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A method for solving poisson's equation in geophysical  
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by

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B E L G I S C H   I N S T I T U U T   V O O R   R U I M T E - A E R O N O M I E

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## **FOREWORD**

The article " A method for solving Poisson's equation in Geophysical and Astrophysical Plasmas " has been presented at the " 17th International Symposium on Rarefied Gas Dynamics , Aachen ( 8-14 July 1990 ) " . It will be published in the proceedings of this symposium .

## **AVANT-PROPOS**

L'article " A method for solving Poisson's equation in Geophysical and Astrophysical Plasmas " a été présenté au " 17th International Symposium on Rarefied Gas Dynamics , Aachen ( 8-14 July 1990 ) " . Il sera publié dans les comptes-rendus de ce symposium .

## **VOORWOORD**

Het artikel " A method for solving Poisson's equation in Geophysical and Astrophysical Plasmas " werd tijdens het " 17th International Symposium on Rarefied Gas Dynamics , Aachen ( 8-14 July 1990 ) " voorgesteld . Het zal in het verslag van dit symposium verschijnen.

## **VORWORT**

Der Text " A method for solving Poisson's equation in Geophysical and Astrophysical Plasmas " wurde während dem " 17th International Symposium on Rarefied Gas Dynamics , Aachen ( 8-14 July 1990 ) " vorgestellt . Er wird im Protokoll dieses Symposiums herausgegeben werden.

A METHOD FOR SOLVING POISSON'S EQUATION IN GEOPHYSICAL  
AND ASTROPHYSICAL PLASMAS

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**ABSTRACT**

The quasi-neutrality (QN) equation is usually solved by an iterative procedure to obtain the electrostatic potential distribution in plasmas. We present, first, a new numerical method to obtain the same result more efficiently. This method is based on the numerical integration of the differential form of the QN equation. In the case when the QN approximation fails to be a valid approximation of Poisson's equation (PS), the latter equation needs to be solved. A robust numerical method to solve PE, with boundary conditions at two different altitudes, has also been proposed. The method consists in integrating numerically by the Quadrature Discretization Method (QDM) a differential form of PE. For boundary conditions corresponding to the QN solution the result coincides with the QN solution.

**RESUME**

L'équation de quasi-neutralité (QN) est habituellement résolue par un processus itératif afin d'obtenir la distribution du potentiel électrostatique dans des plasmas. Nous présentons une nouvelle méthode numérique qui permet d'obtenir cette distribution d'une manière plus efficace. Cette méthode est basée sur l'intégration numérique de la forme différentielle de l'équation de quasi-neutralité. Dans les cas où l'hypothèse de quasi-neutralité n'est plus valide, il devient nécessaire de résoudre l'équation de Poisson. Nous proposons également une méthode numérique robuste qui permet la résolution de l'équation de Poisson avec des conditions aux frontières à deux altitudes différentes. Cette méthode consiste à intégrer numériquement une forme différentielle de l'équation de Poisson par la méthode QDM ( " Quadrature Discretization Method " ). Pour des conditions aux frontières correspondant à la quasi-neutralité, cette méthode fournit la même solution que celle obtenue dans l'hypothèse de la quasi-neutralité.

## SAMENVATTING

De vergelijking van bijna volledige neutraliteit (BVN) wordt, normaalgezien, door een herhalende methode opgelost, om een verdeling van het elektrische potentieel in het plasma te krijgen. We tonen eerst, een nieuwe numerische methode, om, op een zeer werkzame manier, tot hetzelfde resultaat te komen. Deze methode rust op de numerische integratie van de differentiele vorm van de BVN - vergelijking. In het geval waarbij de hypothese van BVN niet juist is, moet de Poissonsvergelijking opgelost worden. Een stevige numerische methode om de Poissonsvergelijking op te lossen, met voorwaarden aan de grens op twee verschillende hoogtes, werd ook voorgesteld. Deze methode integreert, op numerische wijze, een differentiele vorm van de Poissonsvergelijking, door de BVN methode. De oplossing die gekregen wordt, door de voorwaarden aan de grens van de BVN voor te schrijven, is dezelfde als degene die wordt gevonden door de vergelijking van BVN op te lossen.

