

On the impossibility of the convolution of distributions

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ABSTRACT

Certain incompatibilities are proved related to the prolongation of an associative derivation convolution algebra, defined for a subset of distributions, to a larger subset of distributions containing a derivation and the one distribution. This result is a twin of Schwartz' impossibility theorem, stating certain incompatibilities related to the prolongation of the multiplication product from the set of continuous functions to a larger subset of distributions containing a derivation and the delta distribution. The presented result shows that the non-associativity of a recently constructed derivation convolution algebra of associated homogeneous distributions with support in \mathbb{R} cannot be avoided.

RESUMEN

Se prueban algunas incompatibilidades relacionadas con la prolongación de un álgebra de convolución de derivación asociativa, definida para un subconjunto de distribuciones a un subconjunto mayor de distribuciones que contienen una derivación y una distribución. Este resultado es un gemelo del Teorema de Imposibilidad de Schwartz declarando algunas incompatibilidades relacionadas a la prolongación del producto de multiplicación de un conjunto de funciones continuas a un subconjunto mayor de distribuciones conteniendo una derivación y una distribución delta. El resultado presente muestra que la no asociatividad de un álgebra de convolución de derivación construida recientemente de distribuciones homogéneas asociadas con soporte en \mathbb{R} no puede evitarse.

Keywords and Phrases: Generalized function, Distribution, Convolution algebra, Impossibility theorem.

2010 AMS Mathematics Subject Classification: 46F10, 46F30.

1 Introduction

In a series of preceding papers, [3]–[9], the author embarked on an in-depth study of the set $\mathcal{H}'(\mathbb{R})$ of Associated Homogeneous Distributions (AHDs) based on (i.e., with support in) the real line \mathbb{R} , [11]. The elements of $\mathcal{H}'(\mathbb{R})$ are the distributional analogues of power-log functions with domain in \mathbb{R} and contain the majority of the distributions one encounters in (one-dimensional) physics applications (including the δ and $\eta \triangleq \frac{1}{\pi}x^{-1}$ distributions). For an introduction to AHDs, an overview of their properties and possible applications of this work, the reader is referred to [3].

The main result of the above study was the construction of a convolution algebra and an isomorphic multiplication algebra of AHDs on \mathbb{R} . The multiplication algebra provides a non-trivial example of how a distributional product can be defined, for an important subset of distributions containing a derivation and the delta distribution, and how this is influenced by L. Schwartz' "impossibility theorem" [14]. Both constructed algebras are non-commutative and non-associative, but in a minimal and interesting way, see [7], [8].

Schwartz' theorem, stating certain incompatibilities in a distributional derivation multiplication algebra, is well-known. One might be inclined to think that the existence of such incompatibilities is unique to the multiplication product of generalized functions. The aim of this note is to show that this is not the case. We state a similar impossibility theorem related to the prolongation of an associative derivation convolution algebra, defined for a subset of distributions $\mathcal{D}'_+ \subset \mathcal{D}'$, to a larger (not necessarily proper) subset of distributions containing a derivation and the one distribution, [13], [10], [15], [11].

Let us first recall Schwartz' impossibility theorem for the multiplication product.

Let \mathcal{C}^k denote the set of continuous ($k = 0$) or k -times continuously differentiable ($k > 0$) functions from $\mathbb{R} \rightarrow \mathbb{R}$.

Theorem 1.1. *Denote by $(\mathcal{C}^0, +, \cdot; \mathbb{R})$ the algebra over \mathbb{R} , consisting of the set \mathcal{C}^0 together with pointwise addition $+$ and pointwise multiplication \cdot , and 0 its $+$ -identity element.*

(i) *Let $(\mathcal{F}', +; \mathbb{R})$ be any linear space over \mathbb{R} such that $\mathcal{F}' \supset \mathcal{C}^0$.*

(ii) *Let $\cdot : \mathcal{F}' \times \mathcal{F}' \rightarrow \mathcal{F}'$ be an associative multiplication product, which coincides with the one defined on \mathcal{C}^0 and with common \cdot -identity element 1 .*

(iii) *Let $D : \mathcal{F}' \rightarrow \mathcal{F}'$ be a derivation, with respect to \cdot , which coincides with the derivation defined on \mathcal{C}^1 and $x \in \mathcal{C}^1 \setminus \{0\} : Dx = 1$.*

Then, $\nexists \delta \in \mathcal{F}' \setminus \{0\} : x \cdot \delta = 0$.

