18th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 07–09 July 2009

INTRODUCTION TO CLIFFORD ANALYSIS

G. Franssens*

**Belgian Institute for Space Aeronomy*
Ringlaan 3, B-1180 Brussels, Belgium
E-mail: ghislain.franssens@aeronomy.be

Keywords: hypercomplex number, Clifford algebra, Clifford Analysis

Abstract. *The aim of this plenary lecture is to give an introduction to the topics of the special session on "Applicable Hypercomplex Analysis". These topics deal with extensions of the complex numbers to "more complex numbers", called hypercomplex numbers, involving an arbitrary number of imaginary units, and with the function theories that one can build on algebras of hypercomplex numbers. The motivation for studying this subject is the long-standing desire to create a mathematical framework that is capable of modelling the geometric and analytic content of higher dimensional physical phenomena.*

After a basic general introduction to hypercomplex numbers, a more detailed overview of a practical type of hypercomplex analysis, namely Clifford Analysis (CA), will be presented. Clifford Analysis is a part of mathematical analysis where one studies a chosen subset of functions, which take values in a particular hypercomplex algebra, called a Clifford algebra. We will first see what a Clifford algebra is and then the geometrical importance of such algebras will be explained. Thereafter, we will recall the key elements of the familiar Complex Analysis and show how it can be generalized to higher dimensions. The definition of a typical Clifford Analysis, involving a first order vector derivation operator called Dirac operator, will be stated and the physical relevant distinction between Euclidean and pseudo-Euclidean Clifford analyses is discussed. Along the way, the mathematical formulations will be supplemented with interesting physical interpretations, revealing the naturalness of Clifford Analysis and its potential for use in physical applications. Especially, the particular Clifford Analysis based on the Clifford algebra of signature $(1, 3)$ *will emerge as a tailor-made function theory describing electromagnetic and quantum fields in Minkowski space.*

1 INTRODUCTION

We live in a multi-dimensional universe. The actual number of physical dimensions is still a matter of debate among physicists, but of four dimensions we are already certain. This implies that any mathematical framework, if it is to completely and efficiently model the geometric and analytic content of any physical phenomenon, necessarily must incorporate its dimensionality. The long-standing desire for such a mathematical tool has resulted in many attempts to develop mathematical analyses that go beyond the two-dimensional complex numbers and Complex Analysis.

The first question that arises is: "What is an appropriate number system to describe the laws that structure our (and other conceivable) universe(s) ?" This question has been on the table for several centuries and it is still of great interest today. It now appears that Clifford algebra is one of the strongest candidates to qualify as an appropriate number system having physical relevance. In addition, Clifford Analysis, the function theory constructed on top of Clifford algebras, has emerged as a natural tool with applications in physics and engineering.

The development of the number concept has been long and arduous, as is well illustrated by the semantics used in early algebra. To extend the "natural" numbers to "rational", "irrational", "transcendental" and eventually to the "real" numbers took nearly two thousand years. It then required a few more centuries before "imaginary" numbers were combined with real numbers to produce, for the first time, a compound type of number, called a "complex" number. Once the power of the complex numbers was established in algebra, analysis and the geometry of the plane, the exploration of higher dimensional "more complex numbers" began.

It was in his search for a number system that could describe rotations in three dimensions as easily as complex numbers do this for the plane, that the Irish mathematician William Rowan Hamilton in 1843, not without some struggle, eventually discovered the quaternion algebra, [18]. At about the same time, the German mathematician Hermann Günther Grassmann constructed algebras with an outer product, a work he published in 1844, [15]. It was only later in his life that Grassmann realized how quaternions fitted in the framework of his own work. At that moment, the English mathematician William Kingdon Clifford independently arrived at the same unification of both developments. He introduced around 1878 a general set of algebras, which he called geometric algebras and which are now named after him, which contained the complex and quaternion algebras, [5]. Unfortunately, no immediate successor carried his work further and geometric algebras gained little attention for almost a century. Later on, some Clifford algebras were reinvented in quantum mechanics by Wolfgang Pauli and Paul Dirac, in relation with the spin of the electron, [10]. Pauli's algebra of sigma matrices is isomorphic to the real Clifford algebra $Cl_{3,0}$ and Dirac's algebra of gamma matrices is isomorphic to the complexified Clifford algebra $\mathbb{C} \otimes Cl_{1,3}$. Earlier in 1884, in an attempt to construct an algebra of oriented line segments in R^3, the American physicist Josiah Willard Gibbs developed the now familiar vector calculus, [7]. Such a calculus of oriented line segments is extended in a natural way by the Clifford algebra for the underlying linear space, and then also includes higher dimensional objects such as oriented plane segments, etc. In particular, Gibbs' vector calculus in 3-dimensional space is improved (i.e., simplified) and extended by the Clifford algebra $Cl_{3,0}$.

Even today there is still a large proliferation of mathematical systems and nomenclature to express geometrical objects and there interactions: vector calculus, dyadics, exterior differential forms, matrix algebra and tensor coordinate algebra being some of the most common. This diversity reflects a kind of confusion that still exists about geometrical objects of higher dimensionality. This point was made very clear by David Hestenes, who has become one of the most prominent advocates in favour of the use geometrical algebras in physics and mathematics, [19], [21], [20]. His efforts to clarify and simplify the formulation of physics in terms of Clifford algebras have demonstrated the naturalness of these number systems.

In the past 15 years many new contributions, extensions and applications of Clifford Analysis have appeared in the literature. This field is now booming and exciting progress is made in many diverse directions, e.g., [11], [6], [24], [16], [17], [14], [30], [31].

2 HYPERCOMPLEX NUMBERS

There are two main approaches leading to n-dimensional mathematics.

(i) Using n low-dimensional variables, such as real or complex variables, yielding R^n and C^n, respectively.

(ii) Using a single variable that is itself an n-dimensional number.

The topics of the special session on "Applicable Hypercomplex Analysis" mainly belong to approach (ii) (or a combination of (i) and (ii)). Such numbers are called hypercomplex numbers and can be thought of as generalizations of the complex numbers, in the sense that they contain several "imaginary" parts besides a unique real part.

When developing an analysis based on such an algebra, the aim is to select an interesting set of functions, the value of which at any point is a hypercomplex number.

2.1 Three basic types of complex numbers

Before introducing hypercomplex numbers in full generality, let us first take a closer look at the complex number system itself. One can define the following three basic types of complex numbers.

2.1.1 Elliptic complex numbers

The field $\mathbb{C}_{-1} \triangleq \{x + yi : i^2 = -1, \forall x, y \in \mathbb{R}\}$ are just the ordinary *complex numbers*.

Let $z = x + yi$. Conjugation is defined as $\overline{z} \triangleq x - yi$ and the modulus as $|z| \triangleq \sqrt{\overline{z}z} = \sqrt{x^2 + y^2}$. The modulus commutes with multiplication, $|uv| = |u|\,|v|$, and since it is positive definite, it is a norm. The norm $|z|$ models the Euclidean distance of z from the origin. Therefore, $\mathbb{C}_{-1}$ describes the geometry of Euclid's plane. The "unit circle" is $x^2 + y^2 = 1$. Any number of the form $\rho \triangleq \exp(\theta i)$, $\forall \theta \in \mathbb{R}$, has unit modulus. Multiplication by ρ represents a two-dimensional rotation. Non-zero numbers z such that $|z| = 0$ are called null elements. It is clear that there are no null elements in $\mathbb{C}_{-1}$. Any non-zero z has a unique inverse $z^{-1} = \overline{z}/\left|z\right|^2$, hence $\mathbb{C}_{-1}$ is a division algebra. The complex numbers contain no non-trivial idempotents (other than 0 and 1) and no zero divisors, hence $\mathbb{C}_{-1}$ is an integral domain. Consequently, $\mathbb{C}_{-1}$ is a field.

