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Self-oscillations, external forcings,  
and climate predictability

by

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## FOREWORD

This article has been presented at the Conference "Milankovitch and Climate. Understanding the response to orbital forcing". It will be published in the proceedings volume of this Conference.

## AVANT-PROPOS

Ce texte a été présenté au Colloque "Milankovitch and Climate. Understanding the response to orbital forcing". Il sera publié dans les comptes rendus de ce colloque.

## VOORWOORD

Volgende tekst werd voorgedragen op de Conferentie "Milankovitch and Climate. Understanding the response to orbital forcing". Hij zal verschijnen in de mededelingsbundel.

## VORWORT

Dieser Text wurde vor der Konferenz "Milankovitch and Climate. Understanding the response to orbital forcing" vorgetragen. Er wird in die Übertragung dieser Konferenz veröffentlicht.

SELF-OSCILLATIONS, EXTERNAL FORCINGS, AND CLIMATE  

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PREDICTABILITY

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Abstract

A class of nonlinear climate models involving two simultaneously stable states, one stationary and one time-periodic, is analyzed. The evolution is cast in a universal, "normal form" from which a basic difference between "radial" and "phase" variables emerges. In particular, it is shown that the phase variable has poor stability properties which are at the origin of progressive loss of predictability when the oscillator is autonomous. Next, the coupling to an external periodic forcing is considered. New types of solution are found, which correspond to a sharp and reproducible behavior of the phase. In this way predictability is ensured. Moreover, it is shown that the response can be considerably amplified by a mechanism of resonance with certain harmonics of the forcing. The implication of the results on the mechanism of quaternary glaciations is emphasized.

## Résumé

On analyse une classe des modèles climatiques non-linéaires donnant lieu à deux états simultanément stables dont un représente une solution stationnaire et l'autre une solution périodique au cours du temps. Les équations d'évolution sont transformées en une "forme normale" à partir de laquelle une différence fondamentale apparaît entre les variables "radiales" et les variables de "phase". En particulier on montre que l'absence de stabilité prononcée de la phase est à l'origine d'une perte de prédictabilité lorsque l'oscillateur fonctionne de façon autonome. Cependant, la présence d'une perturbation périodique extérieure permet au système d'établir une phase stable et reproductible, tout en amplifiant considérablement sa réponse à travers un mécanisme de résonance. L'intérêt de ces résultats dans l'interprétation des glaciations quaternaire est mis en évidence.

## Samenvatting

Men analyseert een klasse van non-lineaire klimaatmodellen die aanleiding geven tot 2 staten die gelijktijdig stabiel zijn. Eén ervan is stationair, de andere is periodiek en tijdsgebonden. De evolutie vindt plaats in een "normale vorm", van waaruit een fundamenteel verschil ontstaat tussen de "radiaalvariabelen" en de "fasevariabelen". In het bijzonder wordt aangetoond dat het gebrek aan stabiliteit bij de fase de oorzaak is van een progressief verlies van voorspelbaarheid wanneer de oscillator op autonome manier werkt. De aanwezigheid van een externe periodieke storing laat toe dat het systeem een stabiele fase ontwikkelt die tevens kan gereproduceerd worden, dit, terwijl de respons aanzienlijk versterkt wordt d.m.v. een resonantiemechanisme. Het belang van deze resultaten bij de interpretatie van quaternaire ijsvormingen wordt sterk benadrukt.

## Zusammenfassung

Man analysiert eine Klasse von nicht-linearen Klimamodellen die Anlass geben zu 2 Zuständen die gleichzeitig stabil sind. Ein davon ist stationär, die andere ist periodisch und zeitgebunden. Die Evolution findet statt in einer "Normalform"; von dieser entsteht ein fundamentaler Unterschied zwischen den "Radialveränderlichen" und den "Phaseveränderlichen". Besonders ist gezeigt dass der Stabilitätsmangel an Phase die Ursache ist von einem progressiven Voraussagenmöglichkeitverlust wenn der Oszillator autonom arbeitet. Die Anwesenheit von einer externen periodischen Störung erlaubt das System eine stabile Phase zu entwickeln die auch reproduziert werden kann, während die Antwort anscheinlich verstärkt wird mit einem Resonanzmechanismus. Der Belang von diesen Resultaten bei der Interpretation von quaternären Eisformungen ist stark unterstrichen.

## 1. INTRODUCTION

A central question in climate dynamics concerns the relative role of internally generated and external mechanisms of climatic change. It is well known that many climatic episodes, among which quaternary glaciations are the most striking example, present a cyclic character whose characteristic time is strongly correlated with external periodicities, like those of the earth's orbital variations (Berger, 1978). On the other hand, the balance equations of the principal variables predict that the coupling of such external forcings to the system's dynamics is exceedingly small. Hence, it is difficult to understand how such small amplitude disturbances can cause a response in the form of a major climatic change.

In the last few years two types of explanation of this "apparent paradox" have been advanced. In a first attempt by Benzi et al. (1982) and the present author (Nicolis, 1982a), the coupling between an energy balance model giving rise to multiple stable steady states and an orbital forcing perturbing periodically the solar constant has been considered. It has been shown that the internal fluctuations generated spontaneously by the system can amplify dramatically the response to the forcing, provided that the period of the latter is comparable to the mean first passage time between the stable climatic states. As a result the system experiences systematic deviations from the present-day climate to a less favorable climate, which are entrained to the periodicity of the external forcing.

