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B E L G I S C H I N S T I T U U T V O O R R U I M T E - A E R O N O M I E

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FOREWORD

The paper "An iterative method to solve the nonlinear Poisson's equation in the case of plasma tangential discontinuities" will be published in Journal of Computational Physics.

AVANT-PROPOS

L'article intitulé "An iterative method to solve the nonlinear Poisson's equation in the case of plasma tangential discontinuities" sera publié dans "Journal of Computational Physics".

VOORWOORD

Het artikel "An iterative method to solve the nonlinear Poisson's equation in the case of plasma tangential discontinuities" zal gepubliceerd worden in "Journal of Computational Physics".

VORWORT

Der Artikel "An iterative method to solve the nonlinear Poisson's equation in the case of plasma tangential discontinuities" wird herausgegeben in "Journal of Computational Physics".

AN ITERATIVE METHOD TO SOLVE THE NONLINEAR POISSON'S EQUATION IN THE  
CASE OF PLASMA TANGENTIAL DISCONTINUITIES

by

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## Abstract

In order to determine the electric potential in collisionless tangential discontinuities of a magnetized plasma, it is required to solve a non-linear Poisson's equation with sources of charge and current depending on the actual potential solution. This non-linear second order differential equation is solved by an iterative method. This leads to an ordered sequence of non-linear algebraic equations for each successive approximation of the actual electric potential. It is shown that the method holds for transitions with characteristic thicknesses ( $D$ ) as thin as 5 Debye lengths ( $\lambda$ ). For smaller thicknesses, when  $D$  shrinks to  $3\lambda$  or less, the method fails because in that case the iteration procedure does no longer converge. Numerical results are shown for an ion-dominated layer ( $D \sim 10^2-10^3 \lambda$ ), as well as for two electron-dominated layers characterized by  $D \approx 5\lambda$  and  $D \approx 2.5\lambda$ , respectively. In all cases considered in this paper, the relative error on the electric potential obtained as a solution of the quasi-neutrality approximation is of the order of the relative charge density. When the method holds, each successive approximation reduces the relative error on the potential by roughly a factor of 10. For space plasma boundary layers, the quasi-neutrality approximation can be used with much confidence since their thickness is always much larger than the local Debye length.

## Résumé

Afin de déterminer le potentiel électrique dans les discontinuités tangentielles sans collisions d'un plasma magnétisé, il faut résoudre une équation de Poisson non-linéaire, avec des termes sources (charge et courant) qui dépendent du potentiel à déterminer. Cette équation différentielle du second ordre, non-linéaire, est résolue par une méthode itérative. Cela conduit à une séquence ordonnée d'équations algébriques non-linéaires pour chaque approximation successive du potentiel vrai. On montre que la méthode est valable pour des transitions d'épaisseur caractéristique ( $D$ ) aussi mince que 5 longueurs de Debye ( $\lambda$ ). Pour des épaisseurs plus petites, lorsque  $D$  atteint  $3\lambda$  ou moins, la méthode cesse d'être valable, car, dans ce cas, le processus d'itérations ne converge plus. Des résultats numériques illustrent le cas d'une couche ionique ( $D \sim 10^2 - 10^3 \lambda$ ), ainsi que ceux de deux couches électroniques caractérisées respectivement par  $D \approx 5\lambda$  et  $D \approx 2.5 \lambda$ . Dans tous les cas considérés dans cet article, l'erreur relative sur le potentiel électrique, solution de l'équation de quasi-neutralité, est de l'ordre de la densité relative de charge. Lorsque la méthode peut être appliquée, chaque approximation successive diminue l'erreur relative sur le potentiel d'un facteur proche de 10. Pour les couches frontières des plasmas de l'espace, l'approximation de quasi-neutralité peut être utilisée en toute confiance, puisque l'épaisseur de ces couches est toujours beaucoup plus grande que la longueur de Debye locale.

## Samenvatting

Teneinde de elektrische potentiaal te bepalen in de botsingloze tangentiële discontinuïteiten van een gemagnetiseerd plasma, dient in een oplossing gezocht te worden van een niet lineaire Poisson-vergelijking, waarvan de brontermen (lading en stroom) afhangen van de te bepalen potentiaal. Deze niet lineaire differentiaalvergelijking van de tweede orde wordt geïntegreerd door middel van een iteratieve methode. Dit leidt tot een geordende reeks van niet lineaire algebraïsche vergelijkingen voor elke successieve benadering van de werkelijke potentiaal.

Er wordt aangetoond dat de methode geldig is voor overgangen met karakteristieke diktes ( $D$ ) tot 5 Debye lengts ( $\lambda$ ). Voor kleinere diktes, wanneer  $D$  krimpt tot  $3\lambda$  of minder, verliest de methode haar geldigheid, aangezien in dit geval het iteratieproces niet meer convergeert.

De numerieke resultaten worden getoond voor een ion-gedomineerde laag ( $D \sim 10^2 - 10^3 \lambda$ ) en voor een electron-gedomineerde laag, gekarakteriseerd respectievelijk door  $D \approx 5\lambda$  en  $D \approx 2.5 \lambda$ .

In alle gevallen beschouwd in dit artikel is de relatieve fout op de elektrische potentiaal, oplossing van de quasi-neutraliteitsvergelijking, van de orde van de relatieve ladingsdichtheid. Daar waar de methode kan toegepast worden, wordt de relatieve fout op de potentiaal bij elke opeenvolgende iteratie gereduceerd met een factor van de orde van 10. Voor de grenslagen van ruimteplasmas, kan de benadering der quasi neutraliteit met vertrouwen gebruikt worden, daar de dikte van dergelijke lagen altijd veel groter is dan de locale Debye lengte.

## Zusammenfassung

Die elektrische Spannung für Tangentialle-diskontinuitäten in Plasmas, ist Solution des Poisson's Gleichung mit Kwellentermen (Ladung und Ströme) die von diese Spannung selbst abhängen. Diese Differential Gleichung des zweitens Orderns ist bei einer iterativen Methode gelöst worden. Das gibt eine Reie algebraische Gleichungen für jede successive Approximation der Spannung. Wir zeigen dass diese Methode gültig ist für Transitionen die dicker als 5 Debyelänge ( $\lambda$ ) sind ( $D > 5\lambda$ ). Für dünnerer Schikten ( $D \leq 3\lambda$ ), ist diese Methode nicht mehr gültig, um dass die Iterationen nicht mehr konvergieren.

Numerische Resultaten sind erhalten worden für eien Ionen Schicht ( $D \sim 10^2 - 10^3\lambda$ ), sowie für zwei Elektronen Schichten für  $D \approx 5\lambda$  und  $D \approx 2.5\lambda$ . In alle Fälle die wir hier studiert haben, ist der relative Rechenfehler des Spannung, den relativen Rechenfehler der Ladungsdichte gleich. When unsere Methode gültig ist, ist der relative Rechenfheler jeder Approximation, 10 Mahl kleiner als die frühere Approximation. Für Schichten die man im Weltraum findet can die Approximation der Quasi-neutralität meistens benützt werden, dem diese Schichten sind viel breiter als die Debyelänge.

## 1. INTRODUCTION

It is generally considered that in astrophysical and geophysical plasmas the electron density balances almost exactly the ion charge density, i.e. plasmas are quasi-neutral. Nobody questions that this is a very satisfactory approximation in uniform or nearly uniform plasma regions of space.

However, it has sometimes be questioned whether this approximation is still a valid one at the earth's magnetopause [1] or at other sharp boundary layers where the physical properties of the plasma (density, temperature, bulk speed, magnetic field,...) change abruptly from one set of values to another one. The magnetospheric bow shock [2] is another type of boundary where a significantly large electric charge separation might be expected.

The purpose of this work is to verify on a few case studies that the quasi-neutrality equation is indeed a valid zero-order approximation when the boundary layer has a thickness much larger than the characteristic Debye length.

One of the simplest type of boundary layer or plasma discontinuity observed in space is the so-called planar "tangential discontinuity" (TD for short). In the reference frame tied to a TD there is no plasma crossing the layer. In addition, the magnetic field component along the normal to the boundary vanishes. Furthermore, from conservation laws [3], the total plasma and field pressure across a TD does not vary. However, the plasma velocity distribution and the tangential magnetic field (intensity and direction) are both varying over short distances, as at the interface of ferromagnetic domains in solid-state plasmas. TD's have been found to be abundant in the solar wind [4] and observations indicate that the earth's magnetopause can sometimes be considered as a tangential discontinuity [5].



To make theoretical models of steady-state collisionless TD's, the plasma kinetic method has been used by a number of authors [6 to 13]. Actually, the Maxwell's equations for the electric and magnetic fields are combined with the Vlasov equation for the particles. The result is a set of second order non-linear differential equations for the electric and magnetic vector potentials, with sources of charge and current depending on the actual potentials.

The solution of the Vlasov equation is based on the standard methods of constructing distribution functions in terms of the conserved energy and generalized momenta of the particles. It is then a relatively straightforward task to produce solutions which mimic a number of observed steady-state boundary layers [6]. This method is preferable to particle simulation involving a time dependent problem which leads to the set-up of steady-state electric and magnetic structures. Indeed, for typical space plasmas where the number of particles in a Debye sphere is of the order of  $10^9$ , particle simulation involves a large number of particles whose trajectories can be determined with efficiency by only high speed, large scale computers. Although particle simulation is particularly useful whenever a limited number of analytic methods are available, this is not the case for the problem under consideration in this paper for which solutions of the Vlasov equation are straightforward.