It will become clear, further on, that $\mathbb{C}_{-1}$ is isomorphic to the Clifford algebra $Cl_{0,1}(R)$.

2.1.2 Parabolic complex numbers

The commutative associative unital ring $\mathbb{C}_0 \triangleq \{x + yi : i^2 = 0, \forall x, y \in \mathbb{R}\}$ are called *dual numbers.*

Let $z = x + yi$. Conjugation is defined as $\overline{z} \triangleq x - yi$ and the modulus as $\|z\| \triangleq \overline{z}z = x^2$. The modulus commutes with multiplication, $\|uv\| = \|u\|\,\|v\|$, and since it is positive definite, it is a norm. The "unit circle" is the two-bladed straight line $x^2 = 1$. Any number of the form $\chi \triangleq$

$\times$	1	i_1	$\ldots$	i_m
1	1	i_1	$\ldots$	i_m
i_1	i_1	$\in \{-1, 0, +1\}$	TBD	TBD
$\ldots$	$\ldots$	TBD	$\in \{-1, 0, +1\}$	TBD
i_m	i_m	TBD	TBD	$\in \{-1, 0, +1\}$

Table 1: Multiplication table for a hypercomplex number system

$\exp(\nu i)$, $\forall \nu \in \mathbb{R}$, has unit modulus. Multiplication by χ represents a two-dimensional Galileo transformation. Non-zero numbers z such that $\|z\| = 0$ are called null elements ($yi, \forall y \in \mathbb{R}$). Null elements are non-invertible, hence $\mathbb{C}_0$ is not a division algebra, and are zero divisors, so that $\mathbb{C}_0$ is also not an integral domain. Any z such that $\|z\| \neq 0$ has a unique inverse $z^{-1} = \overline{z}/\|z\|$.

The structure $\mathbb{C}_0$ describes the geometry of a superplane, consisting of one bosonic dimension (real part) and one fermionic dimension (imaginary part). With a modified definition of norm, the "unit circle" can be turned into a parabola, [23].

2.1.3 Hyperbolic complex numbers

The commutative associative unital ring $\mathbb{C}_{+1} \triangleq \{x + yi : i^2 = +1, \forall x, y \in \mathbb{R}\}$ are called *hyperbolic complex numbers* or split-complex numbers, [32].

Let $z = x + yi$. Conjugation is defined as $\overline{z} \triangleq x - yi$ and the modulus as $\|z\| \triangleq \overline{z}z = x^2 - y^2$. The modulus commutes with multiplication, $\|uv\| = \|u\| \|v\|$, but since it is not positive definite, it is not a (proper) norm. However, $\|z\|$ models the squared Lorentzian distance of z from the origin. Therefore, $\mathbb{C}_{+1}$ describes the geometry of Minkowski's plane. The "unit circle" is the two-bladed hyperbole $x^2 - y^2 = 1$. Any number of the form $\lambda \triangleq \exp(\alpha i)$, $\forall \alpha \in \mathbb{R}$, has unit modulus. Multiplication by λ represents a two-dimensional Lorentz (boost) transformation. Non-zero numbers z such that $\|z\| = 0$ are called null elements ($(x \pm xi), \forall x \in \mathbb{R}$). Null elements are non-invertible, hence $\mathbb{C}_{+1}$ is not a division algebra, and are zero divisors, so that $\mathbb{C}_{+1}$ is also not an integral domain. Any z such that $\|z\| \neq 0$ has a unique inverse $z^{-1} = \overline{z}/\|z\|$.

Unlike the complex numbers, the split-complex numbers contain nontrivial idempotents ($e_\pm \triangleq (1 \pm i)/2$). Any z can be represented as $z = (x - y)e_- + (x + y)e_+$. With respect to the basis of idempotents, multiplication reduces to $(ae_- + be_+)(ce_- + de_+) = (ace_- + bde_+)$, showing that $\mathbb{C}_{+1}$, as an algebra over the reals, is isomorphic to $R \oplus R$ (hence the name split-complex numbers).

It will become clear, further on, that $\mathbb{C}_{+1}$ is isomorphic to the Clifford algebra $Cl_{1,0}(R)$.

2.2 Definition of a hypercomplex number

The aforementioned three basic types of complex numbers suggests the following construction of an algebra of general hypercomplex numbers.

A (real) hypercomplex number z is an element of an $(m + 1)$-dimensional linear space over $\mathbb{R}$, with basis $\{1, i_1, \ldots, i_m\}$, of the form

$$z = a_0 1 + a_1 i_1 + \ldots + a_m i_m, \tag{1}$$

with $1 \in \mathbb{R}$, $(a_0, a_1, \ldots, a_m) \in \mathbb{R}^m$ and with multiplication defined in Table 1.

In Table 1, TBD stands for a linear combination of basis elements that is "to be determined". Irrespective of the choices made for the non-specified entries in Table 1, these numbers form a closed, unital and distributive algebra. In order for an algebra of hypercomplex numbers to be interesting and/or of practical value, additional properties are usually required. The challenge then consists in discovering which multiplication rules to use.

2.3 Algebra generating sequences

2.3.1 Generalized Cayley-Dickson sequence

A systematic procedure that generates an infinite sequence of algebras of hypercomplex numbers is the following generalized Cayley-Dickson construction.

Denote a hypercomplex number by the couple (a, b) and conjugation by $\overline{a}$.

Consider the recursive process:

(i) for $n = 0$, initialize with $a \in \mathbb{R}$, $\overline{a} = a$ and $(a, 0) = a$,

(ii) $\forall n \in \mathbb{Z}_+$ and with $\epsilon_n \in \{-1, 0, +1\}$, define

$$\overline{(a,b)} = (\overline{a}, -b)\,, \tag{2}$$

$$(a,b) \times (c,d) = \big(a \times c + \epsilon_n d \times \overline{b}, \overline{a} \times d + c \times b\big)\,. \tag{3}$$

This produces a sequence of algebras over $\mathbb{R}$, A_n, each characterized by $(\epsilon_1, \ldots, \epsilon_n)$, with $\dim A_{n+1} = 2 \dim A_n$.

The original Cayley-Dickson sequence corresponds with $\epsilon_n = -1$, $\forall n \in \mathbb{Z}_+$, and generates the algebras: $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\mathbb{S}$, etc. This sequence generates algebras having a conjugate and norm, such that the product of an element and its conjugate equals the square of its norm, and each non-zero element has an inverse.

The drawback of the original Cayley-Dickson scheme is that interesting algebraic properties are quickly lost. E.g.:

a real number is its own conjugate – lost by the complex numbers $\mathbb{C}$,

the complex numbers are commutative – lost by the quaternion numbers $\mathbb{H}$,

the quaternion numbers are associative – lost by the octonion numbers $\mathbb{O}$,

the octonion numbers are alternative – lost by the sedenion numbers $\mathbb{S}$.

From the sedenion numbers on, every Cayley-Dickson algebra is power associative and possess zero divisors.

2.3.2 An alternative sequence

Another procedure that generates an infinite sequence of algebras of hypercomplex numbers with interesting properties is the following alternative construction, [26], [8].