Although this theorem is usually referred to as an impossibility theorem, it does not say that a multiplication product defined on \mathcal{D}' is not possible. It says that a multiplication product defined on \mathcal{C}^0 , with properties as stated in condition (ii), and a derivation defined on \mathcal{C}^1 , with properties as stated in condition (iii), cannot be faithfully prolonged to a superset \mathcal{F}' of \mathcal{C}^0 which contains the delta distribution δ . Notice that \mathcal{F}' is allowed to be a (proper) subset of \mathcal{D}' .

For Schwartz' linear space of distributions $(\mathcal{D}', +; \mathbb{R})$, the multiplication defined for continuous functions is not prolonged to all distributions, so condition (ii) does not hold, and this allows the existence in \mathcal{D}' of a derivation D and a distribution $\delta \neq 0 : x.\delta = 0$ (e.g., $\delta = D^2 \left(\frac{1}{2} |x|\right)$).

In the construction of commutative, associative generalized function algebras, such as by Egorov, [2], Rosinger, [12] or Colombeau, [1], Schwartz' theorem is evaded in a more subtle way. For instance, Colombeau's construction has led to an algebra of generalized functions \mathcal{G} , which does not contain the algebra of the continuous functions as a subalgebra in the usual algebraically exact way. First, the set \mathcal{C}^0 is embedded in \mathcal{G} as a subset $\tilde{\mathcal{C}}^0 \subset \mathcal{G}$ such that \mathcal{C}^0 is "associated" to $\tilde{\mathcal{C}}^0$ in some weak sense, which involves non-standard analysis. Then, Colombeau's product \odot defined on \mathcal{G} , when restricted to $\tilde{\mathcal{C}}^0$, agrees with the usual function product \cdot defined on \mathcal{C}^0 only in his weak sense and the last clause in Theorem 1.1 is circumvented since (i) does not hold. More explicitly, $\forall f \in \mathcal{C}^0$ associated to a $\tilde{f} \in \mathcal{G}$ and $\forall g \in \mathcal{C}^0$ associated to a $\tilde{g} \in \mathcal{G}$ holds that $f.g \in \mathcal{C}^0$ is associated to $\tilde{f} \odot \tilde{g} \in \mathcal{G}$.

We will proof hereafter a twin of Schwartz' impossibility theorem, stating certain incompatibilities in a distributional convolution algebra. We use the notation and definitions introduced in [3].

2 The convolution of distributions

Definition 2.1. Denote by \mathcal{H}' the set of associated homogeneous distributions (AHDs) based on \mathbb{R} and f_m^z a typical element, having degree of homogeneity z and order of association m . Let $\mathcal{D}'_{\mathbb{Z}-}$ stand for the set of all finite sums over \mathbb{C} of elements of $\mathcal{H}'_{\mathbb{Z}-} \triangleq \{f_m^{-k-1} \in \mathcal{H}', \forall k, m \in \mathbb{N}\} \subset \mathcal{H}'$, the set of AHDs based on \mathbb{R} with negative integer degrees.

Theorem 2.2. Denote by $(\mathcal{D}'_{\mathbb{Z}-}, +, *, \mathbb{C})$ the algebra over \mathbb{C} , consisting of the set $\mathcal{D}'_{\mathbb{Z}-}$ together with distributional addition $+$ and distributional convolution $*$, and 0 its $+$ -identity element.

(i) Let $(\mathcal{F}', +; \mathbb{C})$ be any linear space over \mathbb{C} such that $\mathcal{F}' \supset \mathcal{D}'_{\mathbb{Z}-}$.

(ii) Let $* : \mathcal{F}' \times \mathcal{F}' \rightarrow \mathcal{F}'$ be an associative convolution product, which coincides with the one defined on $\mathcal{D}'_{\mathbb{Z}-}$ and with common identity element δ .

(iii) Let $X : \mathcal{F}' \rightarrow \mathcal{F}'$ be a derivation, with respect to $*$, which coincides with the derivation defined on $\mathcal{D}'_{\mathbb{Z}-}$ and $-\delta^{(1)} \in \mathcal{D}'_{\mathbb{Z}-} \setminus \{0\} : X(-\delta^{(1)}) = \delta$.