Numerical difficulties arise however when attempts are made to integrate Poisson's equation for the electric potential - a second order non-linear differential equation - by standard methods (e.g. Runge-Kutta or Hamin). These difficulties arise because the right-hand side of Poisson's equation is a difference between two very large numbers, the electron and ion densities, which are almost exactly equal to each other. A zero-order approximation of the actual electric potential ( $\phi$ ) can however be determined as a solution of the quasi-neutrality equation where the electron density is a very sensitive function of  $\phi$ .

Of course, this zero-order approximation holds whenever the total charge density which is proportional to the Laplacian of  $\phi$  is found a posteriori to be much smaller than the charge density associated with the positively (or negatively) charged particles. Each time this condition is fulfilled, a self-consistent potential is obtained. The aim of this paper is to study the small deviations from quasi-neutrality in TD's. In this case, Poisson's equation for the electric potential must be solved instead of the quasi-neutrality equation. The higher order approximations obtained this way are then compared to the zero-order approximation corresponding to the solution of the quasi-neutrality equation.

In this paper, Poisson's equation has been solved by an iterative method. The procedure leads to successive approximations of the potential starting with the zero-order approximation  $\phi_0$  which is the solution of the algebraic quasi-neutrality equation. The potential in the approximation of order  $n$  ( $n > 0$ ) is the solution  $\phi_n$  of a new algebraic equation obtained by replacing the right-hand side of the quasi-neutrality equation by "a charge density", i.e., a quantity proportional to the Laplacian of  $\phi_{n-1}$ . It has been found that, after only a few iterations, the successive approximations do not differ by more than the precision of the computer (9 significant digits), at least for broad layers (the so-called ion-dominated layers in [7]) whose characteristic thickness is the ion gyroradius. For these ion-dominated layers the zero-order approximation  $\phi_0$  does not differ significantly from higher-order approximations (i.e.,  $|\phi_n - \phi_0| / \phi_n < 3 \times 10^{-6}$ ) and in practice  $\phi_0$  can be considered as very close to the actual potential. For thinner layers with thicknesses of the order of an electron gyroradius (the so-called electron-dominated layers in [7]), it is shown that the iterative process is convergent as long as the characteristic thickness ( $D$ ) does not become smaller than 5 Debye lengths ( $\lambda$ ). The iterative procedure fails however when  $D$  shrinks to about  $3\lambda$  or less. In all cases we have considered so far, the relative error on  $\phi_0$  (with respect to the actual potential) has been found of the order of the relative charge density.

The model used for a collisionless TD will be briefly described in section 2. Although equilibrium configurations of TD's in collisionless plasmas under a wide variety of boundary conditions and assumptions have been discussed in the literature (see references [6] to [13]), the model considered in this paper is a simplified version based on the work of Sestero [7]. Section 3 outlines the iteration process used to solve Poisson's equation. In sections 4 and 5 are displayed the numerical results obtained for layers with various thicknesses, i.e., an ion-dominated layer (section 4) and two electron-dominated layers with  $D \approx 5\lambda$  and  $D \approx 2.5\lambda$  (section 5). Conclusions are summarized in section 6.

## 2. A KINETIC MODEL OF TANGENTIAL DISCONTINUITIES

For the sake of simplicity, we study steady-state, unidimensional planar current layers which are parallel to the (y-z) plane of a cartesian coordinate system. All plasma and field variables are assumed to depend only on the x-coordinate, normal to the layer. The magnetic field  $\underline{B}$  is oriented along the z-axis while the electric field  $\underline{E}$  is parallel to the x-axis. In this model, the z-coordinate is an ignorable coordinate for the motion of a single plasma particle. Therefore, the so-called constants of motion are the energy (H) and the y-component of the generalized momentum (p) :

$$H = \frac{1}{2} m (v_x^2 + v_y^2) + Ze \phi(x) \quad (1)$$

where  $Ze$  is the charge ( $e = 1.6 \times 10^{-19}$  C) of the particle of mass  $m$  and  $v_y$  its velocity, while  $\phi(x)$  is the electric potential;

$$p = mv_y + Ze a(x) \quad (2)$$

where  $a(x)$  is the magnetic vector potential (parallel to the y-axis).

Any function of  $H$  and  $p$ ,  $F(H, p)$ , is a solution of the steady-state Vlasov equation. Consider the following distribution function [7] for a given plasma species :

$$F(H, p) = \delta(p) \eta(H) \quad (3)$$

with

$$\begin{aligned} \delta(p) &= C_{\rho} \text{ if } p \text{ in } ] - (\text{sign } Z) \infty, 0] \\ &= C_{\pi} \text{ if } p \text{ in } [ 0, + (\text{sign } Z) \infty [ \\ &(\text{sign } Z = + 1, \text{ if } Z > 0 \\ &= - 1, \text{ if } Z < 0) \end{aligned}$$

where  $C_{\rho}$ ,  $C_{\pi}$  are arbitrary ( $\geq 0$ ) constants and  $\eta(H)$  a Maxwellian distribution given by :

$$\eta(H) = \alpha \left( \frac{m}{2\pi kT} \right) \exp \left( - \frac{H}{kT} \right) \quad (4)$$

where  $T$  is the asymptotic temperature of the particle species and  $\alpha$  is a parameter which has the dimension of a number density.

The distribution function (3) as a product of an exponential in  $H$  (the Maxwellian  $\eta$  given by equation (4)) by a step function in  $p$  gives conceivably the simplest model describing a sharp transition layer. Note that other choices are possible since for any single-valued distribution function in the whole  $(H, p)$  plane the Vlasov equation a priori admits a solution. Thus the state of the plasma at both ends of a TD does not uniquely determine the transition profile. This is a peculiar feature of the nonlinear Vlasov equation [13]. To remove the nonuniqueness of the distribution functions, consideration of particle accessibility in phase space must be met [14]. This requires the knowledge of the characteristics of the plasma in the boundary source regions together with the transport mechanisms bringing the plasma to the transition itself. This problem being outside the scope of this paper we will be content with Sestero's distribution functions (3). Indeed, to illustrate the departure from charge neutrality in thin boundary layers the choice of the distribution functions is not crucial. Furthermore, the method for solving Poisson's equation developed in the next section would remain conceptually the same for any other choice than the distribution given by (3) and (4).

From (3) and (4), the number density (n) and the current density (j) - parallel to the y-axis - can be computed as a function of  $\phi$  and a. It is found :

$$n = \frac{\alpha}{2} \exp\left(-\frac{Ze\phi}{kT}\right) \left[ C_l \operatorname{erfc}\left(\frac{a}{2^{1/2} R_\alpha B_\alpha}\right) + C_r \operatorname{erfc}\left(-\frac{a}{2^{1/2} R_\alpha B_\alpha}\right) \right] \quad (5)$$

$$j = \frac{1}{2} |Z| e\alpha (C_r - C_l) \left(\frac{2kT}{\pi m}\right)^{1/2} \exp\left(-\frac{Ze\phi}{kT}\right) \exp\left(-\frac{a^2}{2R_\alpha^2 B_\alpha^2}\right) \quad (6)$$

where  $\operatorname{erfc}$  is the complementary error function

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{+\infty} e^{-x^2} dx$$

and  $R_\alpha B_\alpha$  is a constant given by

$$R_\alpha B_\alpha = \left(\frac{m kT}{Z^2 e^2}\right)^{1/2} \quad (7)$$

with  $R_\alpha$  and  $B_\alpha$  being some characteristic asymptotic Larmor radius and magnetic field, respectively, i.e.,

$$R_\alpha B_\alpha = R_l B_l = R_r B_r \quad (8)$$

In (8),  $R_l$  and  $R_r$  are the characteristic Larmor radii at  $x = -\infty$  (on the left-hand side) and  $x = +\infty$  (on the right-hand side), respectively; while  $B_l$  and  $B_r$  are the magnetic field intensities at  $x = -\infty$  and  $x = +\infty$ , respectively.

The Maxwell's equations to solve are

$$\frac{d^2\phi}{dx^2} = -\frac{e}{\epsilon_0} \sum_{\nu=1}^s Z(\nu) n(\nu) \quad (9)$$

$$\frac{d^2 a}{dx^2} = -\mu_0 \sum_{\nu=1}^s j^{(\nu)} \quad (10)$$

where  $\epsilon_0$  and  $\mu_0$  are the vacuum permittivity and permeability, respectively ( $\epsilon_0 = 8.854 \times 10^{-12}$  F/m,  $\mu_0 = 4\pi \times 10^{-7}$  H/m) and  $s$ , the number of particle species.

The electric field ( $E, 0, 0$ ) and the magnetic field ( $0, 0, B$ ) are the derivatives of potentials, i.e.,

$$E = -\frac{d\phi}{dx} \quad (11)$$

$$B = \frac{da}{dx} \quad (12)$$

The magnetic and electric field distributions within the boundary layer are then determined by solving the system of differential equations (9) to (12) with  $n^{(\nu)}$  and  $j^{(\nu)}$  given by (5) and (6), respectively. Equations (10) and (12) will be solved using a Hamming's predictor-corrector scheme while (9) and (11) will be solved by using the iteration process described in the next section. The right-hand sides of equations (9) and (10) are non-linear functions of  $\phi$  and  $a$ . Furthermore, the right-hand side of (9) is a sum of terms which must remain much smaller than each individual term as a consequence of the tendency of plasmas to maintain electric neutrality. For these reasons standard numerical procedures to solve Poisson's equation generally fail to converge.