Again denote a hypercomplex number by the couple (a, b) and conjugation by $\overline{a}$.

Consider the recursive process:

(i) for $n = 0$, initialize with $a \in \mathbb{R}$, $\overline{a} = a$ and $(a, 0) = a$,

(ii) $\forall n \in \mathbb{Z}_+$ and with $\epsilon_n \in \{-1, 0, +1\}$, define

$$\overline{(a,b)} = (a, -b)\,, \tag{4}$$

$$(a,b) \times (c,d) = (a \times c + \epsilon_n d \times b, a \times d + c \times b)\,. \tag{5}$$

This produces a sequence of algebras over $\mathbb{R}$, B_n, each characterized by $(\epsilon_1, \ldots, \epsilon_n)$, with $\dim B_{n+1} = 2 \dim B_n$.

Every algebra B_n has the following properties.

(i) $\overline{(a,b) \times (c,d)} = \overline{(a,b)} \times \overline{(c,d)}$.

(ii) Every element (a,b) has a modulus defined by $\|(a,b)\| \triangleq \overline{(a,b)} \times (a,b) = (a \times a - \epsilon_n b \times b, 0)$ which is in general not a norm.

(iii) Commutative and associative.

(iv) Every element $(a,b) : \|(a,b)\| \neq 0$ has a unique inverse $(a,b)^{-1}$, given by $(a,b)^{-1} = \overline{(a,b)} \,/\, \|(a,b)\|$, and every element $(a,b) : \|(a,b)\| = 0$ is a zero divisor.

This construction produces an infinite sequence of algebras of hypercomplex numbers which inherits the commutative and associative properties from the reals. E.g., if $\epsilon_n = -1$, $\forall n \in \mathbb{Z}_+$, we get the algebras: $\mathbb{C}$, $\mathbb{H}'$, $\mathbb{O}'$, $\mathbb{S}'$, etc. For instance, $\mathbb{H}'$ is determined by the following multiplication table ($z = a + bi_1 + ci_2 + di_3$)

$\times$	i_1	i_2	i_3
i_1	-1	$+i_3$	$-i_2$
i_2	$+i_3$	-1	$-i_1$
i_3	$-i_2$	$-i_1$	$+1$

(6)

and this algebra is isomorphic to the bicomplex numbers (complex numbers over complex numbers).

3 CLIFFORD ALGEBRAS

Another interesting sequence of algebras of hypercomplex numbers, which also have a profound geometrical importance, are the Clifford algebras.

3.1 Introduction

Clifford defined the algebras $Cl_{p,q}$, based on previous work of Hamilton and Grassmann, around 1878 at University College London. Examples of Clifford algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, but not $\mathbb{O}, \mathbb{S}, \ldots$. Clifford algebras naturally form a triangular sequence, with $\dim Cl_{p,q} = 2^n$, $n \triangleq p + q$.

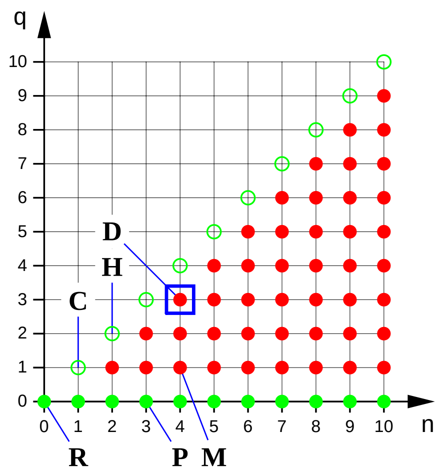

Figure 1: Clifford algebras

This is a truncated schematic overview of the infinite set of Clifford algebras, each dot standing for an algebra. Over each quadratic inner product space of finite dimension n one can define

$n+1$ Clifford algebras, one for each signature with 0 to n negative signs.

3.2 Definition

Let $\mathbf{R}^{p,q} \triangleq (R^n, P)$ denote the n-dimensional linear space R^n together with an inner product given by the canonical quadratic form P of signature (p,q), and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ an orthogonal basis. The universal (real) Clifford algebra $Cl_{p,q}$ over $\mathbf{R}^{p,q}$ is defined by, [2], [25], [1],

$$\begin{aligned} \mathbf{e}_1^2 &= \ldots = \mathbf{e}_p^2 = +1 \text{ and } \mathbf{e}_{p+1}^2 = \ldots = \mathbf{e}_n^2 = -1, & (7) \\ \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i &= 0, i \neq j, & (8) \end{aligned}$$

together with linearity over $\mathbb{R}$ and associativity.

The Clifford product (for two vectors) decomposes into the sum of the inner ($\mathbf{e}_i \cdot \mathbf{e}_j \triangleq P(\mathbf{e}_i, \mathbf{e}_j)$) and outer ($\mathbf{e}_i \wedge \mathbf{e}_j \triangleq \frac{1}{2}(\mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i)$) products,

$$\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j + \mathbf{e}_i \wedge \mathbf{e}_j. \tag{9}$$

Clifford algebras can also be defined over n-dimensional complex space C^n, but we will not consider these here.

3.3 Some properties

Each Clifford algebra $Cl_{p,q}$ is a non-commutative associative unital algebra over $\mathbb{R}$, which naturally forms a graded linear space of dimension 2^n, $Cl_{p,q} = \oplus_{k=0}^{n} Cl_{p,q}^k$. More explicitly, a (real) Clifford number ("cliffor", "multivector") x is a hypercomplex number over $\mathbb{R}$, with $2^n - 1$ imaginary units, of the form (with Einstein's summation convention)

$$x = \underbrace{a1}_{1} + \underbrace{a^i\mathbf{e}_i}_{\binom{n}{1}} + \underbrace{\frac{1}{2!}a^{i_1i_2}(\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2})}_{\binom{n}{2}} + \ldots + \underbrace{\frac{1}{n!}a^{1,\ldots,n}(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n)}_{1}. \tag{10}$$

With $[x]_k$ the k-grade projector, $x = \sum_{k=0}^{n} [x]_k$. A pure grade component $[x]_k$ represents an oriented subspace segment of dimension k, called a k-vector. A Clifford number thus extends the idea of an oriented line segment (i.e., an ordinary vector) by incorporating all possible oriented subspace segments of R^n, with dimensions ranging from 0 to n. E.g., $[x]_2$ represents an oriented plane segment, whereby its $\binom{n}{2}$ components determine its direction in R^n and its magnitude corresponds to its surface area. Each Clifford number encodes a collection of different oriented subspace segments that possibly can exist in R^n. The Clifford product encodes all the natural geometrical constructions that are possible with oriented subspace segments and that result in one or more other oriented subspace segments. E.g., the Clifford product of two 1-vectors (i.e., ordinary vectors) results in a scalar, which represents their scalar product, and a 2-vector, which represents the oriented parallelogram that is naturally constructed from two vectors and is represented by their wedge product. This is the reason why Clifford algebras are also called geometrical algebras. The above interpretation also shows that in order to describe all possible geometrical relations between oriented subspace segments in an n-dimensional space requires numbers with 2^n components.

In particular, the Clifford algebra $Cl_{3,0}$ (numbers with 8 components) is an extension of Gibbs' 3-dimensional vector algebra (numbers with 3 components). The latter can evidently not represent for instance, 2-vectors (bivectors) and is therefore too restrictive to faithfully code certain physical phenomena, such as e.g., a static magnetic field which has the geometric nature of an oriented plane segment in R^3. In Gibbs' vector algebra such quantities are represented by their normal (which is unique in 3 dimensions once a preferred orientation is chosen). However, by neglecting the true geometrical character of such objects and their natural geometrical interactions, a more complicated algebra results (lacking the associative property).

