Group Theoretical Approach to Transversal Electromagnetic Polarization in Radiative Transfer Theory

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September 15, 2012

Abstract

The polarization of a light beam, with electric field vector orthogonal to the direction of propagation, is commonly encoded by a 4-component Stokes vector and a polarization altering medium by a Mueller matrix. We show that, based on group theory, many other formulations of polarization are possible. Consequently it follows that with each representation of polarization corresponds a particular form of the equation of radiative transfer.

1 Introduction

Transversal electromagnetic polarization is considered from a mathematical point of view. We assume that the medium in which light travels is linear, isotropic and reciprocal. In this case, the set of polarization altering operations forms a mathematical structure, called a group. The aim of this communication is to exhibit some of the group theoretical aspects underlying transversal electromagnetic polarization.

2 Transversal Electric Field

The simplest case of a transversally polarized electromagnetic field is that of a plane harmonic wave. When traveling in the z-direction, its electric field is of the form

$$\mathbf{E} = A_x \cos\left(kz - \omega t + \varphi_x\right) \mathbf{u}_x + A_y \cos\left(kz - \omega t + \varphi_y\right) \mathbf{u}_y, \tag{1}$$

with amplitudes $A_x \geq 0, A_y \geq 0$ and phases $\varphi_x, \varphi_y \in \mathbb{R}$.

It is convenient to introduce the complex representative ψ of the real field $\mathbf{E}$,

$$\psi = \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix}, \tag{2}$$

with $\psi_x \triangleq A_x e^{i\varphi_x}, \psi_y \triangleq A_y e^{i\varphi_y} \in \mathbb{C}$. The polarization state of the plane wave is encoded in the Jones vector ψ (modulo a scalar phase factor). With any Jones vector ψ, corresponds a unique Stokes vector $S = [I, Q, U, V]^\top$ with components

$$I = \overline{\psi}_x \psi_x + \overline{\psi}_y \psi_y, \tag{3}$$

$$Q = \overline{\psi}_x \psi_x - \overline{\psi}_y \psi_y, \tag{4}$$

$$U = \overline{\psi}_x \psi_y + \overline{\psi}_y \psi_x, \tag{5}$$

$$V = -i\left(\overline{\psi}_x \psi_y - \overline{\psi}_y \psi_x\right). \tag{6}$$

3 The Group $SL(2,\mathbb{C})$

Any change, undergone by the transversal electric field of a plane wave when it travels through a medium, is given by $\psi' = J\,\psi$ or more explicitly

$$\begin{bmatrix} \psi'_x \\ \psi'_y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix} \text{ with } a,b,c,d \in \mathbb{C}, \tag{7}$$

with ψ', ψ Jones vectors and $J \in GL\,(2,\mathbb{C})$ a Jones matrix. Since

$$0 = I'^2 - \left(Q'^2 + U'^2 + V'^2\right) = |\det J|^2 \left(I^2 - \left(Q^2 + U^2 + V^2\right)\right) = 0, \tag{8}$$

we can take $J \in SL\,(2,\mathbb{C}) \triangleq \{M \in GL\,(2,\mathbb{C}) : \det M = 1\}$.

This makes explicit the appearance of the group $SL(2,\mathbb{C})$, which is the double cover group of the identity component of the Lorentz group $SO_+(1,3)$. This now also reveals that (i) Transversal Electromagnetic Polarization, (ii) Special Relativity and (iii) Quantum Physics share a common mathematical foundation.

4 Group Homomorphisms

Now that we have identified that the group $SL(2,\mathbb{C})$ plays a role in the representation of transversal electromagnetic polarization, we can consider its isomorphy groups. Table A lists all the groups that are isomorphic to $SL(2,\mathbb{C})$.

Group	Polarization Space
$Spin_+\,(1,3)$	**2-dimensional quaternion space** $\mathbb{H}^2$
$Spin_+\,(3,1)$	**4-dimensional real space** $\mathbb{R}^4$
$SV\,(2)$	**2-dimensional hyperbolic quaternion space** $(Cl_{2,0})^2$
$Sp\,(2,\mathbb{C})$	**2-dimensional complex space** $\mathbb{C}^2$
$SL\,(2,\mathbb{C})$	**2-dimensional complex space** $\mathbb{C}^2$
$Spin\,(3,\mathbb{C})$	**3-dimensional complex space** $\mathbb{C}^3$

Table 1: A. Isomorphic spinor groups

In addition, there exist 2-to-1 homomorphisms between each row in table A and each row in Table B.

Group	Polarization Space
$SO_+(1,3)$	$R^{1,3}$ **real space with signature (1,3)**
$SO_+(3,1)$	$R^{3,1}$ **real space with signature (3,1)**
$SM\ddot{o}b\,(2)$	**Extended complex plane** $\mathbb{C} \cup \{\infty\}$
CS^2	**Riemann sphere** S^2
$PSL(2,\mathbb{C})$	**Complex projective line** CP^1
$SO(3,\mathbb{C})$	**3-dimensional complex space** $\mathbb{C}^3$

Table 2: B. Isomorphic vector groups

Each group in Table A and B can be used to encode the composition of polarization altering operations. The second column in both tables gives the (fundamental) representation space on which the group acts. The choice of a particular group to represent transversal electromagnetic polarization thus immediately leads to a fundamental representation of the polarization state. E.g., choosing the group $SO_+(1,3)$ (the identity component of the Lorentz group) shows that polarization can be encoded by a vector in $R^{1,3}$ (Stokes vector).