## ZUSAMMENFASSUNG

Die Gleichung von Quasi-Neutralität (QN) wird normalerweise durch eine iterative Methode aufgelöst, um eine Verteilung vom elektrischen Potential im Plasma zu bekommen. Wir zeigen zuerst, eine neue wirksame numerische Methode, um zum selben Ergebnis zu kommen. Diese Methode wird auf die numerische Integration der differentialen Form der QN Gleichung aufgebaut. Im Fall, wo die Vermutung von QN nicht stimmt, soll die Poissonsgleichung aufgelöst werden. Eine starke numerische Methode um die Poissonsgleichung zu lösen, mit Bedingungen an den Grenzen auf zwei verschiedenen Höhen, wurde auch vorgestellt. Diese Methode integriert numerisch eine differentiale Form der Poissonsgleichung, mit der QN Methode. Die Lösung, die man bekommt, durch Bedingungen an den Grenzen der QN auf zu zwingen, ist dieselbe als die Lösung, die man beim Auflösen der QN Gleichung bekommt..

## I. INTRODUCTION

Photoionization of the components of the terrestrial atmosphere occurs at altitudes greater than approximately 80km, which is the lower level of the ionosphere. At very high altitudes (1000km-2000km), only the lighter constituents such as protons and ionized helium exist together with a population of free electrons. The kinetic theory of this plasma under the influence of a gravitational field and the (self consistent) electrostatic field has been the subject of intense interest among both kinetic theorists and planetary scientists.<sup>1-7</sup>

The calculation of the transport properties of this multicomponent plasma and its stability against escape from the planet has been considered by many authors.<sup>2,4-6</sup> The escape of protons and alpha particles from the topside ionosphere (approximately 1500 km) is referred to as the "polar wind"<sup>2,3</sup> in analogy with the supersonic expansion of the solar atmosphere which is known as the solar wind.<sup>6-8</sup> A major concern with regard the escape of protons and alpha particles from the terrestrial ionosphere is the determination of the self-consistent electrostatic field that plays an important role in the determination of the distribution functions of the ions and electrons.

The quasi-neutrality condition is given by,

$$\sum_i Z_i n_i(h) = 0 \quad (1)$$

where  $Z_i$  and  $n_i(h)$  are the charge and number density profile of the  $i$ th constituent, and  $h$  is the altitude. Equation (1) is generally used to calculate the electrostatic potential distribution  $\phi_E(h)$ , in geophysical and astrophysical plasmas.<sup>7</sup> Indeed, the gravitational force acting with different strengths on the ions and on the electrons produces a charge separation E-field tending to preserve local and global charge neutrality of the plasma.

Equation (1) is a zero order approximation to the solution of Poisson's equation for the electrostatic potential

$$\nabla^2 \phi_E(h) = - (e/\epsilon_0) \sum_i Z_i n_i(h) \quad (2)$$

where  $e$  is the electronic charge. In all cases when the Debye length is much smaller than the characteristic scale length of the system, the electric potential distribution,  $\phi_E(h)$ , obtained by solving Eq. (1) iteratively is a satisfactory zero order solution of Poisson's equation, Eq. (2).

However, at large radial distances, where the geophysical plasma density tends to zero (i.e. where the Debye length becomes arbitrary large) Poisson's equation, which is a non-linear differential equation, must be solved subject to appropriate boundary conditions in order to determine the correct solution. Similarly, at the interface between two regions filled with plasma of different temperatures, densities, electrostatic double layers occur, and the quasi-neutrality approximation gives non-realistic solutions of Eq.(2). The numerical solution of Eq.(2) is generally difficult, since the right hand side of Poisson's equation is very small. It is given by the difference of the electron and ion densities which are very large and almost exactly equal. Truncation errors lead in this case to numerically unstable solutions.