Note that, in this model, if the electron (/ion) velocity distribution remains Maxwellian from  $x = -\infty$  to  $x = +\infty$ , only the ions (/electrons) can be accelerated inside the transition, on a characteristic scale length of the order of a few ion (/electron) Larmor gyroradii. Following Sestero [7], these transitions are called ion (/electron) - dominated layers, respectively.

### 3. AN ITERATION PROCESS TO SOLVE POISSON'S EQUATION

Plasmas being quasi-neutral,  $d^2\phi/dx^2 = 0$ . Therefore, the zero-order approximation of Poisson's equation (9) is the quasi-neutrality equation :

$$\sum_{\nu=1}^s z^{(\nu)} n^{(\nu)} (a_0, \phi_0) = 0 \quad (13)$$

By introducing the quantity  $N_{rst}$  (with the summation extended over all species) defined by

$$N_{rst} = \sum z K_1^r K_2^s K_3^t n \quad (14)$$

with

$$K_1 = - \frac{Ze}{kT} \quad (15)$$

$$K_2 = \frac{1}{kT} = \frac{1}{Z e R_\alpha B_\alpha} \left( \frac{m}{kT} \right)^{1/2} \quad (16)$$

$$K_3 = - \frac{1}{R_\alpha^2 B_\alpha^2} \quad (17)$$

eq. (13) can also be written as

$$N_{000} (a_0, \phi_0) = 0 \quad (18)$$

Coupled with the differential equations (10) and (12), (18) can be solved numerically. The potential  $\phi_0$ - and also the higher order approximations  $\phi_1, \phi_2, \dots$  (see below)- is then obtained by the Brent's method of finding the root of a non-linear equation (this method combines root bracketing, bisection and inverse quadratic interpolation [15]).

We also introduce the quantities

$$J_{rst} = \sum z k_1^r k_2^s k_3^t j \quad (19)$$

and

$$j_{rst} = \sum k_1^r k_2^s k_3^t j \quad (20)$$

By differentiating (18) twice with respect to  $x$ , it is possible to obtain algebraic expressions for the first and second derivatives of  $\phi_0$ . It is found

$$\phi_0' = - B_0 J_{010}/N_{100} \quad (21)$$

$$\begin{aligned} \phi_0'' &= N_{100}^{-1} J_{010} (\mu_0 j_{000} + 2 B_0^2 N_{100}^{-1} J_{110} - B_0^2 N_{100}^{-2} J_{010} N_{200}) \\ &\quad - a_0 B_0^2 J_{011} N_{100}^{-1} = \phi_0''(a_0, \phi_0, B_0) \end{aligned} \quad (22)$$

To carry out those derivatives, we have used the following differentiation rules

$$\frac{\partial N_{rst}}{\partial \phi} = N_{r+1,s,t} \quad (23)$$

$$\frac{\partial N_{rst}}{\partial a} = J_{r,s+1,t} \quad (24)$$

$$\frac{\partial J_{rst}}{\partial \phi} = J_{r+1,s,t} \quad (25)$$

$$\frac{\partial J_{rst}}{\partial a} = a J_{r,s,t+1} \quad (26)$$

and for any quantity depending on  $\phi$ ,  $a$  and  $B$

$$\frac{d}{dx} = \phi' \frac{\partial}{\partial \phi} + B \frac{\partial}{\partial a} - \mu_0 j_{000} \frac{\partial}{\partial B} \quad (27)$$



The next approximation for  $\phi$ , i.e., the first-order approximation  $\phi_1$  is solution of the algebraic equation

$$\sum Z n(a_1, \phi_1) = - \frac{\epsilon_0}{e} \phi_0''(a_1, \phi_1, B_1) \quad (28)$$

The right-hand side of this equation is given by (22). Equation (28) is coupled with the differential equations (10) and (12) to obtain  $a_1$ ,  $\phi_1$  and  $B_1$ .

Higher order approximations for  $\phi$  are solutions of algebraic equations of the form :

$$\sum Z n(a_i, \phi_i) = - \frac{\epsilon_0}{e} \phi_{i-1}''(a_i, \phi_i, B_i) \quad (29)$$

Although the right-hand side of (29) can in principle be obtained by differentiating twice the previous equation for  $\phi_{i-1}$ , the number of terms in the expression for  $\phi_i''$  becomes rapidly large for  $i \geq 2$ . Yet expressions for  $\phi_1'$  and  $\phi_1''$  have been obtained and can be found in the appendix. Fortunately, the determination of expressions for higher order second derivatives is unnecessary because numerical solutions for the second order approximation show that, even in thin electron-dominated layers,  $a_2$  and  $B_2$  do not differ significantly from  $a_1$  and  $B_1$ . Therefore, for orders larger than 2 [ $i \geq 3$  in (29)], (10) and (12) have been decoupled from (29) and the method used to determine  $\phi_2$ ,  $\phi_3, \dots$  can be stated as follows :

1) Coupled with the differential equations (10) and (12), the following equation

$$\sum Z n(a_2, \phi_2) = - \frac{\epsilon_0}{e} \phi_1''(a_2, \phi_2, B_2) \quad (30)$$

controls the second order approximation for  $\phi_2$ . In this equation,  $\phi_1''(a_2, \phi_2, B_2)$  has been obtained by differentiating (28) twice with respect to  $x$ . Its mathematical formulation can be found in the appendix. The coupled equations (10), (12) and (30) are then solved numerically. Their solutions are  $a_2$ ,  $\phi_2$  and  $B_2$ .

2) From the set of values for  $\phi_2(x)$ , the derivatives  $\phi_2'$  and  $\phi_2''$  are determined numerically by a Lagrangian method of interpolating polynomial using 5 or 7 points.

3) From (29), third and higher order approximations for  $\phi$  are successively determined numerically by the Brent's method while keeping the second order approximation  $(a_2, B_2)$  for  $a$  and  $B$ . From these higher order approximations ( $i \geq 3$ ),  $\phi_3''$ ,  $\phi_4'' \dots$ , are also determined numerically by a Lagrangian method of interpolating polynomial.

This iterative method of solving Poisson's equation remains near to the physics sustaining the natural tendency of plasmas to maintain quasi-neutrality. Indeed, each iteration leads to a new potential resulting from the weak charge separation computed in the previous iteration. In a few iterations, the deviation from quasi-neutrality can therefore be easily deduced. When the iterative process is convergent (it is in fact rapidly convergent for broad layers) the solution requires only solving a few non-linear algebraic equations at each value  $x$ . As a constant  $x$ -spacing is not required in this process, the algebraic equation (29) for  $\phi$  can be added to a predictor-corrector scheme for solving the coupled differential equations (10) for the magnetic vector  $a$ . This is particularly interesting when mixed boundary layers (i.e, boundary layers with both electron and ion-gyroradii as characteristic scale lengths) are involved.

Although other methods for solving boundary values differential equations could in principle be applied to this problem, as for instance, the conjugate gradient algorithm [16] or the strongly implicit procedure of Stone [17], these methods usually boil down, at least conceptually, to the solution of large numbers of simultaneous non-linear equations solved by linearization and iteration. Another interesting approach could be the use of the simultaneous overrelaxation (SOR) method [18] for which an initial distribution  $\phi(t,x)$  relaxes to an equilibrium with time derivative vanishing as  $t \rightarrow \infty$ . However, most of these alternative methods use a constant  $x$ -spacing. Therefore, for mixed transition layers, the simultaneous solution of equations (9) and (10)

by these methods should imply a sufficiently small (and constant) x-spacing to be able to describe the very thin electron layers embedded in a broader ion layer. The number of algebraic equations to be solved simultaneously would increase accordingly and the computer storage available to implement these methods should be large enough.

However, the method we have introduced in this paper provides an interesting and completely different numerical procedure which is conceptually simpler and is somewhat more physical. Furthermore, it does not require a large computer storage.

#### 4. SOLVING POISSON'S EQUATION IN AN ION-DOMINATED LAYER

In this section, we consider a TD separating a hot hydrogen plasma from a cooler one. The velocity distribution function for the electrons as a whole is isotropic since an ion-dominated layer is assumed. This layer has plasma boundary conditions characterized by 8 plasma parameters as listed in table I. Across the transition region, the temperature and density of each plasma species are respectively  $\theta(x)$  and  $n(x)$ . Asymptotically,  $\theta(+\infty) = T$ ,  $n(-\infty) = N_\ell$  and  $n(+\infty) = N_r$ . The magnetic field at  $x = -\infty$  is assumed to be 40 nT.

These plasma parameters correspond to two interpenetrated hydrogen plasmas with different characteristics. They correspond to typical magnetospheric plasma populations. The values of the parameters  $C_\ell$  and  $C_r$  [see eq. (3)] are also given in this table. From (3), it can be seen that they are consistent with the fact that the plasma from  $x = -\infty$  (/from  $x = +\infty$ ) is absent at  $x = +\infty$  (/x =  $-\infty$ ).