5 Representations

The elements of the majority of the groups considered here can be represented by matrices. These matrices then act on column vectors, the set of which constitute the representation space. Such a representation is called the fundamental representation. Groups can also be given higher order representations, represented by tensors acting on tensors.

For the particular group $SL(2,\mathbb{C})$, (irreducible) representations are usually characterized by two half integer numbers $(n_1/2, n_2/2)$, $n_1, n_2 \in \mathbb{N}$, and which are called spins in the physics literature. Different representations can encode transversal electromagnetic polarization with various degree of generality, as is illustrated by the following examples.

- $(1/2, 0)$: the fundamental representation and which is equivalent to the Jones calculus,
- $(1/2, 0) \oplus (0, 1/2)$: a reducible representation consisting of a direct sum of a Jones and a conjugate Jones calculus,
- $(1/2, 1/2)$: equivalent to the familiar Stokes vectors description together with (a restricted form of) Mueller calculus,
- $(1, 0) \oplus (0, 1)$: a reducible representation that gives a covariant description of electromagnetic polarization (not necessarily transversal) as a direct sum of a self-dual and anti-self dual electromagnetic field,
- $(1, 1)$: a second order traceless tensor representation, which can accommodate higher-order coherence statistics for non-Gaussian light generating processes.

In particular, the representation $(1/2, 0)$ of the group $SL(2,\mathbb{C})$, can only treat fully polarized light, while the representation $(1/2, 1/2)$ of this same group provides a more general calculus that can also handle partially polarized light.

Formulating a calculus of electromagnetic polarization thus requires selecting a group from Tables A and B and selecting a particular group representation (in terms of matrices or tensors), in order to accommodate the generality of the phenomenon that one wishes to describe. Both choices then determine the corresponding representation space, which fixes the resulting mathematical form in which electromagnetic polarization is to be encoded.

6 Consequences for Radiative Transfer theory

The choice of a mathematical formulation of electromagnetic polarization has an influence on the form of the equation of radiative transfer for polarized light. We here concentrate only on its polarization aspects. To illustrate this influence consider the simplest model of radiative transfer: the infinitesimal formulation of the Lambert-Beer law with polarization, which reads in the Stokes-Mueller formalism,

$$\frac{dS}{ds} = -ES, \tag{9}$$

with S : Stokes vector, E : Extinction matrix and s : distance.

The standard method of solving eq. (9) is by exponentiation. Formally,

$$S(s) = \exp(-Es)\ S(0). \tag{10}$$

By giving attention to the underlying mathematics we can now understand eq. (9), or its corresponding form in terms of any other chosen polarization formalism, as the infinitesimal formulation of a group action on a polarization state object. More precisely, we have that:

- E is an element of a Lie algebra,
- $\exp(-Es)$ is an element of a Lie group (from Table A or B).

However, solving eq. (9) in a correct way requires care, as we have to deal with the following complications.

- One has to use non-commutative Lie algebra.
- One has to take into account the topology of the chosen group. E.g., the Lorentz group is a disconnected group with non simply-connected components. In general, a non-trivial topology may prevent us from reaching the correct Lie group element by exponentiation of a Lie algebra element! Fortunately, for the Lorentz group there exist a proof that, despite the non simply-connectedness, guarantees that each element of the identity component can be reached by exponentiation. Still for the Lorentz group, Stokes vector QUV subspace orientation switching media will further require extra attention.
- One can use insight from mathematics and quantum physics to more efficiently perform the exponentiation. E.g., for the Lorentz group we can exploit the following Lie algebra isomorphism $\mathfrak{so}(1,3)\otimes\mathbb{C}\simeq\mathfrak{sl}(2,\mathbb{C})\oplus\overline{\mathfrak{sl}(2,\mathbb{C})}$.

7 Generalizations

The above mathematical treatment of electromagnetic polarization can be generalized in the following ways.

- In order to arrive at the group $SL(2,\mathbb{C})$ in section 3, polarization states only differing in intensity were considered as being the same state. Including intensity in our description leads to the larger group $R_+\times SL(2,\mathbb{C})$, replacing $SL(2,\mathbb{C})$. This in turn leads to different tables, replacing Tables A and B.
- We could more generally start from Mueller calculus with non-singular matrices, in order to identify our first group, instead of starting from Jones calculus as we did above and which has lead us to the group $SL(2,\mathbb{C})$. This way, we would arrive at the much larger group $Cau_+(1,3)$, i.e., the causality group of time-space (from special relativity).
- We can always use higher order representations of the groups involved.
- We assumed reciprocal media, which are characterizable by non-singular matrices. Dropping this condition brings in also the singular matrices and then we no longer have a group. The resulting, more general, mathematical setting that describes electromagnetic polarization in this case is then Monoid Theory, replacing Group Theory.

8 Bibliography

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