Then, $\nexists 1 \in \mathcal{F}' \setminus \{0\} : \delta^{(1)} * 1 = 0$.

Proof. It follows from the results obtained in [5] that $(\mathcal{D}'_{\mathbb{Z}-}, +, *, \mathbb{C})$ is an associative convolution algebra over \mathbb{C} .

A. Consider the distribution f satisfying

$$X(\delta^{(1)} * f) + 2f = -\delta^{(1)}. \tag{1}$$

By [3, eq. (67)], f is readily seen to be an associated homogeneous distribution based on \mathbb{R} having degree of homogeneity -2 and order of association 1 , hence $f \in \mathcal{D}'_{\mathbb{Z}_-} \subset \mathcal{F}'$.

On the one hand, applying the derivation X to (1) and using the given property $X(-\delta^{(1)}) = \delta$, we get

$$X^2(\delta^{(1)} * f) = -2(Xf) + \delta. \quad (2)$$

On the other hand, since X is given to be a derivation with respect to the commutative convolution product $*$, we have by Leibniz' rule and due to the property $X(-\delta^{(1)}) = \delta$, that

$$\begin{aligned} X^2(\delta^{(1)} * f) &= (X^2\delta^{(1)}) * f + 2(X\delta^{(1)}) * (Xf) + \delta^{(1)} * (X^2f), \\ &= (X^2\delta^{(1)}) * f - 2(Xf) + \delta^{(1)} * (X^2f). \end{aligned} \quad (3)$$

Since it is given that X is a derivation with respect to the commutative convolution product $*$ and that δ is the $*$ -identity, it holds further by Leibniz' rule that

$$X\delta = X(\delta * \delta) = 2(X\delta) * \delta = 2(X\delta),$$

so $X\delta = 0$. This in turn implies that $X^2\delta^{(1)} = X\delta = 0$. Then, (3) simplifies to

$$X^2(\delta^{(1)} * f) = -2(Xf) + \delta^{(1)} * (X^2f). \quad (4)$$

Combining (2) with (4) gives

$$\delta^{(1)} * (X^2f) = \delta, \quad (5)$$

which shows that X^2f is a convolutional inverse of $\delta^{(1)}$. Consequently, X^2f has degree 0 , so $X^2f \notin \mathcal{D}'_{\mathbb{Z}_-}$. As it is given that X is an automorphism of \mathcal{F}' , we must necessarily have that $X^2f \in \mathcal{F}'$.

B. Since it is given that δ is the $*$ -identity, that $*$ in $(\mathcal{F}', +, *, \mathbb{C})$ is associative, that $\delta^{(1)} * 1 = 0$, and using eq. (5), we obtain

$$1 = 1 * \delta = 1 * (\delta^{(1)} * (X^2f)) = (\delta^{(1)} * 1) * (X^2f) = 0 * (X^2f) = 0.$$

Hence, $\nexists 1 \in \mathcal{F}' \setminus \{0\} : \delta^{(1)} * 1 = 0$. □

Theorem 2.2 does not say that a convolution product of distributions is not possible. It says that a convolution product defined on $\mathcal{D}'_{\mathbb{Z}_-}$, with properties as stated in condition (ii), and a derivation defined on $\mathcal{D}'_{\mathbb{Z}_-}$, with properties as stated in condition (iii), cannot be faithfully prolonged to a superset \mathcal{F}' of $\mathcal{D}'_{\mathbb{Z}_-}$ which contains the one distribution 1 . Notice that \mathcal{F}' is again allowed to be a subset of \mathcal{D}' .

For Schwartz' set of distributions \mathcal{D}' , the convolution defined on $\mathcal{D}'_{\mathbb{Z}_-}$ is not prolonged to \mathcal{D}' , so condition (ii) does not hold, and this allows the existence in \mathcal{D}' of a derivation X and a distribution $1 \neq 0 : \delta^{(1)} * 1 = 0$ (e.g., $1 = X^2(-\pi\eta^{(1)})$).

Let \mathcal{F}' now stand for the set of all finite sums over \mathbb{C} of elements of \mathcal{H}' . We constructed earlier in [5]–[7] the convolution product in the algebra $(\mathcal{F}', +, *, \mathbb{C})$. Clearly, $\mathcal{F}' \supset \mathcal{D}'_{\mathbb{Z}}$. Theorem 2.2 (in particular, part B of the proof) shows that the non-associativity of critical convolution products, obtained in [8] for the set \mathcal{H}' , cannot be avoided.