A second attempt at a solution of the problem is to examine to what extent the climatic system is capable of generating self-oscillating dynamical behavior. Saltzman and coworkers (1978, 1980, 1981, 1982) analyzed the interactions between sea ice extent and ocean surface temperature, and revealed the existence of an autonomous oscillator arising from the insulating effect of sea ice on temperature, and from

the negative effect of temperature on sea ice. For plausible parameter values the periodicity turns out to be of the order of  $10^3$  years. This was to be expected, in view of the fact that such a value is an upper bound of variation in time of sea ice extent (Berger, 1981). On the other hand, by analyzing the interactions between the meridional ice-sheet extent and global temperature Ghil and co-authors (1979, 1982a, b) have also identified an autonomous oscillator. The periodicity of the latter turns out to be larger compared to Saltzman's oscillator, with an upper bound of the order of  $10^4$  years for realistic parameter values.

In a recent paper devoted to the implications of self-oscillations in climate dynamics (Nicolis, 1982b), we arrived at the surprising conclusion that an autonomous oscillator subject to its own internal fluctuations or to a noisy environment has poor predictability properties. Specifically, its dynamics can be decomposed into a "radial" part and an "angular" part. It then turns out that the radial variable has strong stability properties whereas the phase variable becomes completely deregulated by the fluctuations. As a result any trace of the oscillatory behavior is wiped out after a sufficiently long lapse of time. It would appear therefore that self-oscillations alone cannot provide a reliable mechanism of cyclic climatic change encompassing both a long time interval and a global space scale.

The purpose of the present work is to show that the coupling between an autonomous climatic oscillator and an external periodic or quasi-periodic forcing can, under certain conditions ensure sharp predictability. The latter will be reflected by the occurrence of cyclic variations of climate displaying a well-defined periodicity. We focus on the role of orbital forcings in quaternary glaciations. As the relevant orbital periodicities are of the order of  $2 \times 10^4$  to  $10^5$  years, it is clear that only the land ice dynamics is expected to couple effectively to such variations. For this reason we subsequently consider the type of models analyzed by Ghil (Ghil and Tavantzis, 1982b) as a prototype of this coupling. In Section 2 we cast these models in a generic, "normal" form



and summarize the results concerning the bifurcation of various kinds of solution. In Section 3 we show how poor predictability can arise in the absence of external forcing. Section 4 is devoted to the effect of such a forcing. We show that, contrary to the autonomous oscillator, the driven one attains, asymptotically, a well defined phase. Moreover, substantial amplification of the response can take place by the occurrence of resonance with some of the harmonics of the forcing. In Section 5 we present some comments on the implications of the results.

## 2. NORMAL FORM AND BIFURCATION ANALYSIS OF GHIL TYPE MODELS

In Ghil and Tavantzis (1982b) a set of two coupled nonlinear differential equations for the evolution in time of the dimensionless variables  $\theta$ ,  $\ell$  descriptive of the mean surface temperature and the meridional ice-sheet extent is analyzed. For plausible values of the physical parameters a stable steady-state climate is shown to coexist with a stable time-periodic one. An intermediate unstable periodic branch separates the above two solutions. The amplitude of the stable periodic solution, which emerges by a mechanism of Hopf bifurcation, corresponds roughly to that obtained from data on quaternary glaciation cycles, whereas its period is in the  $10^4$  year range. For other parameter values more complex behavior becomes possible, culminating in the appearance of infinite period orbits represented in phase space by a separatrix loop which is doubly asymptotic to an unstable steady state. We are not interested here in this latter type of behavior. Instead, we focus on the range of two stable solutions, one of them periodic, separated by an unstable solution.

According to the qualitative theory of ordinary differential equations (Arnold, 1980) any system operating in the vicinity of a Hopf bifurcation and presenting the above mentioned bistable behavior can be transformed, by a suitable change of variables, to the following universal form, known as normal form :

$$\frac{dz}{dt} = z (i\omega_0 + \beta + \gamma |z|^2 - \zeta |z|^4) \quad (1)$$

Here  $z$ ,  $\gamma$ ,  $\zeta$  are complex-valued combinations of the initial variables and parameters;  $\beta$  measures the distance from the Hopf bifurcation; and  $\omega_0$  is the frequency of the periodic solution at exactly the bifurcation point ( $\beta = 0$ ). For simplicity we hereafter take  $\gamma$  and  $\zeta$  to be real, and set without further loss of generality  $\zeta = 1$ . Instead of viewing eq. (1) as equivalent to a pair of equations for the real and imaginary parts of  $z$  (which, as already pointed out, are suitable combinations of the initial balance equations), it now becomes convenient to switch to "radial" and "angular" variables through

$$z = r e^{i\phi} \quad (2)$$

Substituting into eq. (1) and equating real and imaginary parts on both sides we obtain :

$$\frac{dr}{dt} = r (\beta + \gamma r^2 - r^4) \quad (3a)$$

$$\frac{d\phi}{dt} = \omega_0 \quad (3b)$$

This set of equations admits a single steady state solution,

$$r_{so} = 0 \quad (4)$$

It also admits solutions in which  $r$  is a nontrivial root of the right hand side of (3a) and  $\phi$  varies according to (3b) :

$$\phi = \omega_0 t + \phi_0 \quad (5)$$

$\phi_0$  being an (arbitrary) initial phase. In view of eq. (2), these are therefore time periodic solutions of the initial problem. To determine them we consider the biquadratic