From table I and from (5), the number densities  $(n)$  for each plasma species can be written down :

$$n^{(1)} = \frac{1}{2} \alpha^{(1)} \exp \left( \frac{e \phi}{kT^-} \right) \operatorname{erfc} \left( \frac{a}{2^{1/2} R_\ell^- B_\ell} \right) \quad (31)$$

$$n^{(2)} = \frac{1}{2} \alpha^{(2)} \exp \left( \frac{e \phi}{kT^-} \right) \operatorname{erfc} \left( - \frac{a}{2^{1/2} R_r^- B_r} \right) \quad (32)$$

TABLE I Plasma boundary conditions and parameters of velocity distribution functions for the ion-dominated layer illustrated in figure 1.

| $\nu$ | Z   | Species   | T     | $N_{\ell}$       | $N_{\tau}$       | $C_{\ell}$ | $C_{\tau}$ |
|-------|-----|-----------|-------|------------------|------------------|------------|------------|
| 1     | - 1 | electrons | 2500  | 0.5              | 0                | 1          | 0          |
| 2     | - 1 | electrons | 2500  | 0                | 0.15             | 0          | 1          |
| 3     | + 1 | protons   | 12000 | 0.5              | 0                | 1          | 0          |
| 4     | + 1 | protons   | 3000  | 0                | 0.15             | 0          | 1          |
|       |     |           | eV    | $\text{cm}^{-3}$ | $\text{cm}^{-3}$ |            |            |

$$n^{(3)} = \frac{1}{2} \alpha^{(3)} \exp \left( - \frac{e \phi}{kT^{(3)}} \right) \operatorname{erfc} \left( \frac{a}{2^{1/2} R_{\ell}^{(3)} B_{\ell}} \right) \quad (33)$$

$$n^{(4)} = \frac{1}{2} \alpha^{(4)} \exp \left( - \frac{e \phi}{kT^{(4)}} \right) \operatorname{erfc} \left( - \frac{a}{2^{1/2} R_{\tau}^{(4)} B_{\tau}} \right) \quad (34)$$

where  $T^{-} = T^{(1)} = T^{(2)}$  ( $= 2500$  eV)

$$R_{\ell}^{-} B_{\ell} = R_{\tau}^{-} B_{\tau} = \left( \frac{m^{-} kT^{-}}{e^2} \right)^{1/2}$$

Let us now choose the electric potential at  $x = -\infty$  equal to zero, i.e.,

$$\phi(-\infty) = 0 \quad (35)$$

then, the parameters  $\alpha$  in (31) through (34) can be seen to be related to the asymptotic densities and to the value of  $\phi$  at  $x = +\infty$ . Taking the plasma neutrality at  $x = +\infty$  into account, i.e.,

$$N_{\ell}^{(1)} = N_{\ell}^{(3)} = N_{\ell} (= 0.5 \text{ cm}^{-3}) \quad (36)$$

$$N_{\tau}^{(2)} = N_{\tau}^{(4)} = N_{\tau} (= 0.15 \text{ cm}^{-3}) \quad (37)$$

one obtains [assuming that  $B(x) > 0$ , so that  $a(-\infty) = -\infty$  and  $a(+\infty) = +\infty$ ],

$$\alpha^{(1)} = \alpha^{(3)} = N_{\ell} (= 0.5 \text{ cm}^{-3}) \quad (38)$$

$$\begin{aligned} \alpha^{(2)} \exp \left[ \frac{e \phi(+\infty)}{kT^{-}} \right] &= \alpha^{(4)} \exp \left[ - \frac{e \phi(+\infty)}{kT^{(4)}} \right] \\ &= N_{\tau} (= 0.15 \text{ cm}^{-3}) \end{aligned} \quad (39)$$

As the electron velocity distribution as a whole is isotropic, eq. (3) implies that  $\alpha^{(2)}$  must be identical to  $\alpha^{(1)}$ . Therefore, from (38) and (39)

$$\phi(+\infty) = \frac{kT^-}{e} \ln \left( \frac{N_{\tau}}{N_{\ell}} \right) (= -3009.932 \text{ V}) \quad (40)$$

$$\alpha^{(4)} = N_{\tau} \left( \frac{N_{\tau}}{N_{\ell}} \right)^{T^-/T^{(4)}} \quad (\approx 0.055 \text{ cm}^{-3}) \quad (41)$$

The electron density throughout the transition is then

$$n^-(x) = n^{(1)} + n^{(2)} = N_{\ell} \exp \left[ \frac{e \phi(x)}{kT^-} \right] \quad (42)$$

Of course, eq. (42) represents a Maxwellian distribution of electrons in the field of a conservative force (the electric field) acting on them. Consequently, the electrons do not contribute to the current density. This can be seen directly from (6) since  $j^{(1)} = -j^{(2)}$  and  $j^- = 0$ . Therefore, the current density inside the layer is only due to the protons and the characteristic thickness of this transition is the proton gyroradius [ $R_{\ell}^{(3)} = 280 \text{ km}$ ,  $R_{\tau}^{(4)} = 86.5 \text{ km}$  with  $B_{\tau} = 64.70 \text{ nT}$ ].

Figure 1 illustrates the structure of the electric potential, electric field and charge density for different orders of approximation in solving Poisson's equation. The distance across the transition is given in unit of the ion gyroradius at  $x = -\infty$  (i.e.,  $R_{\ell}^+ = R_{\ell}^{(3)} = 280 \text{ km}$ ): The bottom panels demonstrate that, on and after the first order approximation, the electric potential is determined with an relative accuracy  $|(\phi_3 - \phi_1)/\phi_3|$  ( $i = 1, 2$ ) much less than  $10^{-7}$ . The quasi-neutrality approximation ( $\phi_0$ ) still remains a very good approximation, since  $|(\phi_1 - \phi_0)/\phi_1|$  is less than  $3 \times 10^{-6}$ . It can also be seen that, on and after the first order approximation, the charge density ( $Q_i$ ) does not exhibit significant differences between successive approximations. This means that the iterations are converging in cases of ion-dominated layers. Note also that the relative error on  $\phi_0$

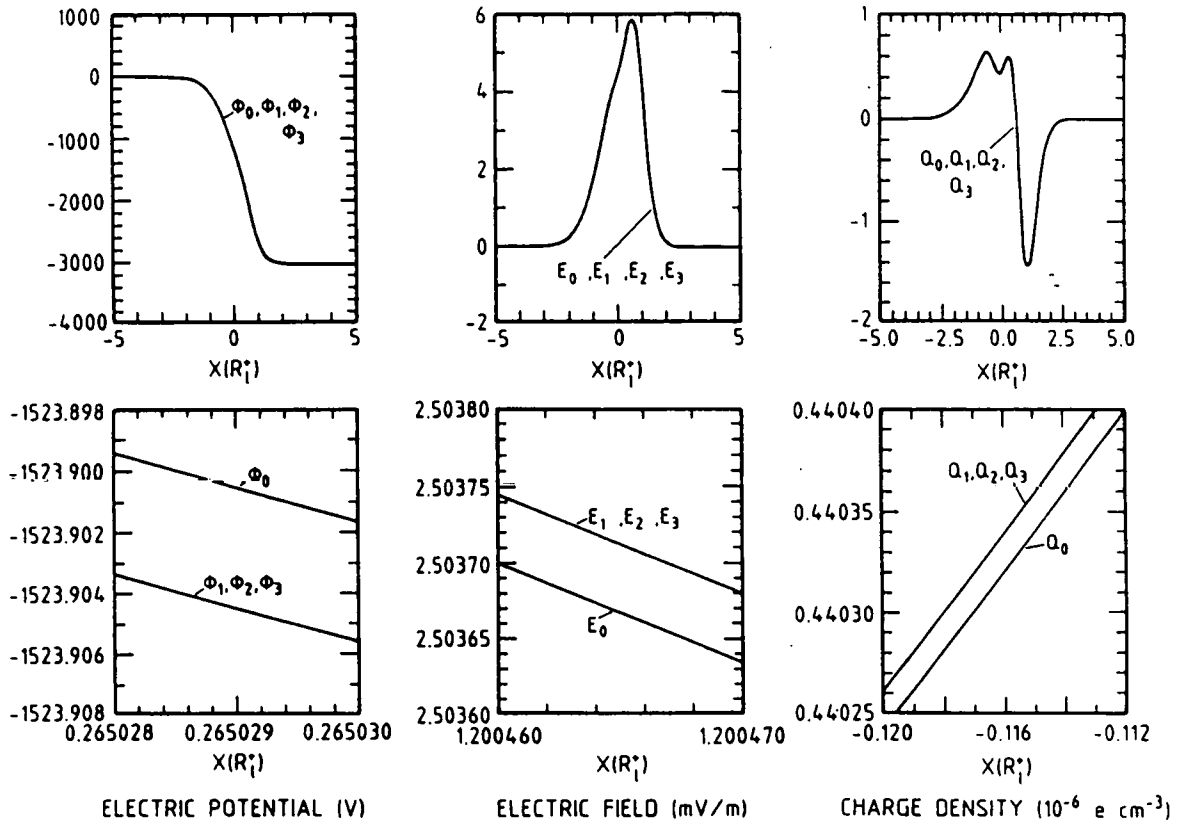


Fig. 1.- Electric structure of an ion-dominated layer. Plasma boundary conditions are given in table I. The magnetic field at  $x = -\infty$  is 40 nT. The ion gyroradius ( $R_l^+ = R_l^{(3)} = 280 \text{ km}$ ) is the characteristic scale length. The left-hand side panels display the electric potential ( $\phi$ ), while the electric field ( $E$ ) and the charge density ( $Q$ ) are represented in the middle and right-hand side panels, respectively. Approximations of the solution of Poisson's equation are displayed up to the third order. The bottom panels are enlargements of a tiny small section of the layer. It can be seen that the successive approximations are very rapidly converging to a solution. This solution still remains very close to the result obtained from the quasi-neutrality approximation or zero-order approximation (within a precision of  $3 \times 10^{-6}$ ). The relative error on  $\phi_0$  can be seen to be of the order of  $Q_0/e n$  ( $n$  being the total electron density).

considered as an approximate solution of Poisson's equation (i.e.,  $3 \times 10^{-6}$ ) is of the order of the relative charge density  $Q_0/en^-$  ( $n^-$  being the total electron density).