The Clifford product of two numbers of pure grade, $x = [x]_k$ and $y = [y]_l$, is given by

$$xy = \sum_{i=|k-l|,2}^{k+l} [xy]_i \,, \tag{11}$$

with the sum stepping by 2.

A Clifford algebra is also a Z_2-graded algebra, consisting of an even part, made up by the direct sum of the even grade subspaces, and odd part, made up by the direct sum of the odd grade subspaces. The even part of $Cl_{p,q}$ is again a Clifford algebra, called the even subalgebra $Cl^e_{p,q}$.

Clifford algebras are isomorphic to matrix algebras in the following ways.

If $p - q \neq 1 \bmod 4$, $Cl_{p,q}$ is a simple algebra, isomorphic to $M(d, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ for some dimension d.

If $p - q = 1 \bmod 4$, $Cl_{p,q}$ is a semi-simple algebra, isomorphic to $M(d, \mathbb{K}) \oplus M(d, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{H}\}$ for some dimension d.

Clifford algebras acquire their unique geometrical meaning by embedding the reals $\mathbb{R} \hookrightarrow Cl_{p,q}$ by its grade 0 part and $\mathbf{R}^{p,q} \hookrightarrow Cl_{p,q}$ by its grade 1 part. These identifications give Clifford algebras more structure than their corresponding matrix algebras.

The following involutions are defined in any Clifford algebra.

(i) The *grade* (or main) involution: $\widehat{\mathbf{e}_i} \triangleq -\mathbf{e}_i$ and $\widehat{(xy)} \triangleq \widehat{x}\widehat{y}$, which induces an algebra automorphism.

(ii) The *reversal* (or transpose) involution: $\widetilde{\mathbf{e}_i} \triangleq \mathbf{e}_i$ and $\widetilde{(xy)} \triangleq \widetilde{y}\widetilde{x}$, which induces an algebra anti-automorphism.

(iii) The *conjugation* involution: $\overline{\mathbf{e}_i} \triangleq -\mathbf{e}_i$ and $\overline{(xy)} \triangleq \overline{y}\overline{x}$, which induces an algebra anti-automorphism.

Conjugation usually serves to define a norm. However, there are signatures (p, q) for which no norm can be defined. Also, an inverse can not be defined $\forall x \in Cl_{p,q}$ in general. Inverses can sometimes be defined for certain elements in the algebra. For instance, those $x \in Cl_{p,q}$: $\widetilde{x}x \in \mathbb{R} \backslash \{0\}$ have a (left and right) inverse $x^{-1} = \widetilde{x}/(\widetilde{x}x)$. In the Euclidean Clifford algebras $Cl_{n,0}$ and the anti-Euclidean Clifford algebras $Cl_{0,n}$ a norm can be defined as $|x| \triangleq \sqrt{[\widetilde{x}x]_0}$ and $|x| \triangleq \sqrt{[\overline{x}x]_0}$, respectively.

3.4 The Clifford group and its subgroups

3.4.1 Clifford group

The *Clifford group* $\Gamma(p, q)$ is the set of invertible Clifford numbers defined by

$$\Gamma(p, q) \triangleq \left\{ s \in Cl_{p,q} : s\mathbf{v}\widehat{s}^{-1} \in \mathbf{R}^{p,q}, \forall \mathbf{v} \in \mathbf{R}^{p,q} \right\}. \tag{12}$$

Its importance stems from the fact that any $s \in \Gamma(p, q)$ induces an isometry of $\mathbf{R}^{p,q}$, i.e., $P(s\mathbf{v}\widehat{s}^{-1}, s\mathbf{v}\widehat{s}^{-1}) = P(\mathbf{v}, \mathbf{v})$. It can be shown that any element of $\Gamma(p, q)$ consists of a finite products of invertible vectors, i.e., $s \in \Gamma(p, q) \Leftrightarrow s = \mathbf{s}_1 \ldots \mathbf{s}_r : \mathbf{s}_i \in \mathbf{R}^{p,q}$ and $P(\mathbf{s}_i, \mathbf{s}_i) \neq 0$.

The map $\chi_{\mathbf{s}} : \mathbf{R}^{p,q} \to \mathbf{R}^{p,q}$ such that $\mathbf{v} \mapsto \mathbf{s}\mathbf{v}\widehat{\mathbf{s}}^{-1}$, with $\mathbf{s} \in \mathbf{R}^{p,q}$, is the reflection of $\mathbf{v}$ with respect to the $(n-1)$-dimensional hyperplane orthogonal to the invertible vector $\mathbf{s}$.

For further convenience, one also defines the subgroup $\Gamma_e(p,q) \triangleq \Gamma(p,q) \cap Cl^e_{p,q}$, whose elements consist of an even finite product of invertible vectors.

3.4.2 Subgroups of the Clifford group

The *Pin group* $Pin(p,q)$ is the normal subgroup of $\Gamma(p,q)$ defined by

$$Pin(p,q) \triangleq \{s \in \Gamma(p,q) : \bar{s}s = \pm 1\}. \tag{13}$$

Its elements consist of a finite product of unit vectors.

The *Spin group* $Spin(p,q)$ is the normal subgroup of $\Gamma_e(p,q)$ defined by

$$Spin(p,q) \triangleq \{s \in \Gamma_e(p,q) : \bar{s}s = \pm 1\}. \tag{14}$$

Its elements consist of an even finite product of unit vectors.

The *Spin plus group* $Spin_+(p,q)$ is the normal subgroup of $Spin(p,q)$ defined by

$$Spin_+(p,q) \triangleq \{s \in Spin(p,q) : \bar{s}s = 1\}. \tag{15}$$

3.4.3 Group coverings

The famous *Cartan-Dieudonné* theorem states that any rotation (resp., anti-rotation) in R^n can be decomposed as at most n reflections, with the number of reflections being even (resp., odd). This result explains why the Clifford group and its subgroups, being reflection groups, are useful to describe rotations and anti-rotations in $\mathbf{R}^{p,q}$.

More precisely, it is found that the Clifford group and its subgroups are the following coverings of the classical indefinite orthogonal groups,

$$\begin{aligned}
\Gamma(p,q)/\mathbb{R}\backslash\{0\} &\simeq O(p,q), & (16)\\
\Gamma_e(p,q)/\mathbb{R}\backslash\{0\} &\simeq SO(p,q), & (17)\\
Pin(p,q)/\{-1,+1\} &\simeq O(p,q), & (18)\\
Spin(p,q)/\{-1,+1\} &\simeq SO(p,q), & (19)\\
Spin_+(p,q)/\{-1,+1\} &\simeq SO_+(p,q). & (20)
\end{aligned}$$

Herein is $O(p,q)$ the indefinite Orthogonal group, $SO(p,q)$ the indefinite Special Orthogonal group, preserving volume orientation and $SO_+(p,q)$ that subgroup of $SO(p,q)$ that preserves both orientations (the identity component of $SO(p,q)$).

Applied to $p=1$, $q=3$, $O(1,3)$ is the full Lorentz group, $SO(1,3)$ is the volume orientation preserving Lorentz subgroup (with determinant $+1$) (called the "proper Lorentz group") (but containing both ortho-chronous and anti-chronous elements) and $SO_+(1,3)$ is the volume orientation and time-orientation preserving Lorentz subgroup (the "proper ortho-chronous Lorentz group").