$$r_s^4 - \gamma r_s^2 - \beta = 0$$

or

$$r_{s\pm}^2 = \frac{1}{2} [ \gamma \pm (\gamma^2 + 4\beta)^{1/2} ] \quad (6)$$

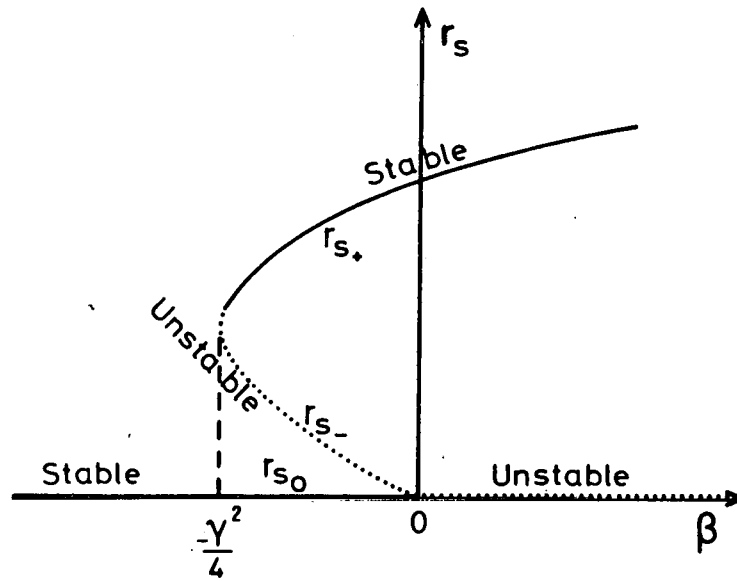
This admits two positive solutions as long as the following inequalities are satisfied :

$$\beta \leq 0$$

$$\gamma > 0$$

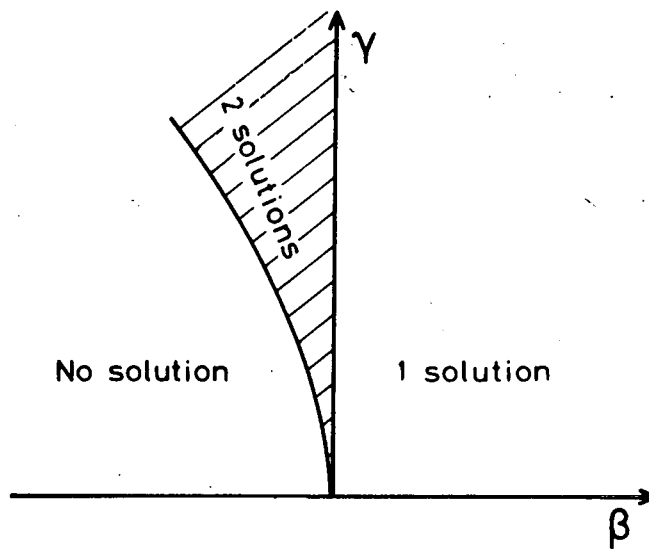
$$\beta \geq -\frac{\gamma^2}{4} \quad (7)$$

Linear stability analysis carried out on eq. (3a) shows that  $r_{s0}$  is stable for  $\beta < 0$  and unstable for  $\beta > 0$ ;  $r_{s+}$  is stable whenever it exists ( $\beta \geq -\gamma^2/4$ ), and  $r_{s-}$  is always unstable. Figures 1a,b summarize the



(a)

Fig. 1a.- Bifurcation diagram for the amplitude of the solutions of eq. (1) as a function of the parameter  $\beta$ . Full and dotted lines denote respectively stable and unstable branches.



(b)

Fig. 1b.- Dashed region indicates the domain of parameter space  $\beta, \gamma$  for which there are two nontrivial periodic solutions of eq. (1).

information concerning these various solutions in the form of a bifurcation diagram, completed by a state diagram in parameter space.

The resemblance between Fig. 1a and the results of Ghil and Tavantzis (1982b) is striking. Actually we can make use of the formal identity of the results to fit some of the parameter values of our model equation. Specifically, in the Ghil-Tavantzis analysis bifurcation occurs when  $\mu$ , the ratio of the heat capacity to the land ice inertia coefficient, is equal to  $\mu = 1.76735$ . Moreover the two periodic solutions -the analogs of our  $r_{s+}$  and  $r_{s-}$ - disappear below  $\mu = 1.7583$ . Comparing with inequalities (7) we see that our equation can fit this model provided that

$$\beta = \mu - 1.76735$$

and

$$\frac{\gamma^2}{4} = 1.76735 - 1.7583$$

or

$$\gamma = 0.1903 \tag{8a}$$

In an example given by Ghil and Tavantzis  $\mu$  is taken to be equal to 1.76 for the present climate. In our notation