### 5. SOLVING POISSON'S EQUATION IN AN ELECTRON-DOMINATED LAYER

The plasma parameters in the case of an electron-dominated layer are given in table II. The magnetic field at  $x = -\infty$  is 40 nT. In this example,  $T^{(1)} = T^{(2)} = T^- = 2500$  eV and  $T^{(3)} = T^{(4)} = T^+ = 5000$  eV.

The expressions for the number densities ( $n$ ) are similar to the ones given by eqs. (31) through (34) with  $T^{(3)} = T^{(4)} = T^+$  and

$$R_\ell^{(3)} R_\ell = R_\ell^{(4)} B_\ell = \left( \frac{m^+ kT^+}{e^2} \right)^{1/2}$$

Assuming (35) to hold and  $B(x) > 0$ , the plasma neutrality at  $x = +\infty$  imposes

$$\alpha^{(1)} = \alpha^{(3)} = N_\ell (= 0.5 \text{ cm}^{-3}) \quad (43)$$

$$\begin{aligned} \alpha^{(2)} \exp \left[ \frac{e \phi(+\infty)}{kT^-} \right] &= \alpha^{(4)} \exp \left[ - \frac{e \phi(+\infty)}{kT^+} \right] \\ &= N_\ell (= 0.4 \text{ cm}^{-3}) \end{aligned} \quad (44)$$

As the ion velocity distribution as a whole is isotropic, (3) and (43) imply that

$$\alpha^{(4)} = \alpha^{(3)} = N_\ell (= 0.5 \text{ cm}^{-3}) \quad (45)$$

Therefore, from (44) and (45)

$$\phi(+\infty) = - \frac{kT^+}{e} \ln \left( \frac{N_\ell}{N_\ell} \right) (= 1115.718 \text{ V}) \quad (46)$$



TABLE II Plasma boundary conditions and parameters of velocity distribution functions for the electron-dominated layer illustrated in figures 2 and 3.

| $\nu$ | Z   | Species   | T    | $N_{\ell}$       | $N_{\mathcal{L}}$ | $C_{\ell}$ | $C_{\mathcal{L}}$ |
|-------|-----|-----------|------|------------------|-------------------|------------|-------------------|
| 1     | - 1 | electrons | 2500 | 0.5              | 0                 | 1          | 0                 |
| 2     | - 1 | electrons | 2500 | 0                | 0.4               | 0          | 1                 |
| 3     | + 1 | protons   | 5000 | 0.5              | 0                 | 1          | 0                 |
| 4     | + 1 | protons   | 5000 | 0                | 0.4               | 0          | 1                 |
|       |     |           | eV   | $\text{cm}^{-3}$ | $\text{cm}^{-3}$  |            |                   |

$$\alpha^{(2)} = N_{\tau} \left( \frac{N_{\tau}}{N_{\ell}} \right)^{T^+/T^-} \quad (= 0.256 \text{ cm}^{-3}) \quad (47)$$

The proton density throughout the transition is then

$$n^+(x) = n^{(3)} + n^{(4)} = N_{\ell} \exp \left[ - \frac{e\phi(x)}{kT^+} \right] \quad (48)$$

As expected, (48) represents a Maxwellian distribution of protons in the field of a conservative force (the electric field) acting on them. Consequently, the protons do not contribute to the current density.

Indeed, from (6),  $j^{(3)} = -j^{(4)}$  and  $j^+ = 0$ . Therefore, the current density inside the layer is only due to the electrons and the characteristic thickness is the electron gyroradius [ $R_{\ell}^{(1)} = 2.98 \text{ km}$ ,  $R_{\tau}^{(2)} = 2.73 \text{ km}$  with  $B_{\tau} = 43.61 \text{ nT}$ ].

Figure 2 illustrates the structure of the electric potential, electric field and charge density up to the 4th order in solving Poisson's equation by the iteration method explained in section 3. The distance across the transition is given in unit of the electron gyroradius at  $x = -\infty$  (i.e.,  $R_{\ell}^- = R_{\ell}^{(1)} = 2.98 \text{ km}$ ). It can be seen that  $|(\phi_4 - \phi_0)/\phi_4|$  is of the order of  $6 \times 10^{-3}$ , while  $|(\phi_4 - \phi_1)/\phi_4|$  is of the order of  $5 \times 10^{-4}$ . It can also be seen that each successive approximation  $\phi_i$  ( $i = 0, \dots, 3$ ) reduces the relative error on  $\phi$ ,  $|(\phi_4 - \phi_i)/\phi_4|$ , by roughly a factor of 10. These successive approximations converge to an asymptotic solution, since the differences between them decrease as the order of approximation increases. This is illustrated in the lower panels : the successive charge densities  $Q_i$  ( $i = 0, \dots, 4$ ) converge to an asymptotic solution located between the  $Q_3$  and  $Q_4$  curves. As previously noted, the relative error on  $\phi_0$ , considered as an approximate solution of Poisson's equation (i.e.,  $6 \times 10^{-3}$ ) is again of the order of  $Q_0/en^-$ .

Fig 1

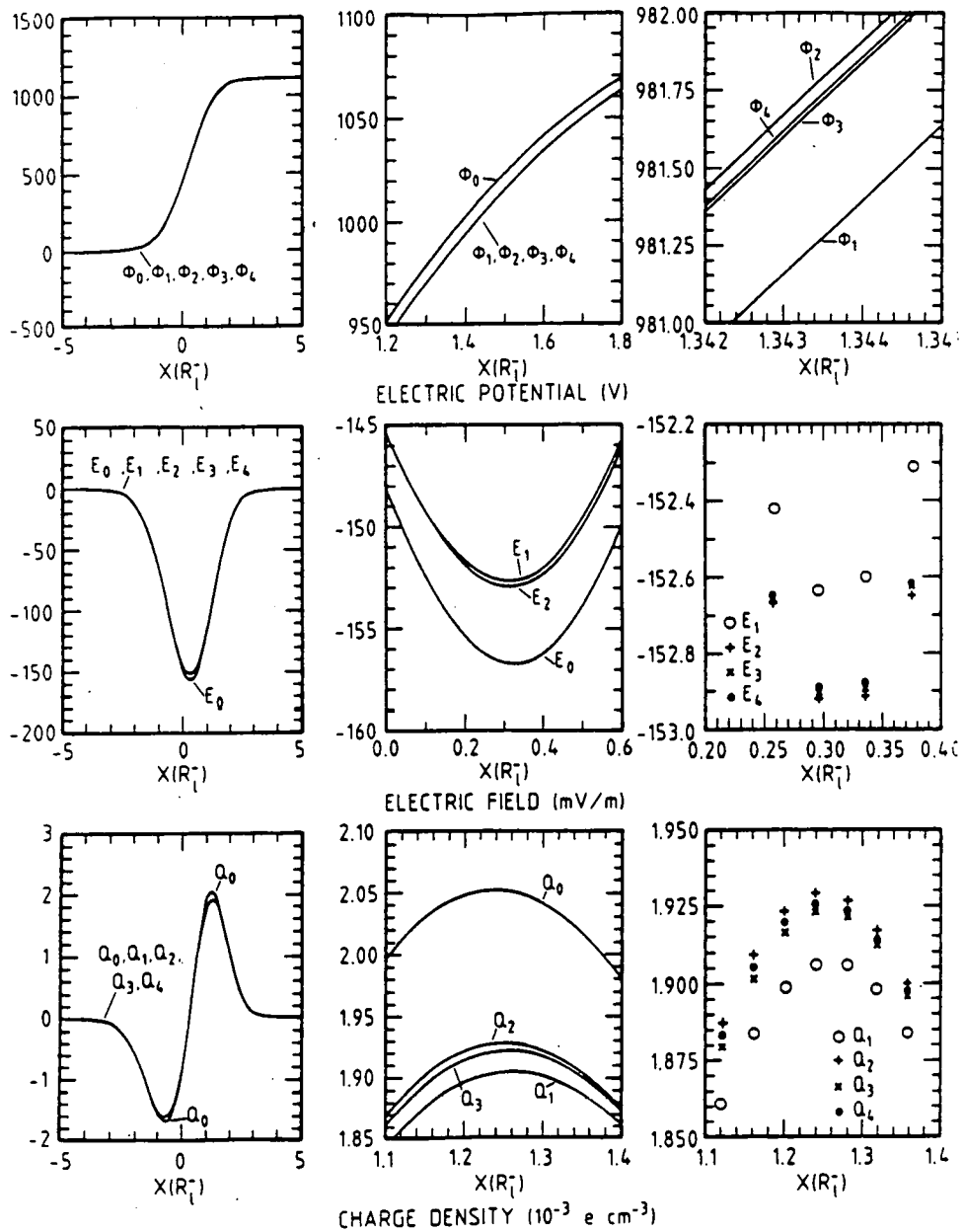


Fig. 2.- Electric structure of an electron-dominated layer. Plasma boundary conditions are given in table II. The magnetic field at  $x = -\infty$  is 40 nT. The electron gyroradius ( $R_L = R_L = 2.98 \text{ km}$ ) is the characteristic scale length. The above panels display the electric potential ( $\phi$ ) while the electric field ( $E$ ) and the charge density ( $Q$ ) are represented in the middle and bottom panels, respectively. Approximations of the solution of Poisson's equation are displayed up to the fourth order. Panels in the second and third columns are enlargements of small or tiny sections of the layer. It can be seen that the successive approximations are converging to a solution. In this example, the characteristic thickness ( $D$ ) is of the order of 5 Debye lengths ( $\lambda$ ). Note that each successive approximation reduces the relative error on  $\phi$  by roughly a factor of 10. The relative error on  $\phi_0$  is still of the order of  $Q_0/en$  ( $n$  being the total electron density). In this example, the solution  $\phi_0$  obtained from the quasi-neutrality approximation holds within a precision of  $6 \times 10^{-3}$ .

