STOCHASTIC ANALYSIS OF ENERGY BALANCE MODELS

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1. INTRODUCTION

We would like to discuss some aspects of the long term response of the earth-atmosphere system, in relation to major climatic changes. In view of this, we will use simple energy-balance models of one space dimension (1-d) like the models developed by Budyko¹), Sellers²) and North³). The important feature of these models is to predict a multiplicity of solutions, whose stability properties change as the parameters take different values. For instance, for values of the solar constant close to the present-day one, in addition to the present day climate one has another stable state corresponding to an ice-covered earth. These two climates are separated by a third one, which is unstable and whose ice boundary is situated at an intermediate latitude.

Most of these qualitative features are well reproduced when the temperature field is developed in Legendre series, and a two mode truncation is performed :

$$T(x, t) = T_{0}(t) + T_{2}(t) P_{2}(x)$$
 (1)

where x is the sine of the latitude, $T_0(t) = \int_{-1}^{+1} dx T(x,t)$ and $T_2(t)$ provides a measure of the equator to pole thermal gradient. One can easily show that T_0 and T_2 obey to the following set of equations :

$$C \frac{dT_0}{dt} = Q H_0(x_s) - (A + B T_0)$$

$$C \frac{dT_2}{dt} = Q H_2(x_s) - (B + 6D)T_2$$
(2a)

with \mathbf{x}_{s} expressed in terms of $\mathbf{T}_{0}^{},~\mathbf{T}_{2}^{}$ through the ice boundary condition :

$$T_0 + T_2 P_2(x_s) = -10^{\circ}C$$
 (2b)

Here C is the heat capacity, A and B are infrared cooling coefficients, Q is the solar constant divided by 4, D the turbulent heat diffusion coefficient and H_0 , H_2 are known functions of the latitude of the ice boundary, x_s . Hereafter we normalize the time scale so that the value of C is equal to unity.

As well known, the characteristic times involved in the deterministic balance equations (2) are, typically, of the order of the year. This is much shorter than all known scales of major climatic changes, of which quaternary glaciations provide one of the most striking examples. In a series of recent papers^{4,5)} we pointed out that the study of <u>statistical fluctuations</u> may provide the missing mechanism of climatic change, and presented detailed calculations for zero-dimensional (0-d) energy balance models. It is the purpose of the present communication to extend our previous results to the more realistic case of 1-d energy-balance models.

A fluctuation can couple to a system in many ways. If it modifies randomly the rate of change of a variable independently of the state of the system, it will be an <u>additive fluctuation</u>. If on the other hand it is associated with the random variation of one of the parameters appearing in our equations (such as Q, D, etc. in eqs. (2)), it will be a multiplicative fluctuation. We discuss these cases successively below.

2. ADDITIVE FLUCTUATIONS

In order to describe the dynamics of fluctuations, we have to extend the meaning of eqs. (2) and speak of stochastic differential equations. In the case of additive fluctuations one has :

$$C \frac{dT_0}{dt} = Q H_0(x_s) - (A + B T_0) + F_0(t)$$

$$C \frac{dT_2}{dt} = Q H_2(x_s) - (B + 6D) T_2 + F_2(t)$$
(3)

where $F_i(t)$ (i = 0,2) are random forces expressing the strength of the stochastic effects. Consequently T_0 , T_2 themselves become random processes. To analyse their characteristics, it is necessary to specify the nature of $F_i(t)$. A common assumption is that they define a multi Gaussian white noise :

$$< F_{0}(t) > = < F_{2}(t) > = 0$$

$$< F_{0}(t) F_{0}(t') > = < F_{2}(t) F_{2}(t') > = q^{2} \delta(t - t')$$

$$< F_{0}(t) F_{2}(t') > = 0$$

$$(4)$$

where q^2 is the variance of the random force F(t). It can then be shown that the pair (T₀, T₂) becomes a Markovian process whose probability distribution is a solution of a <u>bivariate Fokker-Planck</u> equation. Thanks to eqs. (4) the latter can be solved exactly at the steady-state. The solution displays the exponential of a <u>climatic</u> <u>potential</u> U(T₀, T₂)⁴, ⁶), which generates the deterministic evolution in all its details. It is a minimum at the present and the icecovered climates, and maximum at the intermediate, cold climate.