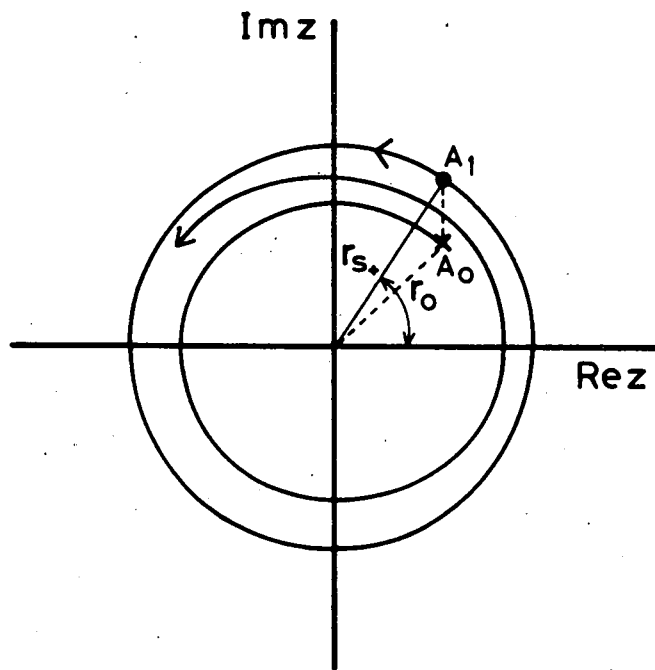
$$\beta_s = - 0.00735$$

(8b)

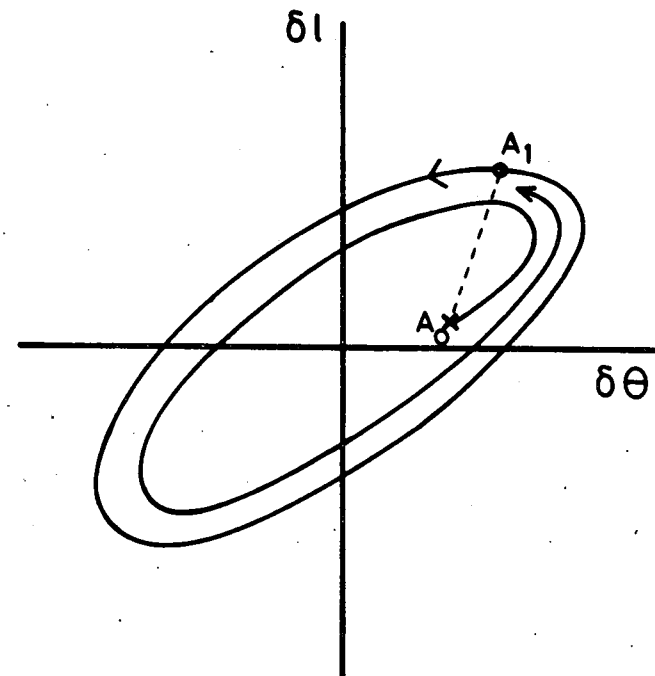
From relations (8) we arrive at the rather interesting conclusion that, for realistic parameter values, we can consider that our model operates close to the bifurcation point  $\beta = 0$ . Moreover, the width of the multiple stable state region as measured by  $\gamma$  is small. As we see in the next Sections these features will allow us to perform a systematic analytical study of the system.

### 3. THE PREDICTABILITY PROBLEM

The decomposition of the dynamics into a radial and phase part achieved by eqs. (3a) - (3b) allows us to obtain, straightforwardly, preliminary information on the predictability properties of our model oscillator. Indeed, let us perform the following thought experiment. Suppose that the system runs on its limit cycle  $r = r_{s+}$ . At some moment, corresponding to a value  $\phi = \phi_1$  of the phase, we displace the system to a new state characterized by the values  $r_0, \phi_0$  of the variables  $r$  and  $\phi$ . According to eq. (3a) and the stability analysis performed in the previous Section the variable  $r$  will relax from  $r_0$  back to the value  $r_{s+}$ , as the representative point in phase space will spiral toward the limit cycle (cf. Fig. 2). On the other hand, according to eq. (5), the phase variable  $\phi$  will keep for ever the memory of the initial value  $\phi_0$ . In other words, when the limit cycle will be reached again, the phase will generally be different from the one that would characterise an unperturbed system following its limit cycle during the same time interval. In as much as the state at which the system can be thrown by a perturbation is unpredictable, it therefore follows that the reset phase of the oscillator will also be unpredictable. In other words, our nonlinear oscillator is bound to behave sooner or later in an erratic way under the action of perturbations. This is tantamount to poor predictability.



(a)



(b)

Fig. 2.- Schematic representations of the evolution following the action of a perturbation leading from state  $A_1$  on the limit cycle to state  $A_0$ . Parts (a) and (b) describe the situation, respectively, in the space of the variables of the normal form and in the space of the variables,  $\delta l$ ,  $\delta \theta$  representing the deviations of the original variables  $l$ ,  $\theta$  from the steady state.

The above surprising property can be further substantiated by a stochastic analysis of the model. As well known any complex physical system possesses a universal mechanism of perturbations generated spontaneously by the dynamics, namely the fluctuations. Basically, fluctuations are random events. It thus follows that the state variables themselves ( $r, \phi$ ) (or  $\theta$  and  $\ell$  in Ghil's original model) become random processes. As we will show shortly this will result in a complete deregulation of the phase variable, whereas the radial variable will remain robust. The end result will be that because of destructive phase interference (cf. eqs. (2) and (5) with a randomly distributed initial phase  $\phi_0$ ), the signal of the system will tend to become flat if an average over a sufficiently long time is performed.