We have found an original mathematical method to avoid this numerical difficulty. This new method, based on the numerical integration of a higher order differential equation derived from Eq.(2), gives accurate and numerically stable solutions for  $\phi_E(h)$ . This method is applicable to the case when the Debye length is small compared to the scale length of variation of  $\phi_E$ . It also gives accurate results in the case when the quasi-neutrality condition, Eq.(1) is a satisfactory first order solution. This method has been applied to the calculation of the ambipolar electric potential distribution in the Earth's ionosphere. Also, a Quadrature Differential Method (QDM) has been used to calculate the solution of the differential equations for boundary conditions given at two different altitudes, and the solutions obtained were compared with the other methods of solution.

## II. ION DENSITY DISTRIBUTIONS

The ion and electron density distributions in planetary ionospheres do not depend only on the gravitational potential  $\phi_g$  given by,  $\phi_g(h) = -GM/(R+h)$ , but also on the polarisation electric field distribution which is induced in the plasma by the gravitational forces acting on the ions and lighter electrons. Indeed the gravitational force for ions is  $m_i/m_e$  times larger than that for the electrons, where  $m_i$  and  $m_e$  are the ion and electron masses respectively. These forces tend to produce a charge separation which induces the polarisation electric field maintaining local quasi-neutrality in the whole ionosphere.

With the assumption that the atmosphere is isothermal and in hydrostatic equilibrium, the density distribution of particles of charge  $Z_j e$  and mass  $m_j$  is given by:

$$n_j(h) = n_j(h_0) \exp[(-m_j \phi_g - Z_j e \phi_E)/kT_j] \quad (3)$$

where  $\phi_E(h)$  represents the electrostatic potential;  $T_j$  are the temperatures of the ions (i) and electrons (e);  $n_j(h_0)$  are their number densities at the reference altitude,  $h_0$ , that we shall take equal to 1000 km appropriate to the Earth's ionosphere. This implies that each constituent is in diffusive equilibrium in the gravitational and electrostatic fields:  $g = -d\phi_g/dh$  and  $E = -d\phi_E/dh$ .

The hydrostatic equations for ions and electrons are then given by :

$$kT_i \, dn_i/dh = -n_i m_i \, d\phi_g/dh - Z_i e n_i \, d\phi_E/dh \quad (4)$$

$$kT_e \, dn_e/dh = -n_e m_e \, d\phi_g/dh - Z_e e n_e \, d\phi_E/dh \quad (5)$$

## III. ELECTROSTATIC POTENTIAL DISTRIBUTION

We obtain the electrostatic field intensity for which the plasma is locally quasi-neutral ( $\sum_j Z_j n_j = 0$ ) by adding Eqs.(4) and (5) after having multiplied their l.h.s and r.h.s. respectively by  $Z_i$  and  $Z_e$ .

$$-(\sum_j Z_j n_j m_j) d\phi_g/dh - (\sum_j Z_j^2 e n_j) d\phi_E/dh = 0 \quad (6)$$

Indeed when the quasi-neutrality condition is verified, the gradients of the particle densities satisfy the relation :  $\sum_i Z_i dn_i/dh = -Z_e dn_e/dh$ . Using the definitions of  $g$  and  $E$ , we find that

$$e E = - \frac{\sum_j Z_j m_j n_j / kT_j}{\sum_j Z_j^2 n_j / kT_j} g \quad (7)$$

This relation is the generalisation for a multi-ionic plasma of the so called Pannekoek-Rosseland (PR) formula which was established by Pannekoek<sup>9</sup> and Rosseland<sup>10</sup> for a fully ionized hydrogen stellar atmosphere. In this case the PR electrostatic potential is simply related to the gravitational potential by :

$$\phi_E(h) = -(m_p - m_e)\phi_g(h)/(2e) + \text{constant}.$$

It is easy to verify that under these conditions the total (gravitational + electrostatic) force and potential energy is the same for the electrons and protons. From Eq. (3), the proton density is strictly equal to the electron density at all altitudes if this is true at the reference altitude  $h_0$ , as imposed by the quasi-neutrality condition.