Having U, one can also express the mean transition time, τ for leaving the present climate and evolve to the other stable state by running across the <u>potential barrier</u> ΔU provided by the difference between the values of U at the intermediate unstable state and the present-day one :

$$au ~ \exp$$
 ($rac{2}{q^2}$ $riangle U$)

The pre-exponential factors involve the curvature of the surface $U(T_0, T_2)$ near the above two states, and are given in detail elsewhere⁷).

For typical values of $\triangle U$ and q^2 , τ is a very long time, of the order of tens of thousands of years. We have therefore found a natural and completely general mechanism operating on a long time scale, comparable to that of glaciations.



<u>Fig. 1</u>.- Curve (a) : time evolution of the mean value \bar{x}_s averaged over 50 samples for an additive noise with q = 6 yr^{-1/2}°K. Curve (b) : time evolution of a typical stochastic sample under the same conditions. In both cases x_s is plotted every 500 yrs. Parameters used throughout this work are the following : $Q = 340 \text{ Wm}^{-2}$, $A = 214.2 \text{ Wm}^{-2}$, $B = 1.575 \text{ Wm}^{-2}$ and D =0.591 Wm⁻². Initial conditions : $T_0 = 14.9^{\circ}$ C, $T_2 = -28.2$, $x_c = 0.96$.

We have studied the transient behavior of the system by simulating numerically the stochastic dynamics, eqs. (3), using 'an improved Euler method. Fig. 1 describes the time evolution of the statistical mean x_s of the ice line averaged over 50 samples, all starting from the present value $x_s(0) \approx 0.96$, for $q = 6 \text{ yr}^{-1/2} \text{°K}$. We see that the present climate is progressively destabilized by the fluctuations (even though it is stable deterministically). At $\tau \approx$ 24,000 yrs, x_s reaches a value of 0.39, close to the value corresponding to the intermediate unstable state. This is in complete agreement with the theoretical prediction, eq. (5). On Fig. 1 we also give a typical stochastic run. We see that the system remains at relatively high values of x_s most of the time, until it undergoes an abrupt transition across the potential barrier and reaches the ice covered earth.





Fig. 2 gives the results of a second series of stochastic simulations for $q = 7 \text{ yr}^{-1/2} \, {}^{\circ}\text{K}$. The transition time τ is now shorter, but remains in complete agreement with eq. (5). A noteworthy feature is the enhancement of the variance of the fluctuations of x as the system runs across the potential barrier.

3. MULTIPLICATIVE FLUCTUATIONS

In addition to the random imbalances it generates and which give rise to the additive fluctuations analyzed in the preceding Section, the climatic system is subject to a complex external environment. As well known, the mean annual solar influx Q is far from keeping a constant value. Moreover, the very nature of turbulence implies that the transport coefficient D should also fluctuate around some mean. Consequently, we set

$$Q = \overline{Q} (1 + F(t))$$

 $D = \overline{D} (1 + F(t))$

with

<
$$F(t) > = 0$$

< $F(t) F(t') > = q^2 \delta(t - t')$

and replace eqs. (2) by the corresponding stochastic differential equations and the associated Fokker-Planck equations. Contrary to the additive fluctuation case, these equations can no longer be solved at the steady state because there exists no climatic potential associated with these situations. We therefore resort, for most of this Section, to numerical simulations of the stochastic equations for T_0 and T_2 .





Fig. 3 describes the main results in the case of fluctuating Q. We plot against time the mean value of x_s for 50 samples and for various values of the variance q². In all cases we see that the fluctuations of Q give rise to a systematic cooling, in agreement with an earlier result for a 0-d_energy balance model⁸). For small q's the cooling is merely a shift of x_s to a lower lever. For higher q's however a tendency of the system to evolve to the ice-covered state is observed.

(6)

(7)

We come now to the fluctuations of D. Fig. 4 describes the results of the numerical simulations. We have plotted this time, the time average of a single typical sample, over a period of 20,000 yrs, against q. We see that for small q there occurs a slight warming. For q beyond about 2% the tendency is inverted and one observes a cooling, which for higher q's attains catastrophic proportions.



<u>Fig. 4</u>.- Time average of a single typical sample, $< x_s >$, over 20,000 yrs for different values of the variance q^2 in the case of a fluctuating coefficient D.

The warming trend for small q has also been confirmed by an analytic calculation based on the truncation of the infinite hierarchy of moments generated by the Fokker-Planck equation, to the second order ones. The truncated system also predicts that for large values of q both the present and the ice-covered climate become <u>stochastically unstable</u>, even though they are deterministically stable. This is in qualitative agreement with the results of the numerical simulations.

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