The analysis of fluctuations follows the same lines as in Nicolis, 1982b. We incorporate the effect of fluctuations by adding random forces  $F_\theta, F_\ell$  to the deterministic rate equations (Hasselmann, 1976). As usual, we assume the latter to define a multi-Gaussian white noise :

$$\begin{aligned}
 \langle F_\theta(t) F_\theta(t') \rangle &= q_\theta^2 \delta(t - t') \\
 \langle F_\ell(t) F_\ell(t') \rangle &= q_\ell^2 \delta(t - t') \\
 \langle F_\theta(t) F_\ell(t') \rangle &= q_{\theta\ell} \delta(t - t')
 \end{aligned}
 \tag{9}$$

This allows us to write a Fokker-Planck equation for the probability distribution  $P(\theta, \ell, t)$  of the climatic variables. In general this equation is intractable. However, the situation is greatly simplified if one limits the analysis in the range in which the normal form (eqs. (1) and (3)) is valid. To see this we first express the Fokker-Planck equation in the polar coordinates ( $r, \phi$ ). We obtain (Baras et al, 1982, Nicolis, 1982b)



$$\begin{aligned}
\frac{\partial P(r, \phi, t)}{\partial t} = & - \frac{\partial}{\partial r} \left\{ (\beta + \gamma r^2 - r^4) r + \frac{1}{2r} Q_{\phi\phi} \right\} P \\
& - \frac{\partial}{\partial \phi} \left\{ \omega_0 - \frac{1}{r^2} Q_{r\phi} \right\} P \\
& + \frac{1}{2} \left\{ \frac{\partial^2}{\partial r^2} Q_{rr} + 2 \frac{\partial^2}{\partial r \partial \phi} \frac{Q_{r\phi}}{r} + \frac{\partial^2}{\partial \theta^2} \frac{Q_{\phi\phi}}{r^2} \right\} P \quad (10)
\end{aligned}$$

in which  $Q_{\phi\phi}$ ,  $Q_{r\phi}$ ,  $Q_{rr}$  are suitable combinations of  $q_\phi^2$ ,  $q_\ell^2$ ,  $q_{\theta\ell}$ ,  $\sin \phi$  and  $\cos \phi$ .

To go further it is necessary to introduce a perturbation parameter in the problem. We chose it to be related to the weakness of the noise terms, and we express this through the scaling

$$\begin{aligned}
Q_{\phi\phi} &= \varepsilon \tilde{Q}_{\phi\phi} \\
Q_{r\phi} &= \varepsilon \tilde{Q}_{r\phi} \\
Q_{rr} &= \varepsilon \tilde{Q}_{rr} \quad , \quad \varepsilon \ll 1 \quad (11)
\end{aligned}$$

We next recall that, in view of the remark made at the end of Section 2, both the bifurcation parameter  $\beta$  and the parameter  $\gamma$  controlling the width of the multiple stable state region can be taken small. We express this by scaling these parameters by suitable powers of  $\varepsilon$ , chosen in such a way that the scaled Fokker-Planck equation still admits smooth and non-trivial solutions. After a long calculation (Nicolis, 1982b and 1982c) we obtain the following two results.

(i) Let us define the conditional distribution  $P(\phi/r, t)$  through

$$P(r, \phi, t) = P(\phi/r, t) P(r, t) \quad (12)$$

Then, to dominant order in  $\varepsilon$ ,  $P(\phi/r, t)$  obeys to the equation

$$\frac{\partial P(\phi/r, t)}{\partial t} = - \frac{\partial}{\partial \phi} \omega_0 (P(\phi/r, t)) \quad (13a)$$

which admits a properly normalized stationary solution

$$P_s(\phi/r) = \frac{1}{2\pi} \quad (13b)$$

(ii) Substituting (13b) into the bivariate Fokker-Planck equation for  $P(r, \phi, t)$  and keeping dominant terms in  $\varepsilon$  we find a closed equation for  $P(r, t)$  of the form :

$$\frac{\partial P(r, t)}{\partial t} = - \frac{\partial}{\partial r} \left( \beta r + \gamma r^3 - r^5 + \frac{Q}{2r} \right) P + \frac{Q}{2} \frac{\partial^2}{\partial r^2} P \quad (14)$$

in which the positive definite quantity  $Q$  is another combination of  $q_0^2$ ,  $q_\ell^2$ ,  $q_{\theta\ell}$ . Eq. (14) admits the following steady-state solution :

$$P_s(r) \sim r \exp \left[ - \frac{2}{Q} (r; \beta, \gamma) \right] \quad (15a)$$

where  $U(r, \beta, \gamma)$  is the kinetic potential generating the equation of evolution for  $r$  :

$$\frac{dr}{dt} = - \frac{\partial U}{\partial r}$$

From eq. (3a) :

$$U(r; \beta, \gamma) = - \beta \frac{r^2}{2} - \gamma \frac{r^4}{4} + \frac{r^6}{6} \quad (15b)$$

Fig. 3 represents the function (15a) in the range  $-\gamma^2/4 < \beta < 0$ . We obtain a distribution in the form of a wine bottle whose upper part has been cut. The projection of the upper edge and of the basis of the bottle on the phase plane are, respectively, the stable and unstable deterministic limit cycles.

The main conclusion to be drawn from the above analysis is that the phase variable has a completely flat probability distribution (cf. eq. (13b)). It may therefore be qualified as "chaotic", in the sense that the dispersion around its average will be of the same order as the average value itself. On the other hand, the radial variable has a stationary distribution (cf. eq. (15a)) such that the dispersion around the most probable value ( $r = 0$  or  $r = r_{s+}$ ) is small. Nevertheless, the mere fact that the probability distribution is stationary rather than time-periodic implies that a remnant of the chaotic behavior of  $\phi$  subsists in the statistics of  $r$  : If an average over a large number of samples (or over a sufficient time interval in a single realization of the stochastic process) is taken, the periodicity predicted by the deterministic analysis will be wiped out as a result of destructive phase interference. This property is at the origin of a progressive loss of predictability.