3.4.4 Lie algebras

The groups $\Gamma(p,q)$, $Pin(p,q)$, $Spin(p,q)$ and $Spin_+(p,q)$ are Lie groups. It is well-known that any element of a Lie group can (locally) be obtained as the exponential of an element of its Lie algebra. One shows that:

(i) the Lie algebra of $\Gamma(p,q)$ is the sub Lie algebra of $Cl_{p,q}$ consists of the *scalars and bivector* space $Cl_{p,q}^0 \oplus Cl_{p,q}^2$;

(ii) the Lie algebra of $Pin(p,q)$, $Spin(p,q)$ and $Spin_+(p,q)$ consists of the *bivector* space $Cl_{p,q}^2$.

E.g., $s = \exp(\frac{1}{2!}a^{i_1 i_2}(\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2})) \in Spin(p,q)$.

3.5 Rotation in Euclidean space

Define $Spin(n) \triangleq Spin(n,0)$, $SO(n) \triangleq SO(n,0)$ and $\mathbf{R}^n \triangleq \mathbf{R}^{n,0}$. A rotation $R_a : \mathbf{R}^n \to \mathbf{R}^n$ such that $\mathbf{v} \mapsto \mathbf{w}$ by an $a \in SO(n)$ is classically calculated as $w^j = a_i^j v^i$. The rotation of $\mathbf{v}$ by an $s \in Spin(n)$ is given by

$$\mathbf{w} = s\mathbf{v}s^{-1}, \tag{21}$$

wherein $s = \exp\left(\frac{1}{2}\theta u\right)$ with $u = \frac{1}{2!}u^{i_1 i_2}(\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2})$ a unit bivector ($u^2 = -1$). The unit bivector determines the plane which is left invariant by the rotation and θ is the rotation angle. To any $a \in SO(n)$ corresponds an $s \in Spin(n)$, both related by

$$a_i^j \mathbf{e}_j = s\mathbf{e}_i s^{-1}. \tag{22}$$

This makes the twofold covering of $SO(n)$ by $Spin(n)$ explicit, as we can use either s or $-s$ in (22).

Using an element of $Spin(n)$ we can not only rotate vectors, but any multivector $x \in Cl_{p,q}$ as $y = sxs^{-1}$. In particular, for any k-vector $x_k = \frac{1}{k!}x^{i_1 \ldots i_k}(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k})$ we have (orthonormal basis),

$$\begin{aligned} y_k &\triangleq sx_k s^{-1}, \\ &= \frac{1}{k!}x^{i_1 \ldots i_k} s\left(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k}\right)s^{-1}, \\ &= \frac{1}{k!}x^{i_1 \ldots i_k} s\left(\mathbf{e}_{i_1} \ldots \mathbf{e}_{i_k}\right)s^{-1}, \\ &= \frac{1}{k!}x^{i_1 \ldots i_k} s\mathbf{e}_{i_1}s^{-1} \ldots s\mathbf{e}_{i_k}s^{-1}. \end{aligned} \tag{23}$$

Using (22) this becomes

$$\begin{aligned} y_k &= \frac{1}{k!}x^{i_1 \ldots i_k} a_{i_1}^{j_1}\mathbf{e}_{j_1} \ldots a_{i_k}^{j_k}\mathbf{e}_{j_k}, \\ &= \frac{1}{k!}\left(a_{i_1}^{j_1} \ldots a_{i_k}^{j_k} x^{i_1 \ldots i_k}\right)\mathbf{e}_{j_1} \ldots \mathbf{e}_{j_k}, \\ &= \frac{1}{k!}\left(a_{i_1}^{j_1} \ldots a_{i_k}^{j_k} x^{i_1 \ldots i_k}\right)\left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_k}\right), \end{aligned} \tag{24}$$

which reproduces the classical transformation of a tensor of order k, $y^{j_1 \ldots j_k} = a_{i_1}^{j_1} \ldots a_{i_k}^{j_k} x^{i_1 \ldots i_k}$.

The rotation operation, when expressed in the form (21), is independent of the dimensionality of the space and of the object to be rotated. It is the generalization to n dimensions of the familiar rule for rotation in the complex plane, expressed by multiplying by $e^{\theta i}$ ($i^2 = -1$). This spinor representation of rotation has found many applications, e.g., in video games, virtual reality, robotics, aeronautics, crystallography, etc.

4 COMPLEX ANALYSIS

We end our exploration of the algebra side and now turn to the question how to select an interesting subset of Clifford algebra valued functions. We first recall how this is done in complex analysis.

4.1 Choice of a subset of functions

Let $\Omega \subseteq \mathbb{C}$ be non-empty, connected and open and $f : \Omega \to \mathbb{C}$ such that $z = x + iy \mapsto f(z) = u(x, y) + iv(x, y)$.

Historically one questioned the existence of functions f, having a unique complex derivative $f'(z_0)$ at z_0, given by $f'(z_0) \triangleq \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ and which is independent of the direction used to calculate the limit, $\forall z_0 \in \Omega$. Somewhat surprisingly, such functions do exist and they constitute the subset M_Ω of *complex monogenic functions* on Ω. It is easy to shown that for f to be in M_Ω it is necessary and sufficient that $u, v \in C^1$ and that they satisfy in Ω the *Cauchy-Riemann (CR) conditions*:

$$\partial_x u - \partial_y v = 0 \text{ and } \partial_y u + \partial_x v = 0, \ \left(\text{or } \left(\partial_x + i\partial_y\right)(u + iv) = 0\right). \tag{25}$$

It was further found that these complex functions have more interesting properties.

(i) Cauchy's theorem. Each $f \in M_\Omega$ satisfies

$$\oint_C f(w)\, dw = 0, \tag{26}$$

for all simple, counter-clockwise oriented, closed curves $C \subset \Omega$ that are inside C. A complex function that satisfies Cauchy's theorem is said to be *complex holomorphic*.

(ii) For each $f \in M_\Omega$ is $f^{(k)} \in M_\Omega$, $\forall k \in \mathbb{Z}_+$ and f is said to be *complex smooth*.

(iii) Each $f \in M_\Omega$ can be represented in Ω by a complex Taylor series

$$f(z + z_0) = \sum_{k=0}^{+\infty} f^{(k)}(z_0) \frac{(z - z_0)^k}{k!}, \tag{27}$$

and f is said to be *complex analytic*.

(iv) Each $f \in M_\Omega : f' \neq 0$ in Ω generates a *conformal map* from $\Omega \to \mathbb{C}$.

When attempting to extend complex analysis to a higher dimensional analysis, it is not a priori clear which of these properties can be preserved. Which property shall we use as criterion to generate an interesting function set in Clifford Analysis ? Trying the various properties, we end up with function sets which are either too small or too large to be interesting. There is one property however, holomorphy, which appears to be a valid criterion to select functions by. For this reason, we take a closer look at this property, to see what it really means.

4.2 Holomorphy

The following is an equivalent formulation of the property of holomorphy.

Theorem 1 *(Cauchy's integral formula). For any $f \in M_\Omega$ holds that*

$$f(z) = \oint_C \left(\frac{i}{2\pi} \frac{1}{z - w}\right) f(w)\, dw, \tag{28}$$

for all simple, counter-clockwise oriented, closed curves $C \subset \Omega$ and $\forall z \in \Omega$ that are inside C.