#### IV. GENERAL SOLUTION OF THE QUASI NEUTRALITY (QN) EQUATION.

For multi-ionic plasmas there is no such simple formula, and the method then used to determine the value of  $\phi_E$  at a given altitude  $h_1$ , is to iterate until that value is found such that the r.h.s. of the quasi-neutrality Eq. (1) is equal to zero with an arbitrarily chosen precision.

The QN equation is a non-linear function of  $\phi_E$ . Common iteration procedures are the bracketing or bisection methods, which can be quite time-consuming in certain cases; furthermore, these numerical methods fail sometimes to converge toward the expected solution. But in general, when the initial value of  $\phi_E(h_1)$  in the iteration procedure is close enough to the solution, convergence is generally



rather fast. This numerical method, well documented in Press et al.<sup>11</sup>, has been used for instance by Lemaire and Scherer<sup>2,6</sup> to determine the electrostatic potential distribution in kinetic polar wind and solar wind models. These numerical methods can become cumbersome and imprecise when first and second order derivatives need to be computed numerically from a series of values of  $\phi_E(h_n)$  corresponding to a set of different altitudes  $h_1, h_2, \dots, h_{N-1}$ . In the next section we propose a more efficient and faster method to determine the solution of the QN equation.

## V. SOLUTION OF QN EQUATION BY NUMERICAL INTEGRATION

Instead of solving the algebraic equation (1) as described above, we found that it is easier and faster to obtain the solution of the QN equation by a different method. This new procedure consists of differentiating analytically the QN Eq.(1), and then integrating numerically the first order differential equation obtained with an appropriate boundary condition at  $h_0$ . With Eq.(3), we find that this differential equation becomes:

$$A(\phi_E) \, d\phi_E/dh = B(\phi_E) \quad (8)$$

where A and B are non-linear functions of  $\phi_E$  :

$$A = (e^2/\epsilon_0) \, R_j [Z_j^2 \, N_j(h)/kT_j] \, \exp[-Z_j e \phi_E / kT_j] \quad (9)$$

$$B = -(e/\epsilon_0) \, R_j [Z_j N_j(h) m_j g / kT_j] \, \exp[-Z_j e \phi_E / kT_j] \quad (10)$$

with

$$N_j(h) = n_j(h_0) \, \exp[-m_j \phi_E / kT_j] \quad (11)$$

The boundary condition at  $h_0$  is imposed to be  $\phi_E = 0$ , in order to satisfy the QN at the reference altitude. The solution of Eq.(8) has been calculated for  $n_{H^+}(h_0) = n_e(h_0) = 800 \, \text{cm}^{-3}$ ;  $T_{H^+} = T_e = 3000 \, \text{K}$  at  $h_0 = 1000 \, \text{km}$  altitude;  $h_{N-1} = 3000 \, \text{km}$  ( $N = 28$ ).