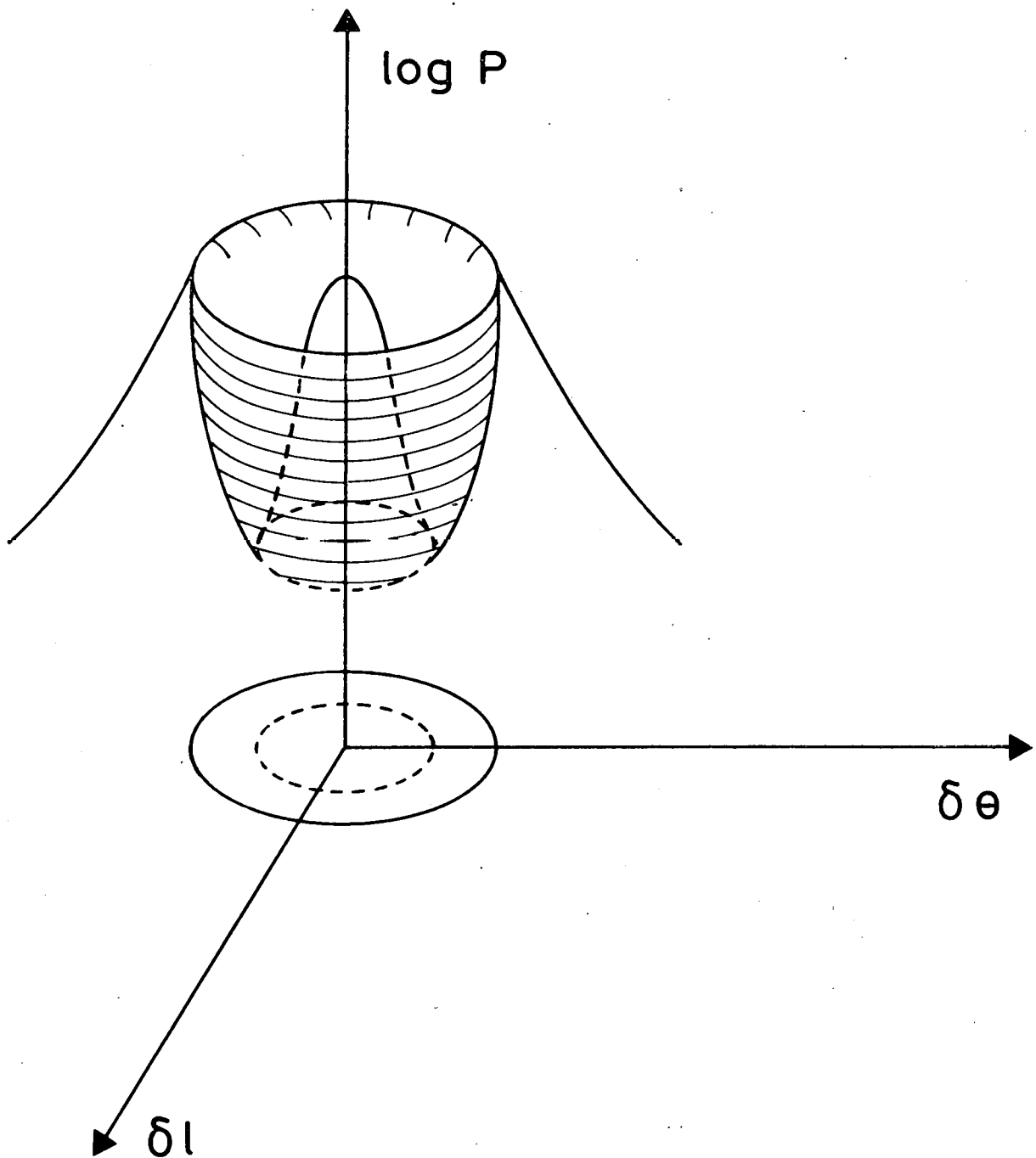


Fig. 3.- Schematic representation of the steady state probability distribution eq. (15a) - (15b) as a function of the excess variables  $\delta l$  and  $\delta \theta$ .

Using eqs. (15) one can also determine the range of parameter values over which the steady state climate ( $r = 0$ ) or the time periodic one ( $r = r_{s+}$ ) are dominant, in the sense that they correspond to the deepest of the two minima of the potential. The results are summarized in Fig. 4. As expected, near the bifurcation point  $\beta = 0$  the time-periodic solution dominates. On the contrary, near the limit point  $\beta = -\gamma^2/4$  the steady-state climate is the most probable one.

The knowledge of the steady-state distribution (15a) is also sufficient for calculating the mean characteristic passage times needed to jump between the two stable states through fluctuations (Nicolis and Nicolis, 1981). We find (Nicolis, 1982c) that for the parameter values given by eqs. (8), this time is only twice the periodicity of the limit cycle itself. This is another manifestation of the poor predictability properties of the system.

#### 4. EFFECT OF A PERIODIC FORCING : PREDICTABILITY ESTABLISHED

We now consider the response of the climatic oscillator described by eqs. (1) and (3) to an external periodic forcing. Our purpose is to show that the forcing provides the synchronizing element that was missing in the autonomous evolution, and ensures in this way the existence of a sharp and predictable signal.

