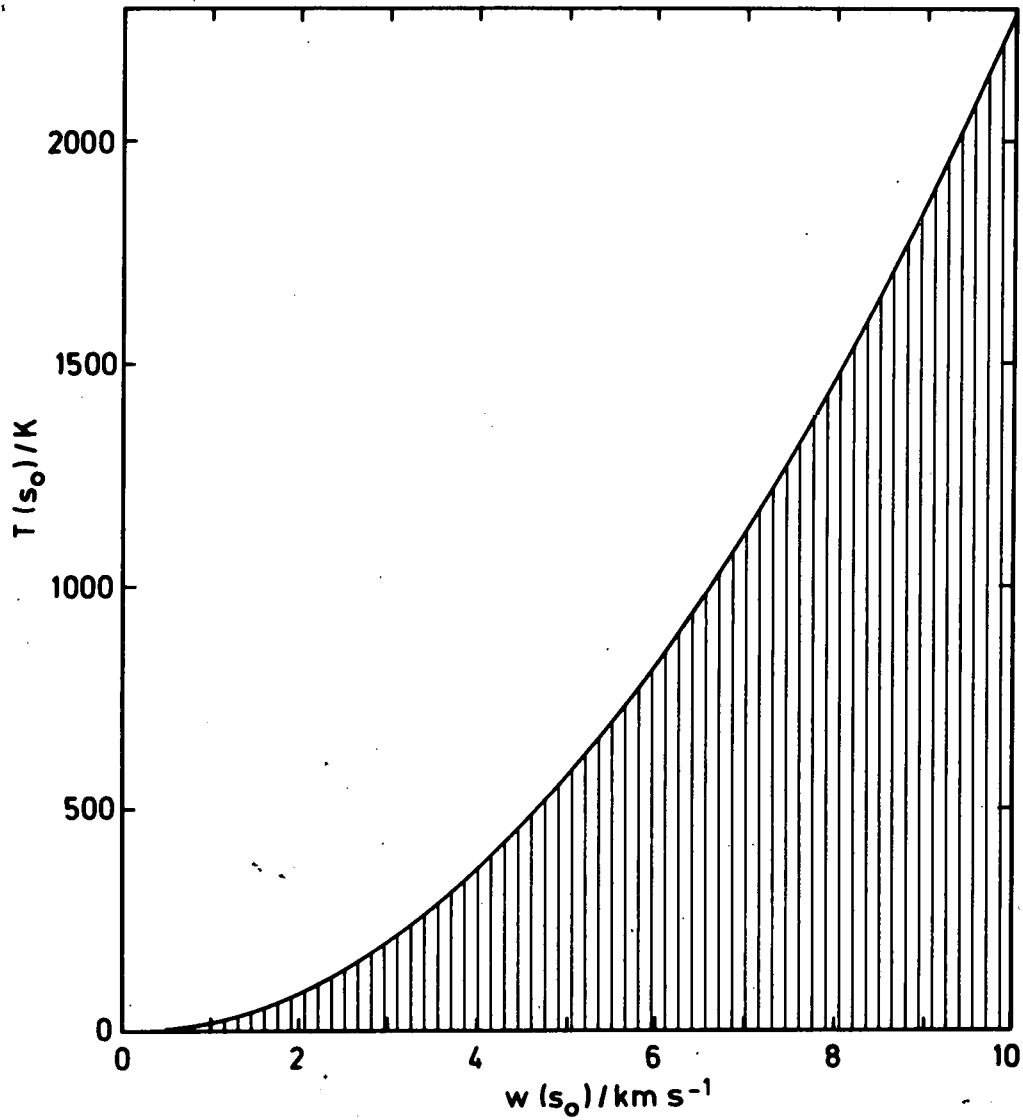Hence the parameters N, $T_{\parallel}$ and $T_{\perp}$ follow immediately from the equations (26), (27) and (30).

Fig. 5.- Only the points [w(s_0), T(s_0)] above the parabola are suitable boundary conditions at the exobase, for kinetic hydrogen polar wind models based on a bi-Maxwellian.

c.   Assuming that $n(s_o)$, $T(s_o)$ and $a(s_o)$ are given, the system of equations (3), (5) and (6) is equivelent to (26),

$$T_\perp = \frac{3T(s_o)}{2 + a(s_o)} \tag{31}$$

and (29). Elimination of $T_\perp$ between (29) and (31) yields

$$T_\parallel = \frac{3\pi}{\pi-2} \ \frac{a(s_o)}{2 + a(s_o)} \ T(s_o) \tag{32}$$

Hence the parameters N, $T_\parallel$ and $T_\perp$ follow immediately from the equations (26), (32) and (31).

5. CASE OF A MAXWELLIAN

In the special case of a Maxwellian (i.e. $T_\parallel = T_\perp = T$, $U = 0$) the velocity distribution (1) depends only on 2 parameters N and T. As mentionned earlier the temperature anisotropy at the exobase can no longer be chosen freely but equals $1 - \frac{2}{\pi}$.

a.   Assuming that $n(s_o)$ and $w(s_o)$ are given the parameters N and $T = T_\parallel$ follow from (26) and (27); $T(s_o)$ can not be chosen but will be given by

$$T(s_o) = \frac{m}{6k} \ (3\pi - 2) \ w^2(s_o) \tag{33}$$

b.   Assuming that $n(s_o)$ and $T(s_o)$ are given the parameters N and T follow from (26) and

$$T = \frac{3\pi \ T(s_o)}{3\pi - 2} \tag{34}$$

$w(s_o)$ can not be chosen but is related to $T(s_o)$ by equation (33).

6. CONCLUSIONS

Although kinetic polar wind models, based on an anisotropic asymmetric Maxwellian depending on four parameters, allow to fix the number density $n(s_o)$, the flow speed $w(s_o)$, the temperature $T(s_o)$ and the temperature anisotropy $a(s_o)$ as boundary conditions at the exobase, these quantities can not be chosen arbitrary, but have to satisfy condition (16) in order to obtain physical acceptable results in the exosphere.

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