The function

$$\frac{i}{2\pi}\frac{\overline{z}}{|z|^2} \tag{29}$$

is called *Cauchy's kernel.*

An expression such as (28) is also called an integral representation of f. It states that it is sufficient to know a complex function, holomorphic in Ω, only at the boundary C of an arbitrary domain inside Ω, in order to reconstruct this function everywhere inside this domain.

5 CLIFFORD ANALYSIS

Clifford Analysis can, from a practical point of view, be divided in two parts:

(i) Clifford Analysis over (anti-)Euclidean spaces. This is now a mature part of mathematical analysis that was developed about 30 years ago by Delanghe, Brackx, Sommen, Souček, et al., [2], [9], [6], [27].

(ii) Clifford Analysis over pseudo-Euclidean spaces. This is a much more complicated theory that is deeply rooted in distribution theory, [34]. For instance, the explicit characterization of the Cauchy kernels of $\mathbf{R}^{p,q}$ requires a detailed study of certain complicated distributions, whose properties are at present still insufficiently known. Much work is to be done to develop this part of mathematical analysis and increase its practical use.

Some authors have studied Clifford Analysis over complex spaces $\mathbf{C}^n$, as a precursor for studying (ii), [3], [28], [29]. The idea is then to take a multi-limit to p real and q imaginary axes in $\mathbf{C}^n$ in order to arrive at $\mathbf{R}^{p,q}$. The relevant distributions based on $\mathbf{R}^{p,q}$ are then obtained as boundary values of the more regular complex distributions based on $\mathbf{C}^n$. The major difficulty of this approach resides in how to deal with the complicated nature and singularities of these boundary value distributions.

5.1 Generalized Cauchy-Riemann condition

Holomorphy in complex analysis is a consequence of the Cauchy-Riemann equations. We therefore try to generalize these equations.

There is a lot of freedom in choosing generalized Cauchy-Riemann conditions. The following are just two possibilities.

(i) Define $D \triangleq \partial_0 + \sum_{i=1}^n \mathbf{e}_i\partial_i$, let $F : \Omega \subseteq R^{n+1} \to Cl_{p,q}$ and consider

$$DF = 0. \tag{30}$$

D is called the generalized *Cauchy-Riemann operator* in R^{n+1}. This is the straightforward generalization of the complex CR conditions.

(ii) Define $\partial \triangleq \sum_{i=1}^n \mathbf{e}_i\partial_i$, let $F : \Omega \subseteq R^n \to Cl_{p,q}$ ($n = p + q$) and consider

$$\partial F = 0. \tag{31}$$

∂ is called the *Dirac operator* in R^n. We will further use choice (ii).

5.2 Clifford Analysis over Euclidean spaces

5.2.1 Cauchy kernel

The distribution $C \in Cl^1_{n,o}$, defined by

$$\partial C = \delta = C\partial, \tag{32}$$

is also called a *Cauchy kernel* for $\mathbf{R}^{n,0}$.

The importance of the Cauchy kernel stems from the fact that the convolution operator $C*$ (resp., $*C$) is a right (resp., left) inverse of the Dirac operator ∂, (since $*\delta = \mathrm{Id} = \delta*$).

The grade 2 part of eq. (32) implies, by Poincaré's lemma, that $C = \partial g$, with g a scalar distribution satisfying

$$\nabla^2 g = \delta. \tag{33}$$

The Cauchy kernel C is then obtained, after operating with the Dirac operator ∂ on a fundamental solution g of the Poisson equation (33), as the regular distribution

$$C\left(\mathbf{x}\right) \triangleq \frac{1}{A_{n-1}} \frac{\mathbf{x}}{\left|\mathbf{x}\right|^n}, \text{ with } A_{n-1} \triangleq \frac{2\pi^{n/2}}{\Gamma\left(n/2\right)}. \tag{34}$$

5.2.2 Holomorphy

The property of holomorphy generalizes to Euclidean Clifford Analysis as is expressed by the following.

Theorem 2 *(Cauchy's integral formula). For any $F : \Omega \subseteq R^n \rightarrow Cl_{n,0} : \partial F = 0$ and any chain $\overline{\Sigma} \subset \Omega$ holds that*

$$F\left(x\right) = \int_{\delta\overline{\Sigma}} C\left(y-x\right) d\sigma_y F\left(y\right), \forall x \in \Sigma, \tag{35}$$

wherein $d\sigma_y$ is the surface element on the orientable boundary $\delta\overline{\Sigma}$.

5.3 Clifford Analysis over pseudo-Euclidean spaces

5.3.1 Cauchy kernel

The distribution $C_{x_0} \in Cl^1_{n,o}$, with parameter point x_0, defined by

$$\partial C_{x_0} = \delta_{x_0} = C_{x_0}\partial \tag{36}$$

is a *Cauchy kernel* for $\mathbf{R}^{p,q}$.

Again, $C_{x_0} = \partial g_{x_0}$, with g_{x_0} a scalar distribution satisfying

$$\Box_{p,q} g_{x_0} = \delta_{x_0}, \tag{37}$$

wherein $\Box_{p,q} = \Delta_p - \Delta_q$ is the generalized d'Alembertian (or wave operator) of signature (p, q).

The Cauchy kernel is no longer a regular distribution and its form depends profoundly on the parity of p and of q. If and only if $(p, q) \in H \triangleq \{(p, q) \in \mathbb{Z}_{o,+} \times \mathbb{Z}_{o,+} : (p, q) \neq (1, 1)\}$, called the Huygens cases, the Cauchy kernel C_{x_0} reduces to the following, somewhat simpler form, [12],

$$C_{x_0} = \frac{\left(x - x_0\right)^S \left(\delta^{((n-2)/2)}_{(P(x-x_0))}\right)_0}{\left(\left(q-2\right)/2\right)_{((n-2)/2)}}. \tag{38}$$

Herein stands $\delta^{(k)}_{(P(x-x_0))}$ for a k-multiplet delta distribution supported on the null-space $P\left(x - x_0\right) = 0$ and $\left(\delta^{((n-2)/2)}_{(P(x-x_0))}\right)_0$ the analytic finite part, the superscript S means spatial conjugation $\left((\mathbf{x}_t, \mathbf{x}_s)^S \triangleq (\mathbf{x}_t, -\mathbf{x}_s)\right)$ and $z_{(k)}$ denotes the falling factorial polynomial. The Huygens cases correspond to universes wherein undisturbed communication is possible, i.e., where initial sharp pulses remain sharp pulses after propagation and are not smeared out by dispersion. Equivalently stated, these are universes where "light" only exists on the null-space, relative to its source point.

5.3.2 Holomorphy

See [13] for the following.

Theorem 3 *For any $F : \Omega \subseteq R^n \to Cl_{p,q} : \partial F = -J$ and a chain $\overline{\Sigma} \subset \Omega$ (satisfying a technical condition) holds that*

$$F(x_0) = \langle C_{x_0}, J\rangle_S + \langle C_{x_0}|_{\delta\overline{\Sigma}}, n\, F|_{\delta\overline{\Sigma}}\rangle_S, \forall x_0 \in \Sigma, \tag{39}$$

wherein n is the outward normal field on the orientable boundary $\delta\overline{\Sigma}$.

Much work is still to be done to fully characterize C_{x_0} and its restriction(s) for arbitrary (p, q).