The solid line in Fig.1 shows the distribution of  $\phi_E$

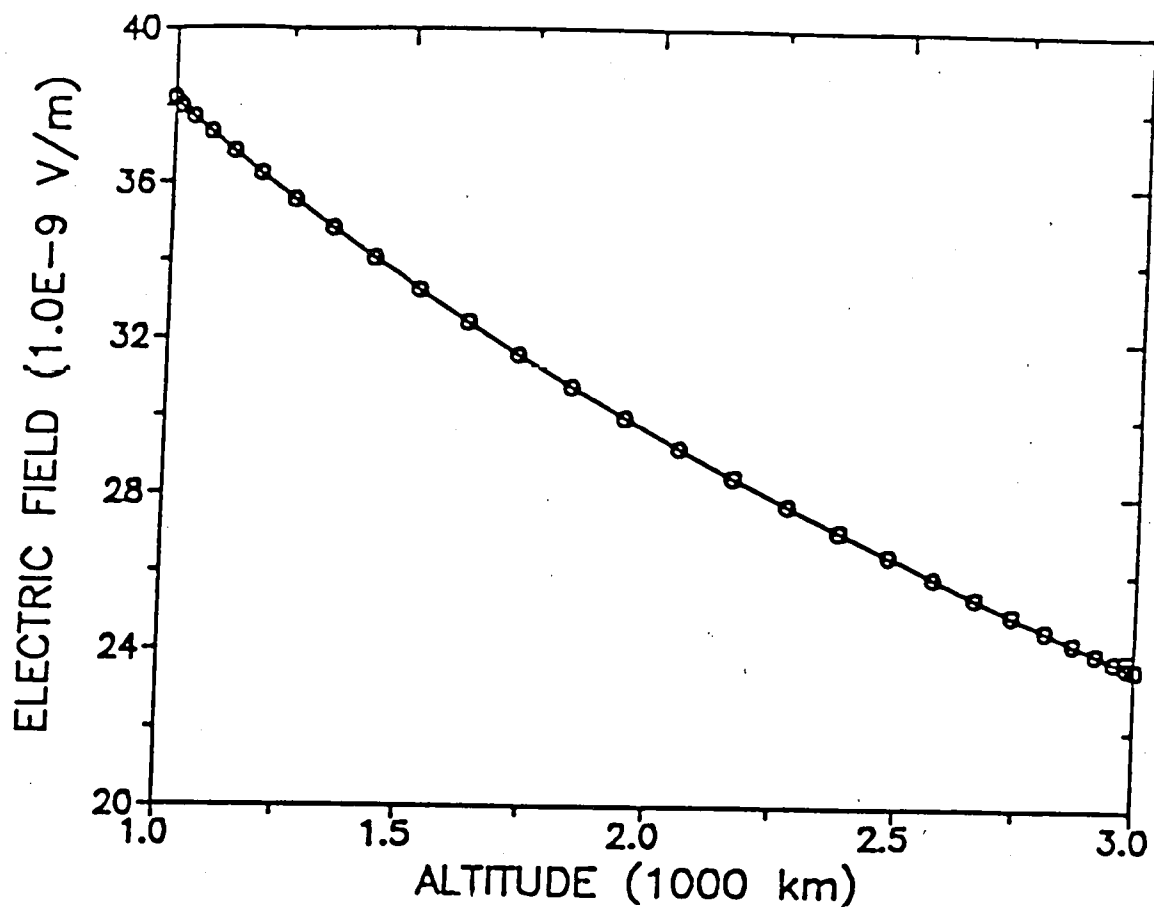


Fig.1. Electrostatic potential distribution corresponding to the quasi-neutrality (QN) solution obtained by solving the algebraic Eq.(1) by an iterative method (solid line); the symbols correspond to the QN solution obtained by integrating the differential Eq.(8) with the Quadrature Discretization Method (QDM); the solution obtained by integrating the 3rd order differential Poisson's Eq.(14) with the QDM is also given by the symbols at 28 quadrature points.

as a function of the altitude; it corresponds to the solution of the QN equation (1) evaluated by any of the algebraic methods described in section IV. The symbols (\*) in Fig.1 correspond to the values of  $\Phi_E(h_n)$  at the 28 quadrature points ( $h_n$ ) which coincide with the zeros of the Legendre Polynomial of 28th order. The Quadrature Discretization Method (QDM) proposed by Shizgal and Blackmore<sup>12</sup> is one of many available numerical method to solve the differential equation (8). Of course, a standard Runge-Kutta or Hamin algorithm can also be used to perform this numerical integration, in this case. The numerical results obtained by any of these algorithms coincide exactly with the solution of Eq.(1) obtained usually by solving this algebraic equation.