The maximum error in the electric field intensity using the quasi-neutrality approximation can be seen to be less than 3%. This maximum error occurs at the center of the transition region, where the gradient of the potential is maximum. Note also that the charge density in this electron-dominated layer is about  $10^3$  times larger than the corresponding charge density computed in the ion-dominated layer illustrated in figure 1. In figures 1 and 2, the Debye length ( $\lambda_D$ ) at  $x = -\infty$  is 526 m. In the electron-dominated layer shown in figure 2, the Debye length at  $x = +\infty$  ( $\lambda_{De}$ ) is 588 m while the characteristic thickness  $D$  lies between  $R_{De}^- = 2.98$  km and  $R_{De}^- = 2.73$  km, i.e.,  $D/\lambda \approx 4.65 - 5.67$ . For such thicknesses ( $D \geq 5 \lambda$ ), the method of successive approximations remains a convergent one and, furthermore, the quasi-neutrality approximation is still a reasonable approximation which can be used for all practical purposes.

In figures 1 and 2,  $E_2$  and  $Q_2$  have been calculated numerically by a Lagrangian method of interpolating polynomial. Third order (figures 1 and 2) and fourth order (figure 2) approximations have been successively determined as solutions of (29) while keeping the second order approximation ( $a_2, B_2$ ) for  $a$  and  $B$ . For the plasma boundary conditions given in table II, figure 3 indicates that  $a$  and  $B$  are not very sensitive to the order of approximation in solving Poisson's equation. Indeed, it can be seen that  $a_2, B_2$  and  $J_2$  (the total current density) do not differ significantly from  $a_1, B_1$  and  $J_1$  (the differences are even much less in ion-dominated layers such as the one illustrated in figure 1). Therefore, on and after the third order, it is quite justified to decouple (29) from equations (10) and (12) as done in this paper.

At this stage, it can be asked whether convergence is still achieved when the thickness  $D$  shrinks to a few Debye lengths  $\lambda$ . The results presented in figure 4 illustrate such a case of a sharper boundary layer. The plasma and magnetic field data are just the same as those presented in table II, except that the number density at  $x = +\infty$  ( $N_e$ ) is now  $0.15 \text{ cm}^{-3}$  instead of  $0.4 \text{ cm}^{-3}$ . We also assumed that the

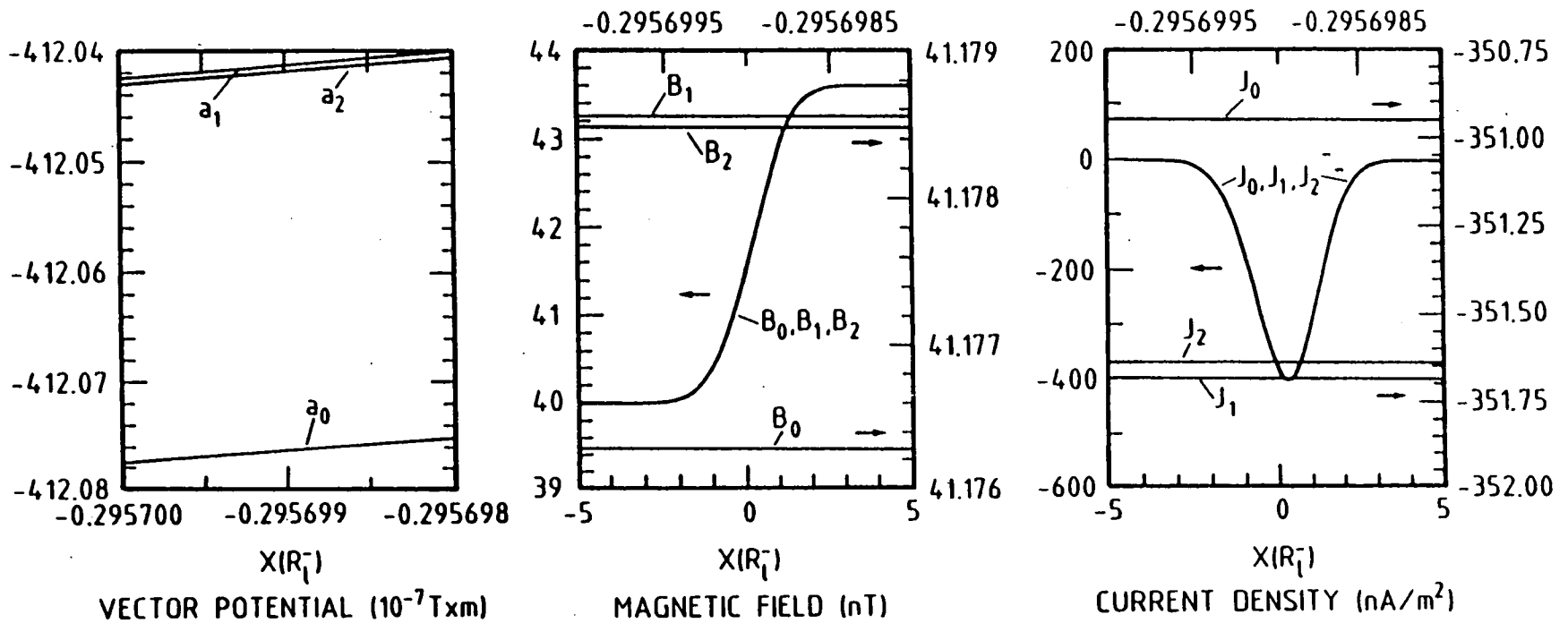


Fig. 3.- An illustration showing that  $a$ ,  $B$  and  $J$  (the total current density,  $J = \sum j$ ) are not very sensitive to the order of approximation in solving Poisson's equation. For the electron-dominated layer considered in this example (same as figure 2), this result is more specifically true on and after the first order approximation. Because  $a$ ,  $B$  and  $J$  do not change significantly when the order of approximation becomes larger than 2, (29) can be decoupled from equations (10) and (12) when considering approximations with order equal to (or larger than) 3.