6 PHYSICAL RELEVANCE

In this section we will see some examples of the appropriateness of Clifford Analysis to describe real world physical phenomena.

6.1 Electromagnetism

Consider as generalized Cauchy-Riemann condition in $Cl_{1,3}$,

$$\partial F = -J, \tag{40}$$

with J a given smooth compact support $Cl_{1,3}$-valued function.

Now assume that F in (40) is a pure grade 2 function (having 6 scalar components) and that J has zero even grades. Writing out eq. (40) in its scalar components reveals that, under these assumptions, it reproduces the Maxwell-Heaviside equations for the electromagnetic (EM) field in vacuum, generated by a charge-current density function J, provided that we identify the EM field with the function F, [22], [33]. The grade 1 part of J contains the electric monopole sources (i.e., the electron charges and currents) and the grade 3 part of J can accommodate any magnetic monopole sources (if they will ever be discovered).

The merit of the model (40) for EM not only lies in its extreme simplicity and compactness, but especially in its analytical tractability, by the methods of Clifford Analysis, for solving EM source problems in vacuum and homogeneous dielectrics. Clifford Analysis over $\mathbf{R}^{1,3}$, with the CR condition (40), so becomes a function theory of EM fields!

The fact that a mathematical product, independent conceived by Clifford and unrelated to the emerging insights in electromagnetism at that time, models so beautifully and efficiently the structure of electromagnetism, is by all the odds a clear indication that physical electromagnetism indeed possesses a deeper number structure. Inversely, one could say that the Clifford numbers in $Cl_{1,3}$ form a kind of “natural” number system for the universe in which we live, especially because this algebra occurs in many other physical contexts (also see below), [19].

6.2 Correspondences

The above physical interpretation can be readily generalized. Choose any Clifford algebra $Cl_{p,q}$, let F be general $Cl_{p,q}$-valued function and J a given smooth compact support $Cl_{p,q}$-valued function. Then eq. (40) becomes a model for a generalized EM in a universe $\mathbf{R}^{p,q}$, with p time dimensions and q space dimensions! Although this may look somewhat farfetched, it

CA	EM
Cauchy-Riemann eq.	Equation of EM
Clifford-valued functions	Generalized EM fields
Holomorphy	Holography
Singularities, Residues	Source fields
Cauchy/Integral theorems	Reciprocity theorems
Riemann-Hilbert problems	EM scattering problems
etc.	etc.

Table 2: Correspondences between CA and EM.

has the benefit that we can now interpret CA as a function theory of generalized EM fields, and use our physical insight in EM to guide us in the development of CA over pseudo-Euclidean spaces. This leads to the correspondences summarized in Table 2.

6.2.1 Holomorphy versus Holography

A particular striking correspondence is the analogy between holomorphy and holography in Table 2.

We have seen that the holomorphic property of Clifford-valued functions satisfying $\partial F = 0$ generalizes to any signature (p, q) and dimension $n = p+q$. A physicist or engineer would probably call this property the possibility to perform holography (popularly called 3-dimensional imaging). That this property would exist for light was postulated by D. Gabor in 1947 and later experimentally verified (in 1963), once the laser as coherent light source became available (Gabor got the Nobel prize for Physics for this in 1971). Remark that our model (40), based on the Clifford algebra $Cl_{1,3}$ together with Cauchy's integral formula, immediately leads to the same conclusion. It is fascinating to see how a mere mathematical reformulation of EM immediately leads to such a far reaching physical insight.

The insight provided by eq. (40) is actually more far reaching than a mere reproduction of Gabor's conjecture. Gabor suggested that, in order to arrive at a perfect imaging, one should register not only the intensity of the light but also its phase. This is only approximately achieved in any holography experiment, since the registration of the phase of light is limited by technological factors. Typically the phase is recorded by registering the intensity of the interference pattern between an incident light beam and the light scattered off the object of which one wants to make a hologram. This may cause the impression that holography is a technical result, resulting from applying interference. The essence of the message, contained in eq. (40), is that the possibility of holography is an intrinsic property of light that resides in the geometrical character of the EM field. This is a very fundamental property of the EM field and has nothing to do with interference. Interference is just a technological aid that is invoked in order to indirectly record phase information by recording intensity information. We need interference in the physical recording of holograms, because we can not record phase information directly. The intrinsic holographic property of the electromagnetic field however is still present in those fields for which the concept of phase cannot be defined. It is present in each field configuration, whether being a static EM field, a non-propagating evanescent field or a very brief transient field. This immediately also implies that there is no such thing as a physical resolution limit (assuming here that we can extend our classical model of EM to the infinitely small), but just a technological resolution limit. The latter is the limit one encounters when using monochromatic waves in holographic imaging and it obviously corresponds to the wavelength of the light used. This wavelength sets a lower limit for the scale of the details that the interference pattern can reproduce. If we would be able to record the time varying value of each EM field component with arbitrary resolution on some boundary surface enclosing some region in three-dimensional space, then eq. (40) tells us that we can reconstruct the full electromagnetic field inside this region with the same arbitrary resolution (provided there are no sources of this field inside this region).

One can argue that an experimental verification of this property, based on current technology, can not be performed and that therefore this property is of marginal practical relevance. Its relevance however lies on the theoretical side, in the sense that this insight shows us the direction along which we must improve our mathematical analysis tools. One could try to construct a hologram for a particular scene, for instance on a closed surface around an object, by computation. If done analytically, the hologram would be known with infinite precision (or resolution). If we could then propagate the information in this hologram into the region outside and inside the hologram, we would have in essence solved an EM scattering problem. The mathematical tools that allow to do exactly this, belong to Clifford Analysis over pseudo-Euclidean space.

It should now be clear that the analytical methods that are appropriate for solving EM field problems are generalizations of the theory of complex holomorphic functions. The algebra of the complex numbers is to be replaced with the Clifford algebra $Cl_{1,3}$ and the complex holomorphic functions with holomorphic Clifford-valued functions over 4-dimensional Minkowski space.

Complex holomorphic functions over the complex plane can be regarded as a mathematical model for the EM field of a kind of (nonphysical) mini electromagnetism, existing in a universe that looks like the 2-dimensional Euclidean plane. The appellation "holomorphic", given by mathematicians in the 19-th century to the main property of the functions occurring in this model, and the independent appellation "holography", given by physicists in the 20-th century to a related physical property of electromagnetic fields in our universe, displays a remarkable and fortunate resemblance.

6.3 Domains of applicability

6.3.1 Clifford Analysis over (anti-) Euclidean space

Here $\partial^2 = \pm\Delta_n$, $\pm\Delta_n g = \delta$ (+: $\mathbf{R}^{n,0}$ or $-$: $\mathbf{R}^{0,n}$) and any Cauchy kernel $C = \partial g$ is a *potential* vector field. We have the following interpretation.

(i) A holomorphic function of grade 1, e.g., $\mathbf{f} \in Cl^1_{3,0} : \partial\mathbf{f} = 0 = \mathbf{f}\partial$, describes in $\mathbf{R}^{3,0}$ an incompressible ($\boldsymbol{\nabla}\cdot \mathbf{f} = 0$)

and irrotational ($\boldsymbol{\nabla} \wedge \mathbf{f} = \mathbf{0}$) fluid without sources nor sinks.

(ii) Elliptical (potential) problems: electrostatics, magnetostatics, Newtonian gravity, fluid problems, etc.

(iii) Is the natural mathematical tool to use in a *static* universe.