The values of the electric field,  $E = -d\Phi_E/dh$ , are given by  $-B/A$ , with A and B defined as functions of h in Eqs.(8) and (9). It can be seen that  $E(h)$  is not obtained here by numerical differentiation, and consequently the results with this method are more accurate than with the algebraic method outlined in section IV.

Similarly, the values of the second derivative of  $\Phi_E$ , which is proportional to the electric charge density, is easier to calculate by differentiating  $B/A$  analytically than by differentiating  $\Phi_E(h_n)$  numerically as indicated in the previous section. The values of the polarisation electric field is generally small in planetary ionospheres :  $E = 10^{-8}-10^{-7}$  V/m. The solid line in Fig.2 shows electric field distribution evaluated as described in section IV by solving iteratively Eq.(1) and differentiating numerically the  $\Phi_E(h_n)$ . The symbols correspond to the results obtained at the quadrature points with the new resolution method outlined in section V. The agreement is perfect.

The distribution of the excess charge density,  $\Delta n/n$ , which is responsible for this polarisation electric field can be determined from

$$\Delta n/n = -\epsilon_0/(e n_e) (d/dh) (B/A) \quad (13)$$

It can be seen that in the case considered, the value of

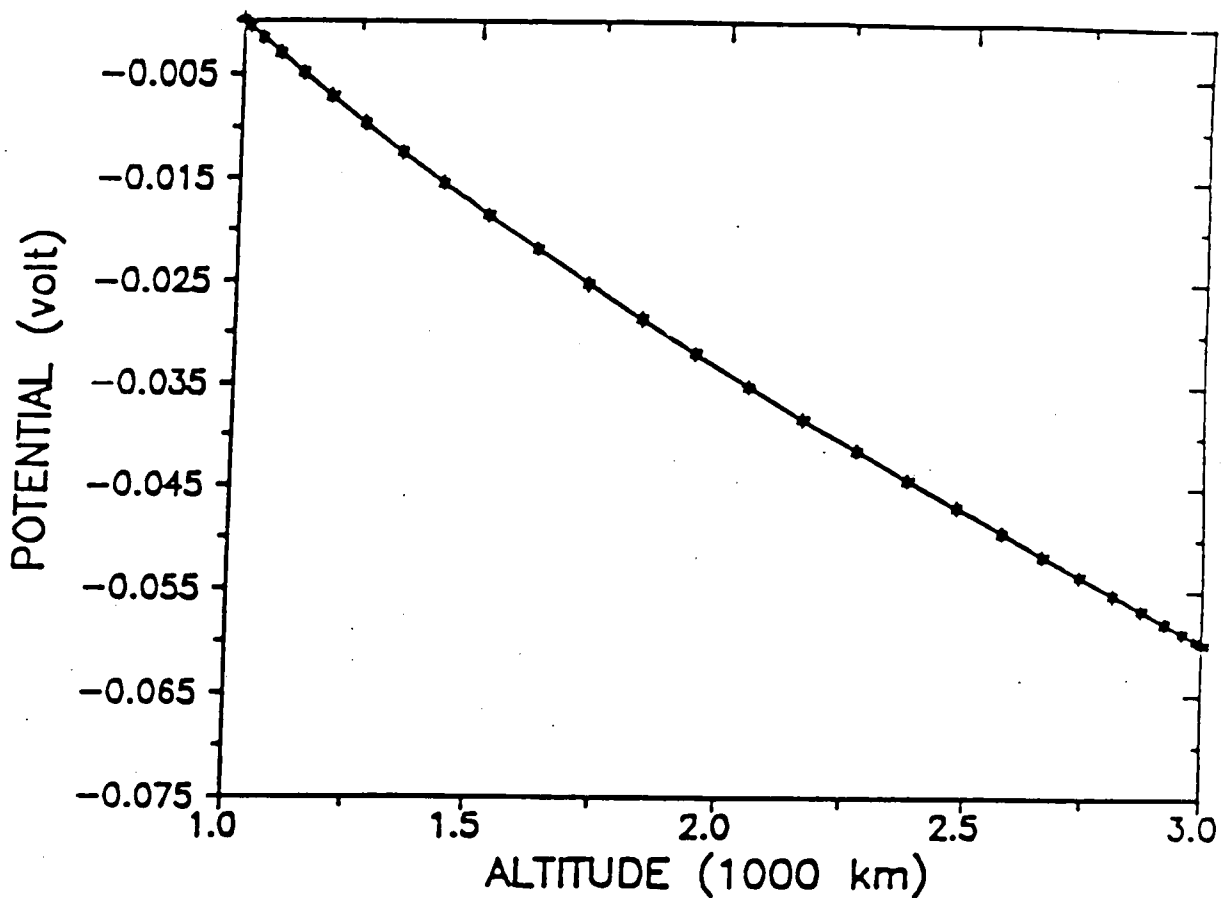


Fig.2. Electric field distribution corresponding to the quasi-neutrality (QN) solution obtained by solving the algebraic Eq.(1) by an iterative method (solid line); the symbols correspond to the QN solution obtained by integrating the differential Eq.(8) with the Quadrature Discretization Method (QDM); the solution obtained by integrating the 3rd order differential Poisson's Eq.(14) with the QDM is also given by the symbols at 28 quadrature points.