We first analyze a purely sinusoidal forcing. The most general coupling with the internal dynamics would be described by an augmented eq. (1), in which both additive contributions as well as contributions multiplied by suitable powers of  $z$  are considered. It is clear however that, as long as the system is in the range of small  $\beta$  and  $\gamma$ ,  $z$  itself would be small. As a result the response will be dominated by the additive part.

To study quantitatively the effect of the forcing it will be convenient to work with the equations for the real and imaginary parts

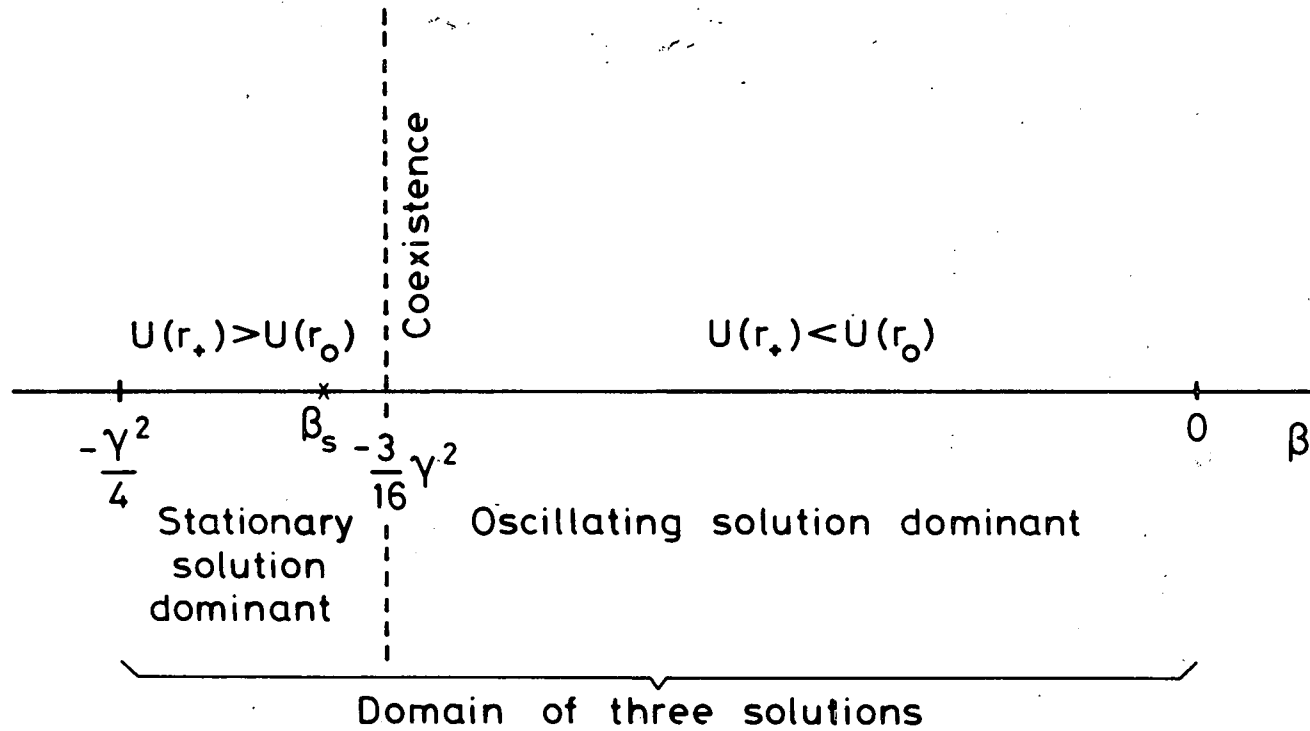


Fig. 4.- Range of values of the parameter  $\beta$  for which the stationary solution or the time periodic solution is the most probable state of the system.

of our variable  $z$ , in view of the subtleties associated with the handling of the phase variable. We can write these equations in the form (cf. eq. (1) with  $z = x + iy$ ) :

$$\begin{aligned}\frac{dx}{dt} &= \beta x - \omega_0 y + \gamma x(x^2 + y^2) - x(x^2 + y^2)^2 + \varepsilon q \sin \omega_1 t \\ \frac{dy}{dt} &= \omega_0 x + \beta y + \gamma y(x^2 + y^2) - y(x^2 + y^2)^2\end{aligned}\quad (16)$$

Here  $\omega_1$  is the frequency of the forcing,  $\varepsilon$  a smallness parameter,  $q$  the coupling amplitude. The absence of coupling in the second eq. (16) simplifies the calculations considerably and will therefore be adopted in the sequel. Note that there is no essential loss of generality implied by such an assumption.

In order to handle eqs. (16) we shall take into account the fact, already utilized in Section 3, that the system operates close to bifurcation ( $\beta \rightarrow 0$ ,  $\gamma \rightarrow 0$ ). Setting

$$\begin{aligned}x &= \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \dots \\ y &= \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 + \dots\end{aligned}\quad (17)$$

we then see that to order  $\varepsilon$  the only terms surviving in (16) are

$$\begin{aligned}\frac{dx_1}{dt} + \omega_0 y_1 &= q \sin \omega_1 t \\ \frac{dy_1}{dt} - \omega_0 x_1 &= 0\end{aligned}\quad (18)$$

A particular solution of (18) is easily found to be

$$\begin{aligned} x_{1P} &= \frac{q\omega_1}{\omega_0^2 - \omega_1^2} \cos \omega_1 t \\ y_{1P} &= \frac{q\omega_0}{\omega_0^2 - \omega_1^2} \sin \omega_1 t \end{aligned} \quad (19)$$

This solution is well-behaved for all values of  $\omega_1$  except those for which there is resonance,  $\omega_1 \cong \omega_0$ . In the context of quaternary glaciations, in which the frequency of the (orbital) forcing is at least two times smaller than  $\omega_0$ , it is unlikely that resonance can be expected. We therefore exclude this possibility for the time being.