Clifford Analyses over (anti-) Euclidean space is a *function theory of holographic static fields*, living in a universe with n time dimensions (or with n space dimensions).

6.3.2 Clifford Analysis over pseudo-Euclidean space

Here $\partial^2 = \Box_{p,q}$, $\Box_{p,q} g_{x_0} = \delta_{x_0}$ and any Cauchy kernel $C_{x_0} = \partial g_{x_0}$ is a *wave* vector field. We now have the following interpretation.

(i) A holomorphic function of grade 1, e.g., $\mathbf{f} = (\rho\mathbf{e}_t, \rho\mathbf{v}) \in Cl^1_{1,3} : \partial\mathbf{f} = 0 = \mathbf{f}\partial$, describes in $\mathbf{R}^{1,3}$ a mass conserved (i.e., $\partial_t\rho - \nabla\cdot(\rho\mathbf{v}) = 0$) 4-dimensional irrotational (i.e., $(\mathbf{e}_t\partial_t, \nabla) \wedge (\rho\mathbf{e}_t, \rho\mathbf{v}) = 0$) fluid without sources nor sinks (with ρ: fluid density and $\mathbf{v}$: fluid velocity).

(ii) Hyperbolic (wave) problems: full EM, acoustics, etc.

(iii) Is the natural mathematical tool to use in a *dynamic* universe.

Clifford Analysis over pseudo-Euclidean space is a *function theory of holographic dynamic fields* living in a universe with p time dimensions and q space dimensions.

6.4 The time-space algebra

The naturalness of $Cl_{1,3}$ for physical applications is further demonstrated by the ease with which one derives all the classical Lorentz-invariant (source) free field equations devised in quantum physics.

(i) $m \neq 0$. By factorization of the Klein-Gordon equation $\partial^2\phi = -m^2\phi$, describing a scalar (spin-0) field with mass m, we get:

(i.1) Dirac's equation describing a spin-$1/2$ field with mass m,

(i.2) Proca's equation for a spin-1 field with mass m.

(ii) $m = 0$. By factorization of the wave equation $\partial^2\phi = 0$, describing a massless scalar field, we get:

(ii.1) The equation for a massless spin-$1/2$ field,

(ii.2) Maxwell-Heaviside's eqs. for a massless spin-1 field (EM).

Furthermore, Clifford Analysis is also useful to describe higher spin fields (spin-$3/2$ Rarita-Schwinger's eq. and generalizations), [4].

All this then raises the question: "Why is $Cl_{1,3}$ so natural ?" A hint at the answer is given by the following observations.

(i) The four dimensionality of our universe.

(ii) The presence of a indefinite quadratic structure of signature $(1,3)$ (metric), manifesting itself as gravity (according to Einstein's general theory of relativity).

(iii) The observation that physical laws are in essence expressing geometrical (symmetry) relationships.

(iv) The fact that Clifford algebras are designed to model all (oriented) geometrical constructions that possibly can exist in a linear space.

So it appears that maybe... Nature's favorite analysis is Clifford Analysis?

REFERENCES

[1] E.F. Bolinder, Clifford Algebra, What is it?, *IEEE Ant. and Prop. Soc. Newsletter*, 18–23, 1987.

[2] F. Brackx, R. Delanghe and F. Sommen, *Clifford analysis*, Pitman, London, 1982.

[3] J. Bureš and V. Souček, Generalized hypercomplex analysis and its integral formulae. *Complex Variables: Theory and Application*, 5, 53–70, 1985.

[4] J. Bureš, F. Sommen, V. Souček, P. Van Lancker, Rarita-Schwinger type operators in Clifford analysis, *J. Funct. Anal.*, 185, 425–455, 2001.

[5] W.K. Clifford, *Mathematical Papers*, Ed. R. Tucker, McMillan, London, 1882.

[6] J. Cnops and H. Malonek, *An introduction to Clifford analysis*, Univ. of Coimbra, Coimbra, 1995.

[7] M.J. Crowe, *A History of Vector Analysis*, Dover, New York, 1993.

[8] C. M. Davenport, Commutative Hypercomplex Mathematics, *http://home.comcast.net/~cmdaven/hyprcplx.htm*, 2003.

[9] R. Delanghe, F. Sommen and V. Souček, *Clifford Algebra and Spinor-Valued Functions*, Kluwer, Dordrecht, 1992.

[10] P.A.M. Dirac, The quantum theory of the electron, *Proc. Roy. Soc.*, A117, 1928.

[11] C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge Univ. Press, Cambridge, 2003.

[12] G.R. Franssens, Cauchy kernels in Ultrahyperbolic Clifford Analysis – Huygens cases, *Proceedings of the Special Session "Clifford and quaternion analysis" of the 7th ISAAC 2009*, Imperial College London, submitted.

[13] G.R. Franssens, Introduction to Clifford Analysis over pseudo-Euclidean space, *Proceedings of the 18th IKM 2009*, Bauhaus University, Weimar.

[14] J. Gilbert and M.A.M. Murray, *Clifford algebras and Dirac operators in Harmonic Analysis*, Cambridge Univ. Press, Cambridge, 1991.

[15] H.G. Grassmann, *Die Lineare Ausdehnungslehre, ein neuer Zweig der Mathematik*, Wiegand, Leipzig, 1844 (English translation: Open Court, Chicago, 1995).

[16] K. Gürlebeck and W. Sprössig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, Birkhäuser Verlag, Basel, 1990.

[17] K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers*, Wiley, New York, 1997.

[18] W.R. Hamilton, *Elements of Quaternions*, Longmans Green, 1866 (reprinted: Chelsea, New York, 1969).

[19] D. Hestenes, *Space-Time Algebra*, Gordon and Breach, New York, 1966.

[20] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometrical Calculus*, Reidel, Dordrecht, 1984.

[21] D. Hestenes, *New Foundations for Classical Mechanics*, Reidel, Dordrecht, 1986.

[22] J.D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1999.

[23] V.V. Kisil, Inventing the Wheel, the Parabolic One, *arXiv:0707.4024*, 2007.

[24] V. Kravchenko and M. Shapiro, *Integral Representations for Spatial Models of Mathematical Physics*, Pitman Research Notes in Mathematics 351, London, 1996.

[25] P. Lounesto, *Clifford Algebras and Spinors*, Cambridge Univ. Press, Cambridge, 2001.

[26] T. Pauwels, Private communication, 2008.

[27] J. Ryan, Basic Clifford analysis, *Cubo Matematica Educacional*, 2, 226–256, 2000.

[28] J. Ryan, Complex Clifford analysis and domains of holomorphy, *J. Austr. Math. Soc. Series A*, 48, 413–433, 1990.

[29] J. Ryan, Complexified Clifford analysis. *Complex Variables: Theory and Application,* 1, 119–149, 1982.

[30] R. Ablamowicz and B. Fauser, Eds., *Clifford Algebras and Their Applications in Mathematical Physics*, Vol. 1 Algebra and Physics, Birkhäuser Verlag, Basel, 2000.

[31] J. Ryan and W. Sprössig, Eds., *Clifford Algebras and Their Applications in Mathematical Physics*, Vol. 2 Clifford Analysis, Birkhäuser Verlag, Basel, 2000.

[32] G. Sobczyk, Hyperbolic Number Plane, *http://www.garretstar.com/HYP2.PDF*, 1995.

[33] J.A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941.

[34] A.H. Zemanian, *Distribution Theory and Transform Analysis*, Dover, New York, 1965.