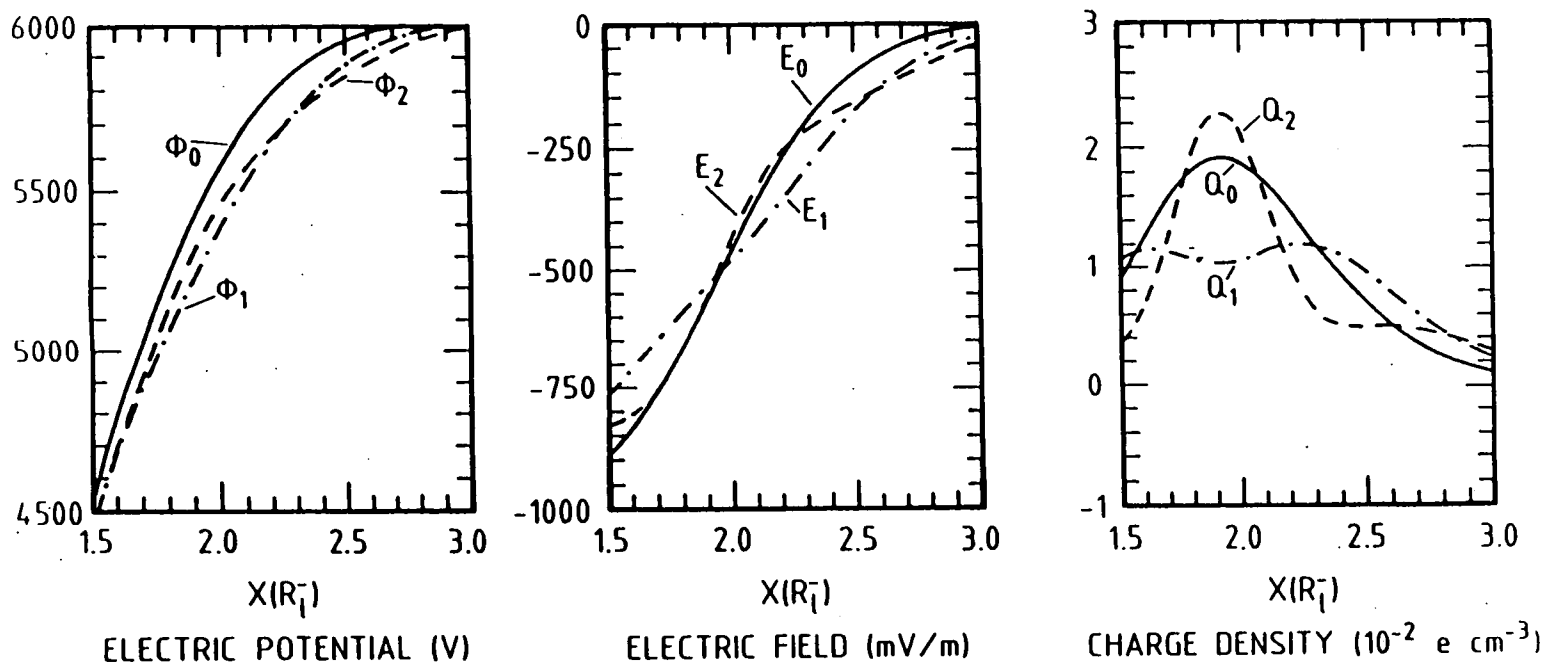


Fig. 4.- Electric structure of an electron-dominated layer. The plasma and magnetic field boundary conditions are identical to those pertaining to figure 2, except that, at  $x = +\infty$ , the number density of the particles of either sign has been lowered ( $N_n = 0.15 \text{ cm}^{-3}$ ). The electron gyroradius ( $R_l^- = R_l^{(1)} = 2.98 \text{ km}$ ) is the characteristic scale length. In this example, the Debye length on the right-hand side of the transition is 960 m, i.e., a small fraction of the electron gyroradius ( $R_n^- = 2.31 \text{ km}$ ). It can be seen that, for positive values of  $x$  ( $x \sim 1.5 - 3 R_l^-$ ), the successive approximations for  $Q_i$  do not converge to a solution, since higher order approximations differ more and more from lower ones. Note that the charge density in this example is 10 times larger than the one illustrated in figure 2.

transition is an electron-dominated layer [ $\phi(+\infty) \approx 6020V$ ,  $\alpha^{(2)} = 0.0135 \text{ cm}^{-3}$ ].

Because the number density at  $x = +\infty$  ( $N_e$ ) has been reduced, the Debye length at the right-hand side of this transition increases ( $\lambda_e = 960 \text{ m}$ ). In this example, the Debye length ( $\lambda_e$ ) is a small fraction ( $R_e^-/\lambda_e = 2.41$ ) of the electron gyroradius at  $x = +\infty$  ( $R_e^- = 2.31 \text{ km}$ ) and it can be seen from figure 4, that for positive values of  $x$  ( $x \sim 1.5 - 3 R_e^-$ ), the successive approximations do not converge to an asymptotic solution, since higher order approximations for  $Q_i$  differ more and more from lower ones. It can therefore be concluded that when  $D$  is of the order of  $1-3 \lambda$ , the method of successive approximations for solving Poisson's equation breaks down. Note also that the charge density in this example is 10 times larger than the one illustrated in figure 2.

It can also be concluded that in the case where the plasma density approaches zero on one side (i.e. plasma-free magnetic field on one side of the TD and field-free plasma on the other) the iterative method proposed above will fail to converge, since when the plasma density approaches zero the Debye length grows indefinitely and the quasi-neutrality approximation always fails to be valid at the outermost edge of the plasma region. This is the case of the classical Ferraro [19] cold plasma sheath model : an early attempt to describe the solar wind-magnetosphere interaction region as a TD between a vacuum geomagnetic field on one side and a cold streaming solar wind on the other. The basic feature of this model was that the random thermal speed of the solar wind was assumed to be small compared to the organized streaming velocity.

An attempt to improve the quasi-neutrality approximation in the case of the cold plasma sheath model was made by Sestero [20]. For that model, which shows large charge separation effects, Sestero found that Poisson's equation could be solved by a Runge-Kutta integration scheme. This result seems to indicate that Runge-Kutta or Hammin's integration

methods could be used when the thickness of the transition becomes smaller than a few Debye lengths, i.e., when charge separation effects cease to be negligible. It can therefore be expected that when the method of successive approximations fails to converge, i.e., when the electric charge density becomes large, usual integration methods such as Runge-Kutta or Hamin's predictor-corrector scheme could successfully be used instead.

## 6. CONCLUSIONS

From this study of Poisson's equation in collisionless TD, it can be concluded that the difficulty in solving this second order differential equation by standard computational methods, such as Runge-Kutta or Hamin schemes, can be avoided in many practical problems by using the so-called quasi-neutrality approximation. For the broad ion-dominated layer illustrated in figure 1 ( $D \sim 10^2 - 10^3 \lambda$ ), the relative error of the quasi-neutral solution is of the order of  $10^{-5} - 10^{-6}$  for the electric potential.

For the thin electron-dominated layer illustrated in figure 2 ( $D \approx 5\lambda$ ), the quasi-neutrality approximation holds to a relative precision of the order of  $10^{-2} - 10^{-3}$ . In this electron-dominated layer, the error on  $\phi$  can be reduced by considering higher order approximations. Each successive approximation reduces the relative error on  $\phi$  roughly by a factor of 10. However, as illustrated in figure 4, when the characteristic thickness of an electron-dominated layer shrinks to a value less than  $3\lambda$ , the method of successive approximations for solving Poisson's equation fails. In these sharp transitions layers, the relative charge density is generally larger than  $10^{-2}$ . In that case, it is expected that Runge-Kutta or Hamin's schemes could be stable integration methods.

In all cases illustrated in this paper, the relative error on  $\phi_0$  is of the order of  $Q_0/en^-$  (the relative charge density). This means that, even in very sharp transitions with  $D \approx 3\lambda$ , the potential obtained



by solving the quasi-neutrality approximation does not differ from the true solution by more than a few percent.

Since, in space plasmas, the Debye length is generally much larger than the characteristic scale lengths of density and field variations, the quasi-neutrality approximation can be used with much confidence in all cases, even in sharp transitions like those involved in the mechanism of generation of discrete auroral arcs as proposed by Evans et al. [21] and Roth et al. [22]. Although the charge-neutrality approximation were used in these studies, the conclusions that were deduced therein remain unaltered, in light of the present study.

The iterative method of successive approximations, that has been developed in this paper to solve Poisson's equation in the case of TD's, is suitable when an improvement in the electric potential accuracy is desirable, particularly when the thickness of the transition region shrinks to  $D \sim 5 \lambda$ . But, even in this case, the correction on the quasi-neutral solution does not exceed 1%.

It has also be shown that the modifications of the electrostatics resulting from the consideration of higher order approximations leave the magnetic field distribution virtually unchanged (see figure 3). Therefore, early in the process of iteration, Poisson's equation can be decoupled from the equation governing the magnetic field. In particular, when the magnetic field coupling is weak, manipulation of the turgid  $\phi_1''$  function in the Appendix can probably be avoided by already ignoring the magnetic field coupling from the second order approximation. In that case, computation of  $\phi_1''$  can be made directly by a Lagrangian method of interpolating polynomial.

When applied to purely electrostatic problems involving solely a non-linear Poisson's equation with source of charge depending on the electrostatic potential, the application of the iteration process is then straightforward. Indeed, as a consequence of the absence of magnetic field coupling, the Laplacian of each  $\phi_n$  can then be computed

directly by the usual Lagrangian method without resorting to explicit expressions for the second derivatives of lower order solutions.

Generally speaking, the iterative method of successive approximations developed in this paper can be easily extended to any problems involving a non-linear Poisson's equation with a right-hand side member depending on the actual potential solution. Compared to other numerical methods, this one is conceptually simpler, has a more physical ground, does not require a constant x-spacing and need no large computer storage.

APPENDIX

The first order approximation  $\phi_1$  is solution of (28), i.e.,

$$\sum Z n(a_1, \phi_1) = - \frac{\epsilon_0}{e} \phi_0''(a_1, \phi_1, B_1) \quad (28)$$

where  $\phi_0''$  is given by (22).