Theorem 2.2 is for convolution algebras what Schwartz’ theorem is for multiplication algebras. However, since the identity element δ of the convolution product $*$ is a distribution, there are no convolution algebras with $*$ -identity for function sets.

A corollary of Schwartz’ theorem states that the set \mathcal{F}' in his theorem contains elements g that satisfy $Dg = 0$ and for which g is not proportional to 1. We have the following analogue for the convolution product.

Corollary 2.3. *The set \mathcal{F}' in theorem 2.2 contains elements g that satisfy $Xg = 0$ and for which g is not proportional to δ .*

Proof. Consider the distribution h satisfying

$$X(\delta^{(1)} * h) + 2h = 0. \tag{6}$$

By [3, eq. (69)], f is readily seen to be an associated homogeneous distribution based on \mathbb{R} having degree of homogeneity -2 , hence $f \in \mathcal{D}'_{\mathbb{Z}} \subset \mathcal{F}'$.

On the one hand, applying the derivation X to (6) gives

$$X^2(\delta^{(1)} * h) = -2(Xh).$$

On the other hand, since it is given that X is a derivation with respect to the convolution product $*$, we have by Leibniz’ rule and by using the property $X(-\delta^{(1)}) = \delta$, given in theorem 2.2, that

$$X^2(\delta^{(1)} * h) = -2(Xh) + \delta^{(1)} * (X^2h).$$

Combining both results gives

$$\delta^{(1)} * (X^2h) = 0.$$

Theorem 2.2 states that in \mathcal{F}' necessarily must hold that

$$X^2h = 0.$$

However, any homogeneous distribution h of degree -2 is of the form, [4, eq. (14)],

$$h = a\delta^{(1)} + b\eta^{(1)}.$$

We can choose $a = 0$ and $b \neq 0$, use $X\eta^{(1)} = \eta \neq \delta$, which is a particular result of [3, eq. (181)], and so obtain a distribution h for which $g \triangleq Xh$ is not of the form $c\delta$. Hence, Xg can be zero for a distribution g that is not proportional to δ . \square

In the non-associative convolution algebra, developed in [5]–[7], we have due to [3, eq. (181)] that $X^2\eta^{(1)} = -1/\pi \neq 0$.

3 Summary

3.1 Multiplication

On \mathcal{C}^0 , a commutative and associative multiplication \cdot can always be defined. Hence, the operation $X \triangleq x \cdot$ (derivation with respect to \cdot) is always possible on \mathcal{C}^0 . The derivation X can be prolonged to any superset \mathcal{F}' , since multiplication of a distribution by a polynomial is always defined. However, the multiplication \cdot can not be prolonged from \mathcal{C}^0 to \mathcal{F}' with all its properties preserved.

Further, the derivation $D \triangleq \delta^{(1)} \cdot$ is not defined everywhere on \mathcal{C}^0 (regarded as a function space). It is however defined everywhere in the subset of regular distributions generated by $\mathcal{C}^0, \mathcal{D}'_{\mathcal{C}^0}$. Then, the derivation D can be prolonged from $\mathcal{D}'_{\mathcal{C}^0}$ to \mathcal{F}' , since convolution of any distribution with a compact support distribution is always defined.

3.2 Convolution

On $\mathcal{D}'_{\mathbb{Z}_-}$, a commutative and associative convolution $*$ can always be defined. Hence, the operation $D \triangleq \delta^{(1)} *$ (derivation with respect to \cdot) is always possible on $\mathcal{D}'_{\mathbb{Z}_-}$. The derivation D can be prolonged from $\mathcal{D}'_{\mathbb{Z}_-}$ to any superset \mathcal{F}' , since convolution of any distribution with a compact support distribution is always defined. However, the convolution $*$ can not be prolonged from $\mathcal{D}'_{\mathbb{Z}_-}$ to \mathcal{F}' with all its properties preserved.

Further, the derivation $X \triangleq x \cdot$ is defined everywhere on $\mathcal{D}'_{\mathbb{Z}_-}$, since multiplication of any distribution by a smooth function is always defined.

Received: February 2012. Accepted: September 2012.

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