$\Delta n/n$  is indeed extremely small:  $[10^{-16} - 10^{-15}]$ . This is generally the case in plasmas where the Debye length  $L_D$  is small compared to the characteristic scale height ( $H$ ) of the density distribution. The Debye length (in cm) is given by :  $L_D = 7 (T/n)^{1/2}$ , where  $T$  is the temperature (in K) and  $n$  the electron number density (in  $\text{cm}^{-3}$ ). In the terrestrial ionosphere where  $T = 3000$  K and  $n(h_0) = 800 \text{ cm}^{-3}$ , the electron density scale  $H = 100$  km, and  $L_D = 13$  cm.

However, along auroral magnetic field lines thin Double Layers (DL) are formed; within these DLs the quasi-neutrality approximation fails to be valid. Within these electrostatic sheaths confined in regions which are only a few Debye lengths thick, Poisson's equation (PE) must be solved instead of its QN approximation.

## VI. RESOLUTION OF POISSON'S EQUATION (PE)

Poisson's equation (2) is a second order differential equation for  $\phi_E$ . Its r.h.s. is a small quantity (proportional to the minute excess charge density) which is unfortunately the difference of large number densities. A small round-off error in the value of the potential  $\phi_E$  can lead in some cases to unphysical diverging solutions, as a consequence of numerical instabilities. Therefore, the PE is difficult to integrate and to solve in the case of plasmas where the Debye length is small compared to the other characteristic scale lengths of the system. This is also the main reason why one is inclined to solve the QN approximation of PE instead of PE directly. In this section, we outline a new technique which enabled us to find numerically stable solutions of PE, in cases where other numerical methods tend to fail.

This mathematical technique consists first in differentiating analytically the two sides of PE (2) and then integrating the third order equation obtained numerically with appropriate boundary conditions. It can be seen that this procedure is similar to that followed above when we differentiated the algebraic QN equation to obtain a first order differential equation for  $\phi_E$ . The third order differential equation derived from PE is given by :

$$d^3\phi_E/dh^3 + A(\phi_E) d\phi_E/dh = B(\phi_E) \quad (14)$$

where A and B are the non-linear functions of  $\phi_E$  given by Eqs.(9) and (10).