By deriving (28) once and twice with respect to  $x$ , one obtain expressions for  $\phi_1'(a_1, \phi_1, B_1)$  and  $\phi_1''(a_1, \phi_1, B_1)$  respectively (note that the derivative rules for  $j_{rst}$  are similar to those given for  $J_{rst}$  : use  $j$  instead of  $J$  in (25) and (26)). It is found

$$\phi_1' = - \frac{J_{010} + \frac{\epsilon_0}{e} [\mu_0 N_{100}^{-1} (j_{000} Y + a_1 j_{001} J_{010}) + B_1^2 G]}{(N_{100} + \frac{\epsilon_0}{e} W)} B_1 \quad (A.1)$$

with

$$Y = 3a_1 J_{011} - 5 N_{100}^{-1} J_{110} J_{010} + 2 N_{100}^{-2} J_{010}^2 N_{200} \quad (A.2)$$

$$\begin{aligned} G = & - N_{100}^{-1} (J_{011} + a_1^2 J_{012}) \\ & + a_1 N_{100}^{-2} (3 J_{110} J_{011} + 2 J_{010} J_{111}) \\ & - N_{100}^{-3} J_{010} (4 J_{110}^2 + 2 a_1 J_{011} N_{200} + J_{010} J_{210}) \\ & + 3 N_{100}^{-4} J_{010}^2 J_{110} N_{200} \end{aligned} \quad (A.3)$$

$$\begin{aligned}
W = & \mu_0 N_{100}^{-1} (j_{000} J_{110} - j_{000} N_{100}^{-1} N_{200} J_{010} + j_{100} J_{010}) \\
& - a_1 B_1^2 N_{100}^{-1} J_{111} \\
& + B_1^2 N_{100}^{-2} (2 J_{110}^2 + 2 J_{010} J_{210} + a_1 J_{011} N_{200}) \\
& - B_1^2 N_{100}^{-3} J_{010} (J_{010} N_{300} + 6 J_{110} N_{200}) \\
& + 3 B_1^2 N_{100}^{-4} N_{200}^2 J_{010}^2
\end{aligned} \tag{A.4}$$

and

$$\phi_1'' = \frac{1}{(N_{100} + \frac{\epsilon_0}{e} W)} \times$$

$$\left[ -\frac{\epsilon_0}{e} S - \phi_1'^2 N_{200} - 2B_1^2 \phi_1' J_{110} + \mu_0 j_{000} J_{010} - a_1 B_1^2 J_{011} \right] \tag{A.5}$$

with

S =

$$\begin{aligned}
& \alpha_1 \phi_1'^2 + \alpha_2 B_1^2 + \alpha_3 (\mu_0 j_{000})^2 + 2 \alpha_4 B_1 \phi_1' - 2\alpha_5 \mu_0 j_{000} \phi_1' \\
& - 2\alpha_6 \mu_0 j_{000} B_1 - \beta_1 \mu_0 j_{000} - \beta_2 \mu_0 (j_{100} \phi_1' + j_{001} a_1 B_1)
\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}
\alpha_1 = & \mu_0 j_{000} N_{100}^{-1} (-2 N_{100}^{-1} N_{200} J_{110} + J_{210} + 2 N_{100}^{-2} N_{200}^2 J_{010} \\
& - N_{100}^{-1} N_{300} J_{010}) + 2\mu_0 j_{100} N_{100}^{-1} (J_{110} - N_{100}^{-1} N_{200} J_{010}) \\
& + \mu_0 j_{200} N_{100}^{-1} J_{010} + B_1^2 N_{100}^{-2} (2a_1 N_{200} J_{111} - a_1 N_{100} J_{211} \\
& - 10 N_{100}^{-1} N_{200} J_{110}^2 + 6 J_{110} J_{210} - 10 N_{100}^{-1} N_{200} J_{010} J_{210} + 2 J_{010} J_{310} \\
& - 2a_1 N_{100}^{-1} N_{200}^2 J_{011} + a_1 J_{011} N_{300} + 9 N_{100}^{-2} N_{200} J_{010}^2 N_{300} \\
& - 8 N_{100}^{-1} J_{010} J_{110} N_{300} - N_{100}^{-1} J_{010}^2 N_{400} + 24 N_{100}^{-2} N_{200}^2 \\
& \times J_{010} J_{110} - 12 N_{100}^{-3} N_{200}^3 J_{010}^2) \tag{A.7}
\end{aligned}$$

$$\begin{aligned}
\alpha_2 = & \mu_0 N_{100}^{-1} (j_{000} J_{011} - 2 a_1 N_{100}^{-1} J_{110} j_{000} J_{011} \\
& + 2 a_1^2 j_{001} J_{011} + a_1^2 j_{000} J_{012} + 2 N_{100}^{-2} J_{110}^2 j_{000} J_{010} \\
& - 2 a_1 N_{100}^{-1} j_{001} J_{110} J_{010} - a_1 N_{100}^{-1} j_{000} J_{111} J_{010} + j_{001} J_{010} \\
& + a_1^2 j_{002} J_{010}) - a_1 B_1^2 N_{100}^{-1} (3 J_{012} + a_1^2 J_{013}) + B_1^2 N_{100}^{-2} (4 a_1^2 \times \\
& J_{110} J_{012} + 4 J_{110} J_{011} + 5 a_1^2 J_{111} J_{011} + 2 J_{010} J_{111} + 2 a_1^2 J_{010} J_{112}) \\
& - B_1^2 N_{100}^{-3} (10 a_1 J_{110}^2 J_{011} + 12 a_1 J_{110} J_{010} J_{111} + 2 J_{010} J_{011} N_{200} \\
& + 2 a_1^2 J_{011}^2 N_{200} + 2 a_1^2 J_{010} J_{012} N_{200} + 4 a_1 J_{010} J_{011} J_{210} \\
& + a_1 J_{010}^2 J_{211}) + B_1^2 N_{100}^{-4} (12 J_{110}^3 J_{010} + 12 a_1 J_{110} J_{010} J_{011} N_{200} \\
& + 6 J_{110} J_{010}^2 J_{210} + 3 a_1 J_{010}^2 J_{111} N_{200}) \\
& - 12 B_1^2 N_{100}^{-5} J_{110}^2 J_{010}^2 N_{200} \tag{A.8}
\end{aligned}$$

$$\alpha_3 = -2a_1 N_{100}^{-1} J_{011} + 4 N_{100}^{-2} J_{010} J_{110} - 2 N_{100}^{-3} J_{010}^2 N_{200} \quad (\text{A.9})$$

$$\begin{aligned} \alpha_4 = & \mu_0 N_{100}^{-1} (-a_1 N_{100}^{-1} N_{200} j_{000} J_{011} + a_1 j_{100} J_{011} \\ & + a_1 j_{000} J_{111} + 2 N_{100}^{-2} N_{200} j_{000} J_{110} J_{010} - N_{100}^{-1} j_{100} J_{110} J_{010} \\ & - N_{100}^{-1} j_{000} J_{210} J_{010} - N_{100}^{-1} j_{000} J_{110}^2 - a_1 N_{100}^{-1} N_{200} j_{001} J_{010} \\ & + a_1 j_{101} J_{010} + a_1 j_{001} J_{110}) \\ & - B_1^2 N_{100}^{-1} (J_{111} + a_1^2 J_{112}) + B_1^2 N_{100}^{-2} (N_{200} J_{011} + a_1^2 N_{200} J_{012} \\ & + 3a_1 J_{210} J_{011} + 5a_1 J_{110} J_{111} + 2a_1 J_{010} J_{211}) \\ & - B_1^2 N_{100}^{-3} (8a_1 N_{200} J_{110} J_{011} + 6a_1 N_{200} J_{010} J_{111} + 4 J_{110}^3 \\ & + 10 J_{010} J_{110} J_{210} + 2a_1 J_{010} J_{011} N_{300} + J_{010}^2 J_{310}) \\ & + B_1^2 N_{100}^{-4} (18 N_{200} J_{010} J_{110}^2 + 6a_1 N_{200}^2 J_{010} J_{011} + 6 N_{200} J_{010}^2 J_{210} \\ & + 3 J_{010}^2 J_{110} N_{300}) - 12 B_1^2 N_{100}^{-5} N_{200}^2 J_{010}^2 J_{110} \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \alpha_5 = & 2 B_1 N_{100}^{-2} (a_1 N_{200} J_{011} - a_1 N_{100} J_{111} \\ & - 6 N_{100}^{-1} N_{200} J_{010} J_{110} + 2 J_{110}^2 + 2 J_{010} J_{210} \\ & + 3 N_{100}^{-2} N_{200}^2 J_{010}^2 - N_{100}^{-1} J_{010}^2 N_{300}) \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned}
\alpha_6 = & - 2 B_1 N_{100}^{-1} (J_{011} + a_1^2 J_{012}) \\
& + a_1 B_1 N_{100}^{-2} (6 J_{110} J_{011} + 4 J_{010} J_{111}) \\
& - B_1 N_{100}^{-3} (8 J_{010} J_{110}^2 + 4 a_1 J_{010} J_{011} N_{200} + 2 J_{010}^2 J_{210}) \\
& + 6 B_1 N_{100}^{-4} J_{010}^2 J_{110} N_{200}
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
\beta_1 = & \mu_0 a_1 N_{100}^{-1} j_{000} J_{011} \\
& - \mu_0 N_{100}^{-2} j_{000} J_{110} J_{010} \\
& + \mu_0 a_1 N_{100}^{-1} j_{001} J_{010} \\
& - B_1^2 N_{100}^{-1} J_{011} \\
& - a_1^2 B_1^2 N_{100}^{-1} J_{012} \\
& + 3 a_1 B_1^2 N_{100}^{-2} J_{110} J_{011} \\
& + 2 a_1 B_1^2 N_{100}^{-2} J_{010} J_{111} \\
& - 4 B_1^2 N_{100}^{-3} J_{010} J_{110}^2 \\
& - 2 a_1 B_1^2 N_{100}^{-3} J_{010} J_{011} N_{200} \\
& - B_1^2 N_{100}^{-3} J_{010}^2 J_{210} \\
& + 3 B_1^2 N_{100}^{-4} J_{010}^2 J_{110} N_{200}
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
\beta_2 = & - 2 a_1 B_1 N_{100}^{-1} J_{011} + 4 B_1 N_{100}^{-2} J_{010} J_{110} \\
& - 2 B_1 N_{100}^{-3} J_{010}^2 N_{200}
\end{aligned} \tag{A.14}$$

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