As well known, the general solution of eqs. (18) is given by the particular solution eq. (19) to which the general solution of the homogeneous equation is added. As the homogeneous equation is simply the harmonic oscillator problem, we finally obtain :

$$\begin{aligned} x_1 &= A \cos \omega_0 t + B \sin \omega_0 t + \frac{q\omega_1}{\omega_0^2 - \omega_1^2} \cos \omega_1 t \\ y_1 &= A \sin \omega_0 t - B \cos \omega_0 t + \frac{q\omega_0}{\omega_0^2 - \omega_1^2} \sin \omega_1 t \end{aligned} \quad (20)$$

At this point A and B are undetermined constants. As a matter of fact their indeterminacy reflects, in part, the lability of the phase variable in the absence of the forcing. Contrary to the autonomous case however, we now have a way to remove this indeterminacy. It suffices to push expansion (17) to a higher order with the additional requirement that  $\beta$  and  $\gamma$  have also to be scaled in terms of  $\varepsilon$  :



$$\beta = \varepsilon^4 \tilde{\beta} + \dots$$

$$\gamma = \varepsilon^2 \tilde{\gamma} + \dots \quad (21)$$

We obtain in this way, to first nontrivial order beyond  $(x_1, y_1)$  :

$$\begin{aligned} \frac{dx_5}{dt} + \omega_0 y_5 &= \tilde{\beta} x_1 + \tilde{\gamma} x_1 (x_1^2 + y_1^2) - x_1 (x_1^2 + y_1^2)^2 \\ \frac{dy_5}{dt} - \omega_0 x_5 &= \tilde{\beta} y_1 + \tilde{\gamma} y_1 (x_1^2 + y_1^2) - y_1 (x_1^2 + y_1^2)^2 \end{aligned} \quad (22)$$

in which  $x_1, y_1$  are given by eqs. (20).

This inhomogeneous set of equations for  $(x_5, y_5)$  admits a solution only if a solvability condition expressing the absence of secular terms (i.e. terms increasing unboundedly in  $t$ ) is satisfied. It turns out (Sattinger, 1973) that this condition expresses the orthogonality of the right hand side of (22), viewed as a vector, to the two eigenvectors of the operator in the left hand side :

$$(\cos \omega_0 t, \sin \omega_0 t)$$

and

$$(\sin \omega_0 t, -\cos \omega_0 t) \quad (23)$$

The scalar product to be used is the conventional scalar product of vector analysis, supplemented by an averaging over  $t$ . The point is

that we dispose of two such solvability conditions, for the two unknown constants A and B. Because of the presence of terms in q in eq. (20) these will be inhomogeneous equations. Both A and B will therefore be fixed entirely, without further indeterminacy. In other words, we have shown that whatever the initial perturbation which may act on the system, the final solution will be perfectly well defined both as far as its amplitude and its phase are concerned. We have therefore shown that the unpredictability pointed out in the previous Section is removed by the presence of the forcing.

Once the solvability condition is satisfied one can compute  $(x_5, y_5)$  from eq. (22). A novel feature then appears since the solution involves trigonometric functions having arguments  $3\omega_1 t$  and  $5\omega_1 t$ . This will give rise to denominators of the form  $(\omega_0 - 3\omega_1)$  and  $(\omega_0 - 5\omega_1)$ . The appearance of resonance becomes now much more plausible, since for an intrinsic period of say  $7 \times 10^3$  years it would be realized by external periodicities of  $2 \times 10^4$  and  $4 \times 10^4$  years. These are known to be present in the orbital forcing. In other words, in addition to establishing predictability we obtain, for free, the enhancement of the response in the form of resonance with certain harmonics of the external forcing.

## 5. DISCUSSION

The principal result of the present paper has been that self-oscillations in climate dynamics are bound to show erratic behavior after the lapse of sufficiently long time, as a result of poor stability properties of the phase variable. On the other hand when the nonlinear oscillator is coupled to an external periodic forcing the response is characterized, under typical conditions, by a sharply defined amplitude and phase. In other words the signal becomes "predictable" in the sense that its power spectrum computed from time series data would be dominated by a limited number of well-defined frequencies. Computer

simulations on a model including land ice dynamics by Le Treut and Ghil (1982a) confirms entirely this point.

In addition to predictability the coupling with the external forcing can also lead to the enhancement of the response, through a mechanism of resonance between the intrinsic frequency and certain harmonics of the forcing. Besides, eqs (20) show that the response to the forcing will in general be quasi-periodic, in view of the fact that the internal and external frequencies need not be rationally related to each other. Again, these features are in complete agreement with the numerical simulations by Le Treut and Ghil (1982a).

In the analysis of Section 4 we adopted a purely phenomenological description. A stochastic analysis incorporating the effect of both fluctuations and external forcing, would certainly provide a more convincing proof of how predictability is ensured in the system. We intend to report on this point in future investigations.

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