The three boundary conditions can either be specified at  $h_0$  or at  $h_{N-1}$  when Runge-Kutta or Hammin algorithms are used to integrate (14). But when a Quadrature Discretisation Method is used boundary conditions can also be defined at different points (altitudes). This is one of the main advantage and power of this alternative numerical algorithm, as compared to the other two methods.

In the following examples the boundary conditions are determined by:  $\phi_E = 0$ , at  $h = h_0$ ;  $\phi_E = \varphi$ , at  $h = h_{N-1}$ ;  $d^2\phi_E/dh^2 = \varphi''$  at  $h = h_{N-1}$ ; where  $\varphi$  and  $\varphi''$  are arbitrary input data for the integration algorithm. When these boundary conditions are chosen to be precisely equal to the corresponding values of the QN solution obtained and discussed in section IV, one obtains almost exactly the same distributions as those deduced above by integrating Eq.(8). The values of the electric potential are so close to that of the QN solution shown in Fig.1 that it is difficult to distinguish them on this graph. The electric field intensity is then also everywhere the same as that of the QN solution shown in Fig.2.

When the boundary condition  $\phi_E$  at  $h_{N-1}$  is taken to be either smaller or larger than the value corresponding to the QN solution, electrostatic potential sheaths are formed at  $h_0$  and  $h_{N-1}$ . Large electric field intensities are then obtained in thin layers a few Debye lengths thick. However, at some distance from these thin layers the electric field intensity tends to be very small as in the QN case. This is an important property of plasmas in general.

## VII. CONCLUSIONS

From this study it is concluded that differentiating analytically the QN equation and integrating it numerically is an easier and more efficient method to obtain the electrostatic potential distribution in a plasma. Therefore

this procedure should replace the classical one consisting of solving the algebraic QN Eq.(1) by bracketing or bisection iterative methods.

Although the QN solution is a very good approximation for the small polarisation electric field intensity in plasmas, to calculate the large charge separation electric potential variation occurring in electrostatic double layers (generally a few Debye lengths thick), Poisson's equation (2) must be solved.

Solving Poisson's non-linear second order differential equation is numerically difficult. A numerically stable and reliable solution has been proposed above. It consists in differentiating analytically the PE (2), and integrating numerically the third order Eq. (14) for appropriate boundary conditions. With this new procedure, we have shown that the QN solution is recovered for specific boundary conditions. But this new method enables one to obtain a more general solution corresponding to double layers.

## REFERENCES

1. Chapman, S., Rarefied Gas Dynamics, 2, pp.69-690, 1961; Shizgal, B., Weinert, U., and Lemaire, J., Rarefied Gas Dynamics, 15, pp.374-383, 1986.
2. Lemaire, J., and Scherer M., Rev. Geophys. Space Phys. 11, pp.427-468, 1973
3. Fahr, H. J., and Shizgal, B., Rev. Geophys. Space Phys. 21, pp.75-124, 1983
4. Schunk, R. W., Rev. Geophys. Space Phys. 15, pp.429-445, 1977.
5. Banks, P. M., Schunk, R. W., and Raitt, J. W., Ann. Rev. Earth Planet. Sci. 4, pp.381-390, 1976.
6. Lemaire J., and Scherer, M., J. Geophys. Res. 76, pp.7479-7490, 1971.
7. Hundhausenm A. J. "Coronal Expansion and Solar Wind", Springer-Verlag, New York, 1972.
8. Parker, E. N., "Interplanetary Dynamical Processes, John Wiley, New York, 1963.
9. Pannekoek, A., Bull. Astron. Inst. Neth. 1, pp.107-110, 1922.
10. Rosseland, S., Mon. Not. R. Astron. Soc. 84, pp.720-725, 1924.
11. Press, W. H., Flannery, B. P., Tenkofsky, S. A. and Vetterling, W. T., "Numerical Recipes", Cambridge University Press, Cambridge, 1988.
12. Shizgal, B., and Blackmore, R., J. Computational. Phys. 55, 313-327, 1984.